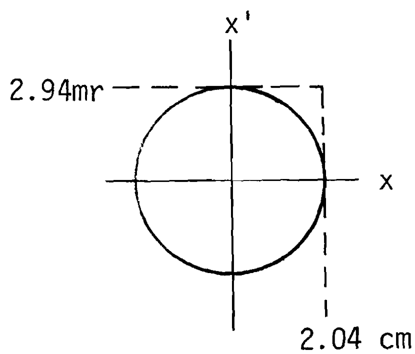
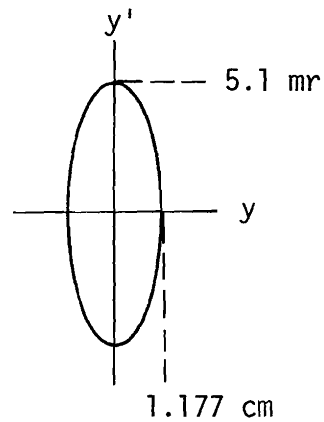
CALCULATIONS OF MAJOR 3RD ORDER GEOMETRIC ABERRATIONS FOR FINAL TRANSPORT LINE

Eugene Colton
Argonne National Laboratory

Karl Brown and Jack Peterson have presented a Final Transport Line for 10 GeV U^{+4}. The system consists of three 1/2 wave quadrupole sections and three dipoles. Furthermore, the system has been chromatically corrected to second-order utilizing two families of sextupoles.

The system was designed to produce a final 4mm radius spot in the center of a 5m radius reaction chamber for a beam with a geometrical emittance of 60×10^{-6} π m-rad in both transverse phase planes; the starting ellipses in the phase planes are upright, viz





$x_o = 2.04$ cm $\qquad y_o = 1.177$ cm

$x'_o = 2.94$ mr $\qquad y'_o = 5.1$ mr

The final transfer matrices, e.g. $\begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$

are $\begin{pmatrix} -0.197 & 1.11 \times 10^{-3} \\ 1.41 \times 10^{-2} & -5.07 \end{pmatrix}$ in the x plane (Horiz.)

$$\text{and} \quad \begin{pmatrix} -0.3413 & 1.6 \times 10^{-4} \\ 7 \times 10^{-3} & -2.93 \end{pmatrix} \quad \text{in the y plane (Vert.).}$$

where the primes denote differentiation with respect to s. So the system is point to point and waist to waist ($S = C' = 0$).

The third-order aberration coefficients for the system can be calculated by integrating the first-order functions over the transport.

An aberration coefficient for a quadrupole system is given by (see Steffen p. 52)

$$q = S(s_e) \int_0^{s_e} C(s)\, f(s)\, ds - C(s_e) \int_0^{s_e} S(s)\, f(s)\, ds \tag{1}$$

where s_e is the end of the beam (the central trajectory propagates in the +s direction), and f is the driving coefficient for the aberration . Since the system is point to point ($S(s_e) = 0$), we rewrite Eq. 1

$$q = -C(s_e) \int_0^{s_e} S(s)\, f(s)\, ds \tag{1'}$$

The Transport run indicates that the large beam envelopes occur at very large S values (i.e., are built up only by the initial angles), therefore, we only consider the aberrations due to the initial angles x_0' and y_0' and not those involving the initital beam sizes x_0 and y_0. The problem is reduced to a modest exercise of evaluating four aberrations instead of twenty.

The terms are

$$(x|x_0'^3) = -C_x(s_e)\, \frac{3}{2} \int_0^{s_e} S_x^{\,2} \left(KS_x'^2 - \frac{1}{18} K'' S_x^{\,2}\right) ds$$

$$(x|x_0' y_0'^2) = -C_x(s_e) \int_0^{s_e} S_x \left[\frac{S_x}{2} \left(KS_y'^2 - \frac{1}{2} K'' S_y^{\,2}\right) - S_y S_y' \left(KS_x' + K' S_x\right)\right] ds$$

$$(y|y_0'^3) = C_y(s_e)\, \frac{3}{2} \int_0^{s_e} S_y^{\,2} \left(KS_y'^2 - \frac{1}{18} K'' S_y^{\,2}\right) ds$$

$$(y|x_0'^2 y_0') = C_y(s_e) \int_0^{s_e} S_y \left[\frac{S_y}{2} \left(KS_x'^2 - \frac{1}{2} K'' S_x^{\,2}\right) - S_x S_x' \left(KS_y' + K' S_y\right)\right] ds$$

where K = 0 in a field free region

and $K = \frac{g}{B\rho}$ in a quadrupole where g is the field gradient.

