FILAMENTATION OF A CONVERGING HEAVY ION BEAM*

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Abstract

A major concern in the use of heavy ion beams as igniters in pellet fusion systems is the vulnerability of the beam to the transverse flamentation instability. The undesirable consequence of this mode is the transverse heating of the beam to the extent that convergence on the pellet becomes impossible. This work considers the case of a beam injected into a gas filled reactor vessel, where finite pulse length and propagation distance play an important role in limiting growth. Two geometries are analyzed: a non-converging case where the radius at injection is nearly equal to the desired radius at the pellet, and a converging case in which the injection radius is large and the beam is pre-focused to converge at the target. It is found that a cold beam will be severely disrupted if the product of the magnetic plasma frequency and the propagation distance is much larger than unity. This product may be lowered by dividing the energy of the original beam into many (≈50) individual beams arranged to converge simultaneously at the pellet, however this represents a significant engineering complication. Even if this product is large, growth may be limited to about six e-foldings if enough transverse velocity spread is added that the latter half of the pulse propagates in pinched equilibrium. The disadvantage of this mode is, however, that much of the pulse is lost to thermal expansion.

Introduction

Among the critical issues confronting the use of heavy ion beams as pellet igniters is whether such a beam can propagate through the reactor vessel to finally achieve a spot size of the order of 1 mm on the pellet. Assuming the beam can be directed to strike the target, this final spot size will not be obtainable if the beam has been subjected to any of various instabilities which transversely heat it at the expense of its longitudinal energy. The most serious of these appears to be the filamentation instability, in which modes of transverse wave number k can grow with characteristic times $\tau_m = 4\pi\sigma/c^2k^2$, where σ denotes the conductivity of the background plasma. The use of a low pressure ($n_g \approx 10^{12}$) vessel environment and/or many (~ 50), simultaneous, low current beams would allow each beam to be magnetically stiff and hence stable over the entire distance of flight. This approach, however, places constraints on the reactor system and suggests the examination of unstable growth in the high pressure ($n_g \approx 10^{16}$) regime. This report is a summary of the results of a more comprehensive treatment given elsewhere[1,2] of filamentation growth in converging and non-converging beams of heavy ions in a background plasma of finite conductivity.

The basic mechanism of the filamentation instability is that a beam which is given small transverse mode structure will separate into small beamlets as the result of the attraction of parallel currents. If the beam is moving in a background plasma such that charge neutralization can occur, each beamlet will continue to self-pinch until its magnetic pressure $B^2/8\pi$ becomes equal to the transverse thermal pressure γn $(1/2\ mv_{th}^2)$. If we define η to be the ratio of thermal pressure to pinch pressure for a filament in the instant just after the perturbation, we conclude that $\eta \approx 1$ should be required for successful propagation. For $\eta < 1$ the filament will pinch at a rate, (Ω) proportional to the square root of the ratio of the magnetic force per unit length to the mass per unit length. A maximum growth rate occurs for a filament whose radius is less than or equal to the magnetic skin depth of the pulse and takes the value $\Omega_{max} = \Omega_b = (4\pi q^2 n/\gamma mc^2)^{1/2}$

where Ω_b is just the magnetic plasma frequency divided by γc. If the beam is to propagate a distance L, then the total number of e-foldings of growth is $\alpha = \Omega_b L$. In light of these observations, we can take α and η as two dimensionless characterizations of the beam and its filamentary tendencies.

