# STABILITY OF LONGITUDINAL MOTION IN INTENSE ION BEAMS

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Inertial confinement fusion using high energy heavy ion beams requires focussing of the igniting ion beams in longitudinal, as well as transverse, space at the pellet target. The focussing requirements set limits on the size of the beam emittances at the target, and obtaining sufficiently small emittances at the target requires sufficient stability in beam transport and acceleration from source to target, and an analysis of that stability is necessary for heavy ion fusion (HIF) accelerator design. Theoretical analysis is necessary since practical accelerator experience with high intensity non-relativistic ion beams has been limited. This analysis is particularly important for the case of a heavy ion induction linac, since previous induction linacs have been electron accelerators, and the highly relativistic electrons have negligible longitudinal motion. In this paper we present some results of our analysis of the of the stability of longitudinal motion.

## I. Equations of Motion

The equations of longitudinal motion which we use are obtained by solution of Maxwell's equations with simplifying assumptions. We assume that the transverse (x-y) and longitudinal (z) motions of particles in the beam are completely decoupled with the beam length much greater than the beam radius. We choose the longitudinal distance from the center of the bunch z and the position of the center of the bunch s as the dependent and independent variables. We will assume the motion is non-relativistic and that the center of the beam bunch is not accelerating but moves with constant speed $\beta c$. If the beam pipe is perfectly conducting, we find the following equation of motion (in MKS units):

$$\frac{d^2z}{ds^2} \cong z'' = -\frac{q^2e^2}{M\beta^2c^2}\,\frac{g}{4\pi\varepsilon_o}\,\frac{d\lambda}{da} + \frac{qe}{M\beta^2c^2}\,E_z \qquad (1)$$

where e is the proton charge, q is the ion charge state, M is the ion mass, $\lambda$ is the number of ions per unit length, and g is a geometric factor of order unity. For the particular case of an ion at the center of a constant transverse density round beam of radius a inside a round pipe of radius b, $g = 1 + 2\,\ell n\,(b/a)$. We assume that transverse variations simply produce some average g, which we treat as constant. In equation 1, we have added an external bunching field $E_z$ to the space charge self-field.

Analysis of design studies of HIF accelerators indicates that the assumption of perfectly conducting walls may not be adequate. If we assume a resistive coupling per meter R', a term

$$\left.\frac{d^2z}{ds^2}\right|_{\text{resistive}} = -\frac{q^2e^2}{MBc}\,\lambda\,R' \qquad (2)$$

must be added to equation 1. In sections II and III we will assume that the walls are perfectly conducting (R' = 0) and in later sections we will consider the effects of non-zero R'.

## II. Envelope Equation for Longitudinal Motion

Unperturbed longitudinal motion of a beam bunch through a transport system can be calculated using the envelope equation derived before.[1] This envelope equation applies to a bunch transported through a system with linear bunching fields; that is

$$E_z(s,z) = \frac{dE}{dz}(s)\,.\,z \qquad (3)$$

The equation of motion (1) is rewritten as:

$$z'' = \frac{qe}{M\beta^2c^2}\,\frac{dE(s)}{dz}\,z - \frac{q^2e^2}{M\beta^2c^2}\,\frac{g}{(4\pi\varepsilon_o)}\,\frac{\partial\lambda}{\partial z} \qquad (4)$$

$$\equiv -K(s)\quad z - A\,\frac{\partial\lambda}{\partial z}$$

The particle distribution which is a solution to the Vlasov equation with this equation of motion is:

$$f(z, z', s) = \frac{3N}{2\pi\epsilon_L} \sqrt{1 - \frac{z^2}{z_o^2} - \frac{z_o^2}{\epsilon_L^2}\left(z' - \frac{z_o'}{z_o} z\right)^2} \quad (5)$$

defined wherever the square root is real (f = 0 otherwise), and where N is the total number of ions in the bunch, $\epsilon_L$ is the longitudinal emittance, and $z_o$ is the envelope amplitude. This distribution has a parabolic particle density:

$$\lambda(z, s) = \int f(z, z's)dz' = \frac{3N}{4z_o}\left(1 - \frac{z^2}{z_o^2}\right) \quad (6)$$

and $z_o$ is a solution of the envelope equation:

$$z_o'' = \frac{d^2 z_o}{ds^2} = \frac{\epsilon_L^2}{z_o^3} + \frac{3}{2}\frac{AN}{z_o^2} - K(s)\, z_o \quad (7)$$

where the initial conditions ($z_o$ (s=0, $z_o$(s=0)) may be chosen arbitrarily.