we integrate the terms involving K" once by parts and assume K' = 0 at s = 0 and at s_e

$$\int_0^{s_e} K'' S_x^4 \, ds = -4 \int_0^{s_e} K' S_x^3 S_x' \, ds$$

$$\int_0^{s_e} K'' S_x^2 S_y^2 \, ds = -2 \int_0^{s_e} K' (S_x S_x' S_y^2 + S_x^2 S_y S_y') \, ds$$

The final expressions for the aberration coefficients

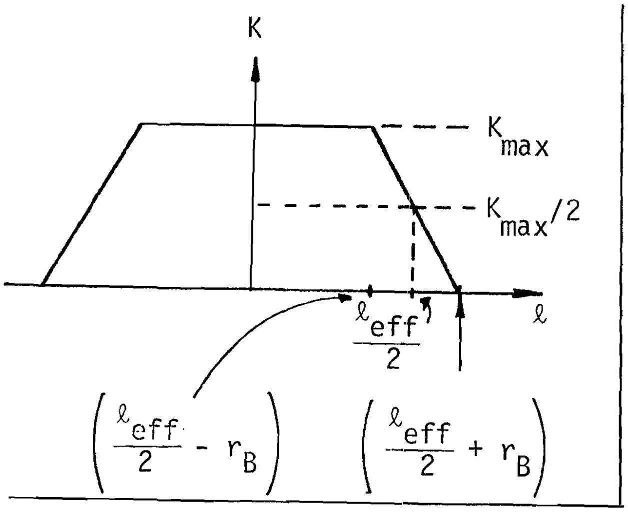
$$(x|x_o'^3) = -\frac{3}{2} C_x(s_e) \int_0^{s_e} [K S_x^2 S_x'^2 + \frac{2}{9} K' S_x^3 S_x'] \, ds \tag{2}$$

$$(x|x_o' y_o'^2) = -\frac{1}{2} C_x(s_e) \int_0^{s_e} [K(S_x^2 S_y'^2 - 2S_x S_x' S_y S_y') + K'(S_x S_x' S_y^2 - S_x^2 S_y S_y')] \, ds \tag{3}$$

$$(y|y_o'^3) = \frac{3}{2} C_y(s_e) \int_0^{s_e} (K S_y^2 S_y'^2 + \frac{2}{9} K' S_y^3 S_y') \, ds \tag{4}$$

$$(y|x_o'^2 y_o') = \frac{1}{2} C_y(s_e) \int_0^{s_e} [K(S_x'^2 S_y^2 - 2S_x S_x' S_y S_y') + K'(S_x^2 S_y S_y' - S_x S_x' S_y^2)] \, ds \tag{5}$$

Equations (2)-(5) are integrated to give the 3rd order aberrations - contributions are non-zero only within the magnetic elements. We ignore the three dipoles of the system and integrate over the twelve quadrupoles. For simplicity we approximate the fringe field behavior of a quadrupole by linear functions i.e., the K value is assumed to behave as shown



The gradient is constant with value K_{max} for $|\ell| < \frac{\ell_{eff}}{2} - r_B$ where ℓ_{eff} = quadrupole effective length and r_B = bore radius and the gradient changes linearly when

$$\frac{\ell_{eff}}{2} - r_B < |\ell| < \frac{\ell_{eff}}{2} + r_B$$

Of course we maintain the same integrated strength as for a square edge magnet:

$$\int_{-\infty}^{\infty} K d\ell = K_{max}\,\ell_{eff} = K_{max}\,[(\ell_{eff} - 2r_B) + \frac{1}{2}(2r_B + 2r_B)],$$

so in the central regions $K = K_{max}$, $K' = 0$ and in fringe regions

$$K' = -\frac{K_{max}}{2r_B}\,\frac{\ell}{|\ell|}$$

$$K = \frac{K_{max}}{2r_B}\,[(\frac{\ell_{eff}}{2} + r_B)] + \ell K'$$

Each quadrupole was subdivided into fifty intervals and Eqs. (2)-(5) were evaluated numerically for an interval

$$\Delta + (\ell_{eff} + 2r_B)/50 \qquad \text{and} \qquad \phi = \Delta\sqrt{K}$$

The S and S' functions were taken through each section of quadrupole by the standard transform

$$\begin{pmatrix} S \\ S' \end{pmatrix} = \begin{pmatrix} \cos\phi & \frac{\sin\phi}{\sqrt{K}} \\ -\sqrt{K}\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} S_0 \\ S'_0 \end{pmatrix} \qquad \text{for a focussing quad}$$

$$\begin{pmatrix} S \\ S' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \frac{\sinh\phi}{\sqrt{K}} \\ \sqrt{K}\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} S_0 \\ S'_0 \end{pmatrix} \qquad \text{for a defocussing quad}$$

where the starting values of the S and S' functions were taken from the TRANSPORT output at $\ell = -\ell_{eff}/2$ and transformed back to $\ell = -(\ell_{eff}/2 + r_B)$.