Non-Converging Case

In the previous treatment of the non-converging beam[1] the equilibrium distribution function $f_0 = n_0 F(\underset{\sim}{v})\, H(\tau)\, H(\tau_p - \tau)$ and the equilibrium vector potential A_0 are perturbed by the form

$$(f_1,\ A_1) \propto \exp\left[i(\underset{\sim}{k}\cdot\underset{\sim}{r} - \Omega z - \omega\tau\right] = \exp\left[i(\underset{\sim}{k}\cdot\underset{\sim}{r}) + g\right], \tag{1}$$

where z is the longitudinal variable measured from the reactor wall into the chamber and $\tau = t - z/\beta c$ is a convenient transformed time such that $\beta c\tau$ is a longitudinal distance into a pulse of duration τ_p as measured from the head. We have assumed $\underset{\sim}{\nabla} A_0 = 0$ and have let $H(\tau)$ represent the step function. The resulting dispersion relation is

$$\bullet i\omega\tau_m = 1 + \Gamma(k,\Omega) \tag{2}$$

where
$$\Gamma(k,\Omega) = \Omega_b^2 \int d^2\underset{\sim}{v}\ \frac{F(\underset{\sim}{v})}{(\Omega - \underset{\sim}{k}\bullet\underset{\sim}{v}/\beta c)^2}\quad , \tag{3}$$

giving a growth exponent of

$$g(\Omega,\tau,z) = i\Omega z - (\tau/\tau m)\ (1 + \Gamma) \tag{4}$$

Several velocity distributions have been studied, the most convenient being the single pole approximation to the Maxwellian:

$$\Gamma_s \equiv \left(\frac{\Omega}{\Omega_b} + i\delta\right)^{-2} \tag{5}$$

where $\delta(k) = kv_{th}/(\sqrt{2}\,\Omega_b\beta c)$. The dispersion relation becomes

$$i\omega\tau_m = 1 + \left(\frac{\Omega}{\Omega_b} + i\delta\right)^{-2} \tag{6}$$

If τ is held fixed and the growth factor g, given by Eq. 4, is maximized with respect to z we obtain

$$g_{max} = \left(\frac{1}{\delta^2} - 1\right)\frac{\tau}{\tau_m} \leq \frac{\tau_p}{\delta^2\tau_m} \tag{7}$$

It is interesting to note here that the mode number dependencies of τ_m and δ leave g_{max} independent of k at fixed τ. Further, if we assume a parabolic profile for the beam current density J_b and a conductivity independent of r and arrange the thermal velocity so that the beam pinches half-way back from the head (specifically we require that $\eta = 1$ at $\tau = \tau_p/2$), then

$$g_{max} \leq \frac{\tau_p}{\delta^2\tau_m} = \frac{6}{\eta} \tag{8}$$

If $\eta = 1$ we obtain marginally severe growth but have sacrificed the first half of the pulse to rapid expansion.

Converging Beam

We consider next a geometry in which the pulse converges from a large radius at z = 0 to the chosen target radius at z = L. We perturb away from straight line converging particle orbits. The unperturbed system is characterized by $\underset{\sim}{\nabla} A_0 = 0$ and distribution function f_0 which is of finite extent and uniform in the tranverse plane, f_0 is normalized to density n_w at Z = o. The problem of characterizing f_0 is simplified by noting that

$$\underset{\sim}{V} = \underset{\sim}{v}\,(1 - z/L) + \frac{\beta c}{L}\ \underset{\sim}{r} \tag{9}$$

is an integral of the unperturbed motion which displays the assumed convergence at z = L; thus we consider $f_0 = n_w F(\underset{\sim}{V})$. The density at arbitrary z is then

$$n_o(z) = \int d^2 \underset{\sim}{v}\, f_o = \int d^2\underset{\sim}{V}\, n_w\, F(\underset{\sim}{V}) = \frac{n_w}{(1-\frac{z}{L})^2}\int d^2\underset{\sim}{V}\ F\,(\underset{\sim}{V}) = \frac{n_w}{(1-\frac{z}{L})^2}\cdot \tag{10}$$

The mean squared thermal velocity is then

$$v_{th}^2(z) = |\,\underset{\sim}{v} - <\underset{\sim}{v}>|^2 = \frac{1}{(1-\frac{z}{L})^2}\int d^2\underset{\sim}{V}\ V^2 F(\underset{\sim}{V}) = \frac{v_{th}(0)}{(1-\frac{z}{L})^2}\ , \tag{11}$$

where v_{th}^2 (0) is the initial mean squared thermal velocity assumed independent of $\underset{\sim}{r}$.