This solution can be, and has been, used to check computer programs which integrate the Vlasov equation numerically, such as the code of Neil, Buchanan, and Cooper.[2]

An analysis of perturbations of this distribution can also be used to evaluate longitudinal transport stability analytically.

## III. Stability of Space Charge Perturbations

Following techniques previously developed by L. Smith and others for analyzing transverse stability, [3, 4, 5] and analysis of the stability of space charge perturbations of the distribution of section II has been presented and in this section we summarize the results of the analysis.[6]

We first consider the case of the stationary distribution, the particular solution of equations 4–7 in which K(s) is constant, and $z_o$ is chosen such that $z_o''(0) = z_o'(0) = 0$. Our unperturbed distribution is ($v \equiv z'$):

$$f_o(z,v) = \frac{3N}{2\pi v_o z_o} \sqrt{1 - \frac{z^2}{z_o^2} - \left(\frac{v}{v_o}\right)^2} \quad (8)$$

Stability is determined by adding a small perturbation $f_p$ (z, v, s) to $f_o$ (z, v) and solving the linearized Vlasov equation for $f_p$ (z, v, s) and $\lambda_p$ (z, s) with our solutions of the form

$$f_p(z, v, s) = f_n(z, v)\, e^{-i\omega_n s}$$

$$\text{and} \quad \lambda_p(z, s) = \lambda_n(z) e^{-i\omega_n s} = \int f_n(z,v)\, dv\, e^{-i\omega_n s} \quad (9)$$

Instability exists where $Im(\omega) \neq 0$. As reported in reference 6, the solutions have the following properties:

$$\lambda_n(z) \propto P_n\left(\frac{z}{z_o}\right) \quad \text{(Legendre polynomials)}$$

and $\omega_n$ is a solution of

$$\frac{\nu^2 + \omega p^2}{\omega_n^2} = \omega_p^2 \sum_{m=-n}^{n} \frac{1}{4^n} \binom{2m}{m}\binom{2n-m}{n-m} \times$$

$$\frac{1}{\omega^2 n - ((n-2m)\nu)^2} \quad (10)$$

with $\omega_p^2 \equiv \frac{3}{2}\frac{AN}{z_o^3}$, $\nu^2 \equiv K$.

It can be shown that all $\omega_n$ which are solutions of (10) are real, which indicates that small space charge perturbations are stable, unlike the transverse case.[4]

The analysis has been extended to the case where K(s) is periodic, and it is found that instabilities can exists where the eigenfrequency of the normal mode $\omega_n$ and the period of K(s) are near resonance.

The largest such resonances are:

1. A second order resonance (n = 2) which can occur if the phase shift of individual particle longitudinal motion over a period of K(s) is between 90° and 104° at zero current. This has a growth rate of ~ 1.1 per period.

2. A fourth order resonance ($n = 4$) which can occur for longitudinal phase shifts between 45° and 57° per period. The growth rate is ~ 1.03.

For most accelerators considered to date, such as the HIF induction linac, the longitudinal phase shift per period of structure is quite small, so periodic space charge instabilities can occur only in very high order n and the analysis indicates that these instabilities become vanishingly small. The only possible exception proposed to date would be a bunching ring with a very large ($\gtrsim 30°$) longitudinal phase shift per turn at peak field. Such bunching rings should be designed to avoid the largest low order resonances.

## IV. Resistive Wall Instability

Faltens[7] has suggested that particle motion, particularly in an induction linac, may show significant resistive coupling. In that case the equations of motion are modified as described in section 1. The self forces are given by

$$z'' = -\frac{q^2e^2}{M\beta^2c^2}\frac{g}{4\pi\epsilon_0}\frac{\partial\lambda}{\partial z} - \frac{q^2e^2R'}{M\beta c}\lambda$$

$$\equiv -A\frac{\partial\lambda}{\partial z} - B\lambda$$

The value of R' depends upon the current, acceleration and efficiency requirements of the induction linac; Faltens[8] suggests that for HIF it will be of the order of 100 Ω/m.