Equations (2) to (5) reduce to a summation over the contributions from twelve quadrupoles

$$q = \sum_{i=1}^{12} \Delta q_i \qquad \text{for each aberration}$$

The contributions from each quadrupole to the four aberrations, as well as the final results are listed in Table II. The coefficients are small as calculated by this method - the major contributors to $(x|x_0'^3)$ are quadrupoles 3, 6 and 11 where the beam is 50 cm wide horizontally. For the extreme rays

$x'_0 = 2.94$ mrad, $\qquad y'_0 = 5.1$ mrad

$$\Delta x_1 = (x|x'^3_0)\, x'^3_0 = 4.67 \text{ mm}$$

$$\Delta x_2 = (x|x'_0 y'^2_0)\, x'_0 y'^2_0 = 1.59 \text{ mm}$$

$$\Delta y_1 = (y|y'^3_0)\, y'^3_0 = 0.31 \text{ mm}$$

$$\Delta y_2 = (y|x'^2_0 y'_0)\, x'^2_0 y'_0 = 1.58 \text{ mm}$$

In view of the expected distribution functions, most x'_0, y'_0 values will be small; since the aberration contributions are cubic in angle the geometric aberrations cause a negligible increase in final focus size. The results as found in this quick analysis should be verified with a ray tracing program.

TABLE I. QUADRUPOLE DATA

QUADRUPOLE	ACTION	g(T/m)	ℓ_{eff}(m)	r_b(m)
1	HF	1.8966	1.0	0.15
2	VF	-3.6847	1.0	0.15
3	HF	5.8519	2.0	0.3
4	VF	-7.1923	2.0	0.3
5	VF	-7.1923	2.0	0.3
6	HF	5.8519	2.0	0.3
7	VF	-3.6847	1.0	0.15
8	HF	1.8966	1.0	0.15
9	HF	1.8097	1.0	0.15
10	VF	-3.5971	1.0	0.15
11	HF	5.8879	2.0	0.30
12	VF	-7.8457	2.0	0.30

TABLE II. ABERRATION CONTRIBUTIONS

	ABERRATION $(mm/(mr)^3)$			
QUADRUPOLE	$(x\|x_o'^3)$	$(x\|x_o'y_o'^2)$	$(y\|y_o'^3)$	$(y\|x_o'^2y_o')$
1	$2.08x10^{-8}$	$-0.73x10^{-8}$	$-3.33x10^{-8}$	$+6.0x10^{-8}$
2	$1.414x10^{-4}$	$6.40x10^{-4}$	$+4.73x10^{-4}$	$+1.17x10^{-3}$
3	$5.88x10^{-2}$	$3.79x10^{-3}$	$4.59x10^{-5}$	$+5.45x10^{-3}$
4	$-7.654x10^{-5}$	$2.20x10^{-3}$	$+2.37x10^{-4}$	$+4.78x10^{-3}$
5	$-0.30x10^{-3}$	$2.18x10^{-3}$	$+2.36x10^{-4}$	$+4.85x10^{-3}$
6	$5.82x10^{-2}$	$3.78x10^{-3}$	$+4.63x10^{-5}$	$+5.52x10^{-3}$
7	$1.428x10^{-4}$	$6.41x10^{-4}$	$+4.73x10^{-4}$	$+1.16x10^{-3}$
8	$2.099x10^{-8}$	$-0.73x10^{-8}$	$-3.31x10^{-8}$	$+6.02x10^{-8}$
9	$1.976x10^{-8}$	$0.76x10^{-8}$	$-3.18x10^{-8}$	$+5.61x10^{-8}$
10	$1.467x10^{-4}$	$6.54x10^{-4}$	$+4.77x10^{-4}$	$+1.19x10^{-3}$
11	$6.657x10^{-2}$	$4.16x10^{-3}$	$+4.88x10^{-5}$	$+5.94x10^{-3}$
12	$1.511x10^{-4}$	$2.73x10^{-3}$	$+3.03x10^{-4}$	$+5.79x10^{-3}$
TOTAL	0.1839	$2.078x10^{-2}$	$2.34x10^{-3}$	$3.585x10^{-2}$