To parallel previous work we have

$$\beta c \frac{\partial f_1}{\partial z} + \underset{\sim}{v} \bullet \underset{\sim}{\nabla} f_1 = - \frac{q\beta}{\gamma M}\underset{\sim}{\nabla} A_1 \bullet \underset{\sim}{\nabla}_v\ f_o\ , \tag{12}$$

$$\nabla^2 A_1 - \frac{4\pi\sigma}{c^2}\frac{\partial A_1}{\partial \tau} = - \frac{4\pi J_{b1}}{c} = -\ 4\pi q\beta \int d^2\underset{\sim}{v} f_1\ . \tag{13}$$

We proceed in the conventional way by formally solving Eq. (12) for f_1:

$$f_1 = f_{10} - \frac{q\beta}{\gamma M} \int_0^z \frac{dz'}{\beta c} (\nabla A_1 \cdot \nabla_v f_o) \Bigg|_{z'} \tag{14}$$

where f_{10} is the initial disturbance carried along the unperturbed orbits. Recall that both $\underset{\sim}{V}$ and $\underset{\sim}{v}$ were constants of the motion; hence if we introduce $u(z) = (1 - \frac{z}{L})^{-1}$ we can write Eq. (9) as

$$\underset{\sim}{V} = \underset{\sim}{v}/u(z) + \frac{\beta c}{L} \underset{\sim}{r} = \text{constant} \tag{15}$$

Then any point along an unperturbed orbit can be described by

$$\underset{\sim}{r}(z')\, u(z') = \underset{\sim}{r}\,(z)\, u(z) + \frac{L}{\beta c} \underset{\sim}{V}\, [u(z') - u(z)] \tag{16}$$

We then select a perturbation with $\underset{\sim}{r}$ dependence of the form

$$(A_1, J_{b_1}) \propto \exp(i\underset{\sim}{k}\cdot\underset{\sim}{r}u) = \exp\,[i\underset{\sim}{k}\cdot\underset{\sim}{r}_o u + \frac{i\underset{\sim}{k}\cdot\underset{\sim}{V}L}{c} (u' - u)] \tag{17}$$

to obtain the perturbed distribution function with proportionality constants $\overline{A_1}$ and $\overline{J_{b_1}}$:

$$f_1 - f_{10} = -\left(\frac{qn_w L}{\gamma Mc}\, i\underset{\sim}{k} \cdot \frac{\partial F}{\partial \underset{\sim}{V}}\right) \int_1^u \frac{du'}{(u')^2}\, \overline{A}_1'\, \exp\,[iu\underset{\sim}{k}\cdot\underset{\sim}{r} + \frac{i\underset{\sim}{k}\cdot\underset{\sim}{V}L}{\beta c} (u'-u)]\;. \tag{18}$$

The perturbed current is then

$$J_{b_1} - J_{b_{10}} = q\beta c \int d^2\underset{\sim}{v}\ (f_1 - f_{10})$$

$$= - \left(\frac{q^2\beta n_w L u^2}{\gamma m}\right) \int d^2\underset{\sim}{V}\ i\underset{\sim}{k}\bullet \frac{\partial F}{\partial \underset{\sim}{V}} \int_1^u \frac{du'}{(u')^2}\, \overline{A}_1'\, \exp\left[iu\underset{\sim}{k}\bullet\underset{\sim}{r} + \frac{i\underset{\sim}{k}\bullet\underset{\sim}{V}L}{c\beta}(u'-u)\right] \quad (19)$$