To show the effects of resistive coupling we use an approximate analysis previously presented by L. Smith.[9] We start with an unperturbed distribution with constant density in z and with a step function in z':

$$f_0(z, z') = \frac{N'}{2\Delta}\Big(S(z' + \Delta) - S(z' - \Delta)\Big)$$

and consider perturbing waves of the form

$$f_1(z, z's) = f_1(z')\, e^{i(kz-\omega s)} \qquad (11)$$

We then solve the linearized Vlasov equation to find $\omega(k)$.

$$-i(\omega - kz')\, f_1(z') + \frac{\partial f_0}{\partial z'}(-iAk - B)\int f_1(z')dz' = 0 \qquad (12)$$

The result is: $\omega^2 = k^2\Delta^2 + k^2AN' - ikBN'$

or

$$\omega^2 = k^2\Delta^2 + k^2\frac{N'q^2e^2}{M(\beta c)^2}\frac{g}{4\pi\epsilon_0} - i\,k\,\frac{q^2e^2}{M\beta c}N'R' \qquad (13)$$

Instability can occur since $\mathrm{Im}(\omega) \neq 0$.

With parameters suitable for fusion induction linacs, the waves of equations 11-13 show some interesting characteristics:

1. The requirement of small energy spread for final focussing sets $\Delta$ quite small, so that in equation 13 the velocity dependent term $k^2\Delta^2$ is negligible to a first approximation. As a corollary to this, the wave velocity ($\mathrm{Re}(\omega/k)$) of disturbances in the bunch is much greater than individual particle velocities.

2. The space charge term of equation (13) $k^2AN'$ is usually larger than the resistive term $ikBN'$ in absolute value, if $R' < 200$ Ω/m. With this approximation, we have $\mathrm{Re}(\omega) \cong \pm\sqrt{k^2AN'}$ and we find that the wave velocity ($\mathrm{Re}(\omega/k)$) is independent of k, so that propagating wave packets do not disperse but travel together coherently.

3. Also with space charge dominant we have the relation

$$\mathrm{Im}(\omega) \cong \frac{1}{2}\frac{k\,B\,N'}{|k|\sqrt{A\,N'}} \cong -\frac{k}{|k|}\frac{1}{2}\,qeR'\sqrt{\frac{4\pi\epsilon_0 N'}{g\,M}}$$

so that the magnitude of the growth parameter is independent of k. Waves change in amplitude as $e^{\mathrm{Im}(\omega)s}$ and the sign of

$Im(\omega)$ is correlated with $Re(\omega/k)$ so that for $Re(\omega/k) > 0$ ("fast" wave) we have $Im(\omega) > 0$ and the wave decays, while with $Re(\omega/k) < 0$ ("slow" wave) we have $Im(\omega) > 0$ and the wave grows.

Typical parameters for HIF can be substituted into equation (13) to find sample values of $Re(\omega/k)$ and $Im(\omega)$. For example, with $R' = 100\,\Omega/m$, $N' = 3 \times 10^{13}$ ions/m., $q = 4$, $g = 2$, $M = 238\ m_p$, and $\beta = .33$, we find

$$\begin{aligned} &1)\quad Re(\tfrac{\omega}{k}) = 7.4 \times 10^{-3} \\ &2)\quad |Im(\omega)| = 2 \times 10^{-3}\ m^{-1} = (500\ m)^{-1} \end{aligned} \qquad (14)$$

The growth distance is about 500 m., which is less than the total length of the HIF induction linac (a few km.), but it is a substantial fraction of it.

This wave motion in a beam bunch can be simulated numerically. In figures 1 we show wave propagation in a perturbed beam bunch, calculated using the program of Neil, Buchanan and Cooper, which numerically integrates the longitudinal Vlasov equation. In this case an initial disturbance at the center of the bunch splits into forward-going "fast" and backward "slow" wave packets which decay and grow respectively. The behavior agrees closely with equation (13) and the discussion above.

## V. Effect of the Resistive Coupling on Beam Stability

In the previous section, we demonstrated that a resistive coupling can lead to growth of perturbations in a beam bunch with HIF parameters. We need to determine the amount of growth by resistive coupling which can be permitted without endangering HIF performance. To estimate this, we must include the effects of the finite bunch size, which means that a propagating disturbance will reach the end of the bunch in a finite time.

A naive expectation is that a growing "slow" wave will reach the end of the bunch, be immediately reflected to a decaying "fast" wave by the external bunching field, and therefore produce no net instability. With this expectation, we can set a limit on the allowable growth by requiring that an individual wave packet not grow by more than some factor F in traversing the bunch length $L_B$. This requirement can be written as:

$$G = \left| Im(\omega) \cdot \frac{L_B}{Re(\omega/k)} \right| \cong \frac{4\pi\varepsilon_0}{2g}\, \beta c R'\ L_B < \ell n\ |F| \qquad (14)$$

For the sample case of section 4, with $L_B = 10$ m., we have $G \cong 2.5$ or $F > 12$. This amount of growth may be tolerable provided that initial wave packet perturbations are limited to a few per cent.

Numerical simulation seems to indicate that longitudinal motion does not fit this naive picture. To observe wave packet reflection at the bunch end, we calculate a case with $R' = 0$ so that waves neither grow nor decay. In this example (shown in figures 2) the disturbances propagate to the bunch ends from the center in about 800 m., then remain localized at the ends for 800 m. while particle motions reverse, and then propagate back toward the center. Wave packet reflection is substantially delayed.

The same type of delayed reflection exists in numerical simulation with $R' \neq 0$. However substantial wave packet distortion occurs on reflection and this distortion is not fully understood. Future analysis will attempt to understand this reflection distortion, and to determine its importance in describing the stability of longitudinal transport.