We may reverse the order of integration and integrate by parts on $\underset{\sim}{V}$ to get

$$\frac{4\pi}{ck^2u^2}\,(\overline{J}_{b_1} - \overline{J}_{b_{10}}) = \alpha^2 \int_1^u du'\ A'(u-u')\ \hat{F}, \quad (20)$$

where $\alpha^2 = (4\pi q^2 n_w/\gamma M c^2)\ L^2 = \Omega_w^{\ 2} L^2$,

$A = \overline{A}_1/u^2$,

$\underset{\sim}{\lambda} = (u-u')\ \dfrac{L\underset{\sim}{k}}{\beta c}$

$\hat{F} = \int d^2\underset{\sim}{V}\ F(\underset{\sim}{V})\ \exp(-i\underset{\sim}{\lambda}\bullet\underset{\sim}{V})$.

If we now apply the assumed form of Eq. (17) and the definitions above to Eq. (13) we obtain for the perturbed field

$$u^2A + \tau_m \frac{\partial}{\partial\tau}\ A = \frac{4\pi}{ck^2u^2}\,\overline{J}_{b_1} \quad . \quad (21)$$

Finally if we define an amplitude χ proportional to the perturbed density, and a source S as

$$\chi = \frac{4\pi}{ck^2u^2} \bar{J}_{b_1} \text{ and } S = \frac{4\pi}{ck^2u^2} \bar{J}_{b_{10}} ,$$

then Eqs. (20) and (21) can be written as

$$\chi = S + \alpha^2 \int_1^u du' \, A' \, (u-u') \, \hat{F} \quad , \tag{22}$$

$$u^2 A + \tau_m \frac{\partial}{\partial \tau} A = \chi \tag{23}$$

<u>Cold Converging Beam</u>

We first solve Eqs. (22) and (23) for the cold limit, where $F(\underset{\sim}{V}) = \delta(\underset{\sim}{V})$ gives $\hat{F}(\underset{\sim}{\lambda}) = 1$. Then differentiation of Eq. (22) twice gives

$$\frac{\partial^2}{\partial u^2} \chi = \alpha^2 A \quad . \tag{24}$$

If we further let $\tau_m \rightarrow 0$ either because $\sigma \rightarrow 0$ or $k \rightarrow \infty$ then Eq. (23) becomes

$$u^2 A = \chi \quad . \tag{25}$$

The solution is $\chi_{\pm} = u^{\ell}$ where

$$\ell = \frac{1}{2} \pm \left(\frac{1}{4} + \alpha^2 \right)^{1/2} = \frac{1}{2} \pm \left(\frac{1}{4} + (\Omega_w L)^2 \right)^{1/2} \quad . \tag{26}$$

The cold converging beam thus displays algebraic growth with distance rather than the exponential growth found for the straight beam. The magnitudes of growth, however, are quite similar over the distances of interest.

When τ_m is finite growth is reduced. This can be seen by application of the Laplace transform in the variable :

$$\tilde{\chi} = \int_0^\infty \chi \, d\tau \, e^{-p\tau}$$

and similarly for A. The system (22) and (23) combine to give

$$\frac{\partial^2}{\partial u^2} \tilde{\chi} = \frac{\alpha^2}{u^2 + p\tau_m} \tilde{\chi} \quad .$$

The WKB solutions, good for large α, are

$$\tilde{\chi}_\pm = \exp\left(\pm \alpha \int_1^u \frac{du'}{(u'^2 + p\tau_m)^{1/2}} \right) = \left[\frac{u + (u^2 + p\tau_m)^{1/2}}{1 + (1 + p\tau_m)^{1/2}} \right]^{\pm\alpha} . \quad (28)$$

We take the initial conditions to be

$$\chi(u=1) = \begin{Bmatrix} 0 & \tau < 0 \\ 1 & \tau > 0 \end{Bmatrix} \text{ and } \frac{\partial}{\partial u} \chi(u=1) = 0 \quad .$$