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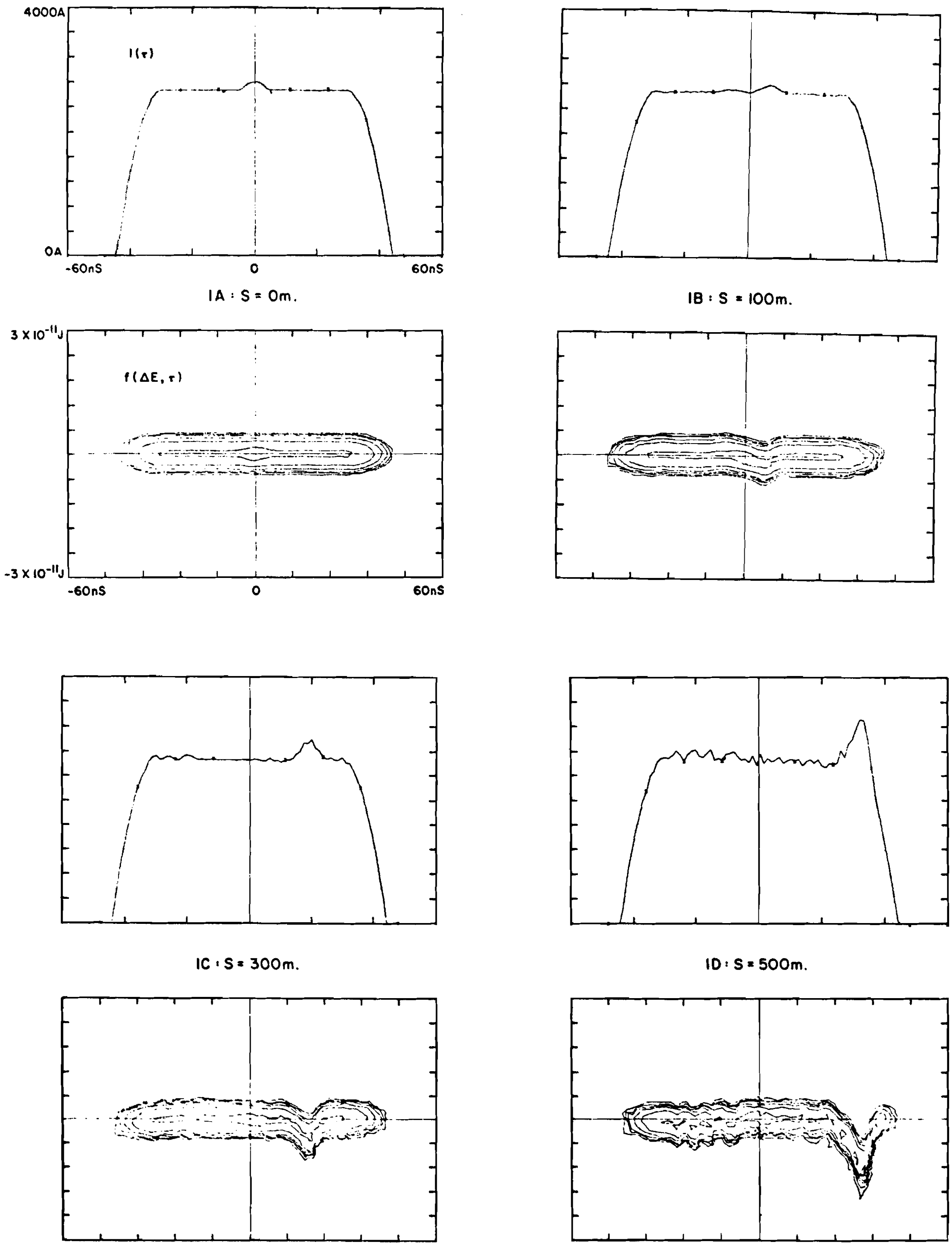
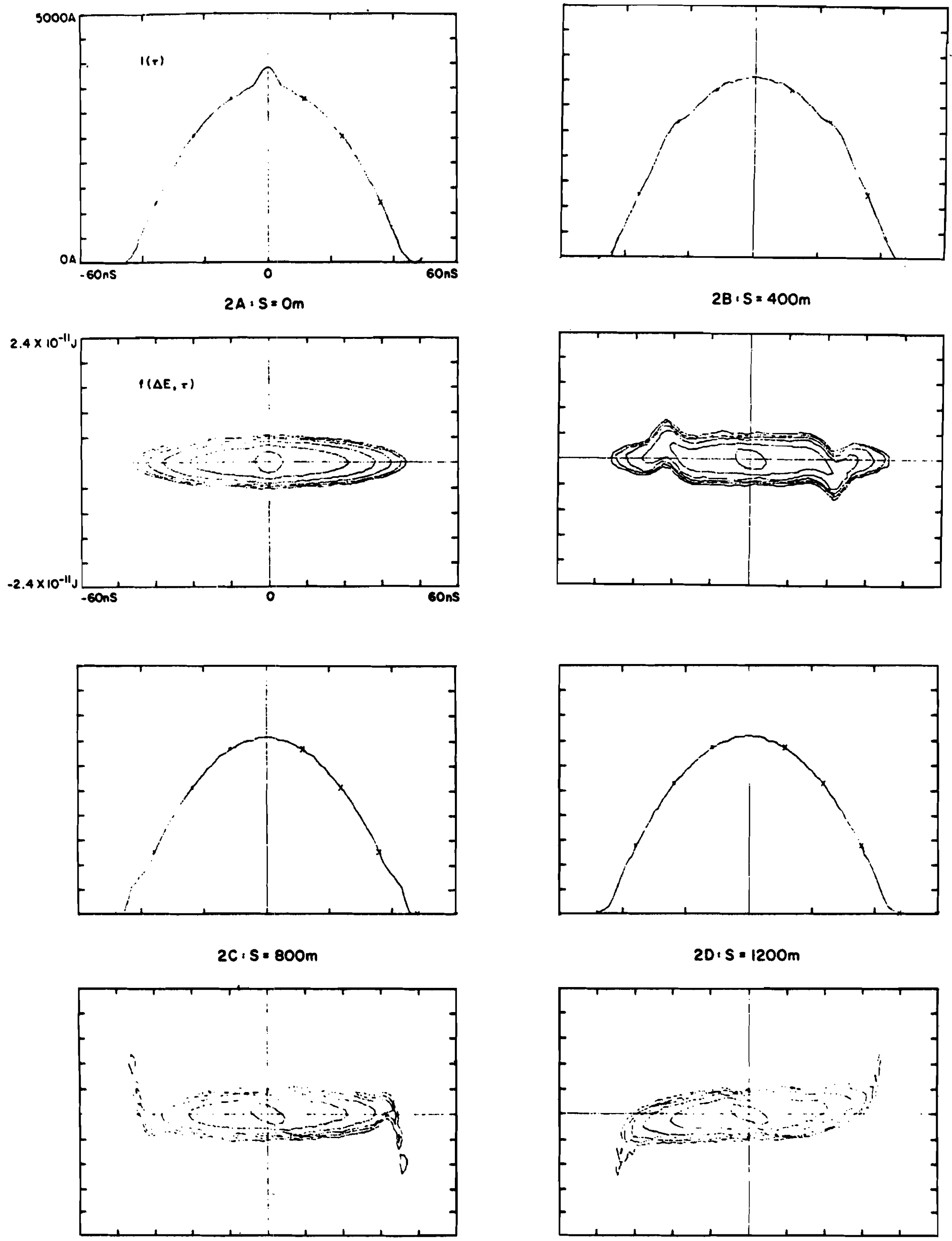
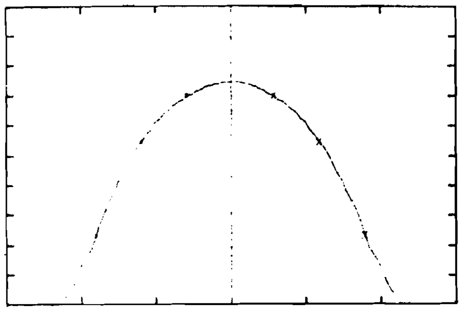
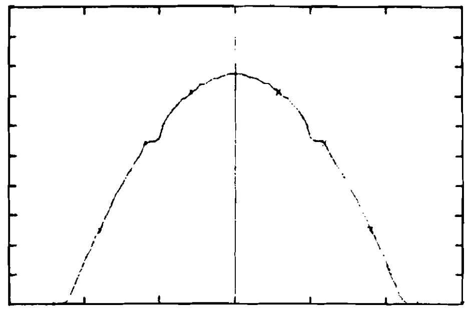



1:A–D. Wave propagation in a perturbed beam bunch. In this case we have R' = 200 Ω/m., $N'_{max}$ = 3 x $10^{13}$ ions/m., q = 4, g = 2, M = 238 $m_p$, and β = .35. An initial disturbance shown in A (s = 0 m.) separates into "fast" and "slow" waves at B (s = 100 m.) with the "fast" wave rapidly decaying and the "slow" wave growing at C (s = 300 m.). At D (s = 500 m.) the waves have reached the ends of the bunch.



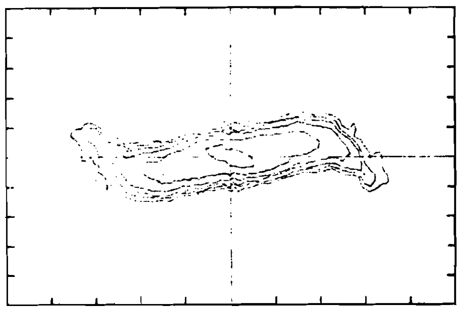

2:A–F Wave propagation and reflection with R' = 0. The other parameters (N', q, g, M, β) are the same as in Figures 1, A–D, except the beam bunch is parabolic. In this case the "fast" and "slow" waves travel to the ends of the bunch from s = 0 m., to s = 800 m. (A, B, C). From s = 800 to s = 1600 (C, D, E) the beam bunch reflects the distrubance. The reflected waves (reversed in sign and direction) appear clearly in figure F (s = 2000 m.). Reflection is not instantaneous in any usual approximation.

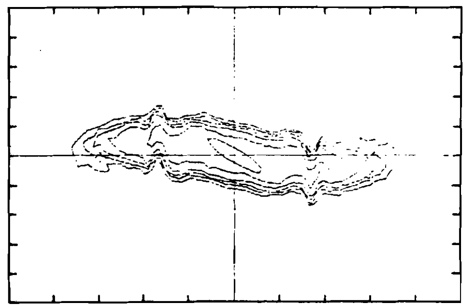




2E : S = 1600m

2F : S = 2000m





Each of these figures (1(A–D), 2(A–F)) contains two plots. The upper plots graph the current I as a function of position $\tau$, where $\tau = -\beta cz$. The lower plots are contour plots of the distribution function $f(\Delta E, \tau)$ which is proportional to $f(z', z)$. In both plots the horizontal axis is position $\tau$.