We can approximate the inversion integral in the limit of $\frac{u^2 \tau}{\tau_m} \gg 1$ and $\frac{\tau}{\tau_m} \ll 1$ with a saddle analysis to obtain $\chi \propto \exp[g]$ where

$$g \approx \frac{\alpha}{2}\,[1 + \ln\,(\frac{8}{\alpha})\,] + \alpha \ln\left[u\,(\frac{\tau}{\tau_m})^{1/2}\right] \quad . \tag{29}$$

The gross consequences of a non-constant background conductivity may be obtained by using a model in which conductivity rises as the inverse square of beam radius

$$\tau_m = \tau_m(u=1)\,u^2 = \tau_w\,u^2 \quad . \tag{30}$$

This is viewed as an attempt to model generation of conductivity proportional to beam intensity. Eq. (27) becomes

$$\frac{\partial^2}{\partial u^2}\,\tilde{\chi} = \frac{\alpha^2}{u^2(1 + p\tau_w)}\,\tilde{\chi} \;, \tag{31}$$

with WKB solutions

$$\chi_{\pm} \approx \exp\left[\pm\,\frac{\alpha \ln u}{(1 + p\tau_m)^{1/2}}\right] \tag{32}$$

A saddle analysis of inversion integral yields $\chi \propto \exp\,(g)$ where

$$g \approx 3\left[\frac{\alpha}{2}\left(\frac{\tau}{\tau_w}\right)^{\frac{1}{2}} \ln u\right]^{2/3} - \frac{\tau}{\tau_w} \tag{33}$$

which is exactly the cold non-converging result with $\Omega_b z \rightarrow \alpha \ln u$.

Warm Converging Beam

Solution of Eqs. (22) and (23) for arbitrary choices of $F(\underset{\sim}{V})$ is in general quite tedious, but we may obtain the overall features by the use of a single pole form of $F(\underset{\sim}{V})$. This procedure yields

$$\hat{F}_L = \exp\left[- K(u-u')\right] \tag{34}$$

where K is a constant proportional to the thermal velocity. It may be shown that an arbitrary $F(\underset{\sim}{V})$ can be approximated in single pole form with

$$-\frac{1}{K^2} = \frac{\beta^2 c^2}{k^2 L^2} \int d^2\underset{\sim}{V} \, \frac{\underset{\sim}{k} \cdot \frac{\partial F}{\partial \underset{\sim}{V}}}{\underset{\sim}{k} \cdot \underset{\sim}{V}} \tag{35}$$

It is this fact that lends wide applicability to this somewhat unphysical form, for now Eq. (22) can be written as

$$\chi = S + \alpha^2 \int_1^u du' A'(u-u') e^{-K(u-u')} \quad . \tag{36}$$

When the factor exp [K(u-1)] is absorbed into χ, S, and A, the equation returns to its cold form (already solved.) We can write the solutions

$$\chi = \chi_{cold} \, e^{-K(u-1)} \quad . \tag{37}$$

For a Maxwellian profile $K = \alpha\delta$.

In the limit of low conductivity ($\tau_m \to 0$), we use the results of Eq. (26) with α large to get:

$$\chi \simeq u^{\alpha} e^{-\alpha\delta(u-1)} \tag{38}$$

Note that, in contrast to the straight beam case in this limit, the growth here reaches a maximum at $u = \frac{1}{\delta}$. This feature appears consistently in the warm beam analysis and is due to transverse phase mixing of beam particles. As a result we should expect total growth of a warm beam to be somewhat less than the growth of the corresponding cold beam case. For instance, if we consider the case of τ_m finite we can proceed as before with the exception that, following the saddle point evaluation, we maximize the warm growth rate

$$g_{warm} = g_{cold} - \alpha\delta(u-1) \tag{39}$$

with respect to u, k, and τ. We find that growth is a maximum for k=o, u=∞, $\tau=\tau_p$ but with the product uk finite. Defining the quantity $Q = \alpha\eta/24$, we find

$$g_{max} = \frac{6}{\eta} 2Q \left[Q - (Q^2+1)^{1/2} + \ln\left(\frac{(Q^2+1)^{1/2}+1}{Q}\right)\right]$$

$$= \frac{6}{\eta} \begin{cases} 1 - \frac{1}{12Q^2} \text{ for } Q \gg 1 \\ \\ 2Q[\ln(\frac{2}{Q}) - 1] \text{ for } Q \ll 1 \ . \end{cases} \tag{40}$$

Observe that growth is less than the cold beam value of $6/\eta$ for all values of α and η.

Finally, for $\tau_m \propto u^2$ we proceed as before, but find that g_{max} can be written only as an implicit function of Q. Maximum growth still occurs at $\tau = \tau_p$ however, and is found to be everywhere less than that for the converging, constant conductivity case treated above. In particular, for $Q \ll 1$ we can approximate

$$g_{max} \approx \frac{6}{\eta} 2Q \ln(Q^{-1}) \left[1 - \frac{2}{\ln(Q^{-1})} - \frac{\ln\ln(Q^{-1})}{\ln(Q^{-1})}\right] \ . \tag{41}$$

It is obvious that the last two expressions are similar, at least in leading approximation, hence we adopt the general form

$$g_{max} = \frac{6}{\eta} f\ (\alpha\eta/24) \tag{42}$$

where the specific form of f depends on the precise situation. With the above results of Eqs. (40) and (41) we can characterize f in two limits

$$\begin{aligned} f\ (\alpha\eta \rightarrow \infty) &= 1 \\ f\ (\alpha\eta \rightarrow 0) &= \frac{\alpha\eta}{12}\left\{\ln\frac{1}{\alpha\eta} + C\right\} \quad . \end{aligned} \tag{43}$$

The constant C depends on the specific model but can be taken as approximately 2.2. Clearly, a non-converging beam must have $\eta \approx 1$ in order to propagate in equilibrium, giving only a few e-folds of growth during transit. For a converging beam, however, η increases proportional to σ/q^2 where q is the effective ion charge. For the beam to pinch at the pellet, η must be unity there. At the wall then, η must be much smaller and bounded by

$$\eta_w \leq \eta_{pinch} = (\sigma/q^2)_{min}/(\sigma/q^2)_{pellet} \tag{44}$$

where the minimum is taken over the converging profile. There is considerable growth, then, early in transit and it can be approximated when the conductivity ratio is high from Eq. (43):

$$g_{max} = \frac{\alpha}{2}\left\{\ln\left[\frac{(\sigma/q^2)_{pellet}}{(\sigma/q^2)_{min}}\right] + 2.2\right\} \tag{45}$$

Values of α less than unity could not yield large effects under extreme assumptions on conductivity development. Conversely it seems unlikely that $\alpha > 5$ could be tolerated even if σ/q^2 varied by only a factor of 10. One suspects that $\alpha \approx 3$ represents an effective bound for the converging beam. From our definition of $\alpha = \Omega_b L$ we should expect, then, a maximum transported energy of

$$W \leq (49\ \text{MJ}) \left(\frac{A}{Z}\right)^2 \beta^3 \left(\frac{\tau_p}{1\ \text{ns}}\right) \left(\frac{R_o}{L}\right)^2 , \qquad (46)$$

where A and Z are the atomic mass number and stripped charge for the beam particles, R_o is the rms radius of the beam at z=o and L is the chamber radius. The beam is assumed to have a parabolic profile with edge $a_o = \sqrt{3}\, R_o$ and central density $n = 2I_b/q\beta c\pi a^2$. If we take τ_p = 10 ns, A/Z = 4, $R_o/L = 10^{-2}$, and $\beta = 0.3$ we find

$$W \leq 0.021\ \text{MJ} \quad .$$

A total of 50 beams would be required to put 1 MJ on the target placing formidable complications in the way of reactor system design.

References

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