LONGITUDINAL DYNAMICS IN HEAVY-ION INDUCTION LINACS FOR INERTIAL-FUSION POWER PLANTS

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I. INTRODUCTION

All induction linacs thus far constructed have accelerated relativistic electrons. Because these move with almost the speed of light, there were few longitudinal dynamics problems to solve. In contrast, heavy-ion induction linacs for inertial-fusion power plants will handle ions in the range $0.1 \lesssim \beta \lesssim 0.5$ ($\beta = v/c$); the controlled variation of bunch length during acceleration is not only a very useful new property but also introduces a new set of problems.

In this paper we first point out that the requirements for pellet ignition in commercial-scale power plants, and the limitations of final focusing lens systems, place stringent bounds on the allowable random spread of longitudinal velocities, or equivalently on the longitudinal "temperature". We evaluate these bounds; the result is that the shapes of the applied voltage pulses must, on the average, fit those required to compress the bunches and cancel their space-charge repulsion like a glove fits a hand, within very tight tolerances.

Next we show that at any point within a bunch there is a characteristic longitudinal velocity with which small "signals" (departures from the desired number λ of ions per unit length) are propagated. This velocity turns out to be of the same order of magnitude as that of ions at bunch ends (relative to a bunch center) due to bunch compression in accelerator designs currently being studied. Both of these velocities are about an order of magnitude greater than the maximum random velocity spread allowed by pellet and final lens requirements. Therefore the longitudinal propagation of small error signals (density waves) may with good approximation be studied as if the ions had a negligible longitudinal temperature, in analogy with the propagation of waves in a cold plasma.

Next it is shown that the linearized wave equation for these small disturbances may be dealt with by the method of separation of variables for a wide class of bunches whose variation of λ along the bunch and variation of length during acceleration are characterized by two independent arbitrary functions. The requirement for separation is that the bunch behavior is scaling; that is, the bunch shape in terms of λ depends only on the ratio $z/z_o(t)$ with z the distance from the bunch center and $z_o(t)$ the varying bunch half-length. [Normalization to a fixed total number of ions in the bunch requires that λ be also inversely proportional to $z_o(t)$.] Although the bunch behavior desired in current designs is not exactly of this scaling form, solutions obtainable from the separated equations are sufficiently general to give a good understanding of the dynamic response of a variety of bunch shapes to small errors in initial conditions and in applied voltage, and to provide a basis for comparison with computer simulation results.

Such computer simulations of longitudinal bunch dynamics will follow in time the variations of two-dimensional longitudinal phase space density. Neuffer[1,2] has derived a particular form of non-stationary distribution function, in which λ is a parabolic function of z. In the concluding section of the present paper we show how to generalize this type of distribution so as to provide not only an arbitrarily varying rate of bunch compression but also (in principle) an arbitrary scaling bunch shape. Examples illustrating this wider class of non-stationary distributions are presented.

II. IMPLICATIONS OF LONGITUDINAL PHASE SPACE CONSTRAINTS

1. Velocity Spread and Longitudinal Temperature

Present conceptual designs of induction linac systems for commercial-scale power production by inertial-confinement fusion are based on the assumption that after acceleration an individual ion bunch will be split in transverse phase space into many beamlets but will not be chopped longitudinally into segments. Therefore the longitudinal phase area of a bunch may not exceed that acceptable at a final lens system or at the traget.

This requirement may be written, for ions of mass $m = Am_p$, as

$$\frac{(\varepsilon_L)_{linac}}{(\varepsilon_L)_{final}} = \frac{\frac{1}{2}(L\Delta p_z)_{linac}}{\frac{1}{2}(L\Delta p_z)_{final}} = \frac{L_s\gamma_s^3\, Am_p(\Delta v_z)_s}{(\beta_f c\tau_f)[\beta_f\gamma_f Am_p c(\Delta p_z/p_z)_f]} < 1,$$

with bunch length L, velocity spread $\pm\Delta v_z$, and γ varying along the linac; the subscript s denotes evaluation at a distance s along it. The final beam duration time τ_f, relative momentum spread $\pm(\Delta p_z/p_z)_f$, β_f, and γ_f are evaluated at final lens or target. Our purpose here is to obtain estimates, so we neglect relativistic corrections and factors such as $4/\pi$ related to phase space shapes (e.g., elliptical vs. rectangular). We use the relations

$$L_s = \beta_s c\tau_s = \beta_s cQ/I_s, \quad Q = qeE/T_f$$

with Q the electric charge per bunch, I the mean electric current, E the delivered energy, qe the charge per ion, and T the kinetic energy per ion. Then

$$(\Delta v_z/v_z)_s \;<\sim\; (T_f^2/T_s E)(\Delta p_z/p_z)_f(\tau_f/e)(I_s/q).$$

It is convenient to express T_f, T_s in GeV, E in megajoules, and τ_f in nanoseconds, and to define $\mathcal{I}_s = I_s/q$ as the particle current in particle-Amperes. Then

$$(\Delta v_z/v_z)_s <\sim 10^{-6} \left[\frac{T_f^2(\text{GeV})\,\tau_f(\text{nsec})\,(\Delta p_z/p_z)_f}{E(\text{MJ})}\right]\left[\frac{\mathcal{I}_s(\text{Part.-Amp})}{T_s(\text{GeV})}\right]$$

in which the first bracket's value is fixed by the system design, and that of the second varies along the linac. We employ Neuffer's estimate,[3] modified by a hoped-for sextupole improvement factor $F \geqslant 1$, for allowable momentum spread at the final lens;

$$(\Delta p_z/p_z)_f \;<\sim\; \tfrac{1}{2}(r_s/X)F$$

with spot radius r_s and quadrupole lens bore radius X. In the numerical estimates below we use the value 2×10^{-3} F (r_s = 1 mm, X = 25 cm).

Another measure of this constraint is the maximum allowed disordered ion longitudinal kinetic energy as seen in the moving frame, which may be expressed as a longitudinal "temperature" Θ_z in electron-Volts:

$$\Theta_z \;\tilde{=}\; \tfrac{1}{2}Am_p(\Delta v_z)^2 .$$

This quantity must be limited, using the units above, by

$$(\Theta_z)_s <\sim 10^{-3}\left[\frac{T_f^2(\mathrm{GeV})\,\tau_f(\mathrm{nsec})\,(\Delta p_z/p_z)_f}{E(\mathrm{MJ})}\right]^2 \left[\frac{\mathcal{J}_s^2(\mathrm{part.-Amp})}{T_s(\mathrm{GeV})}\right] \ \mathrm{eV}.$$

As an illustration we take $T_f = 20$ GeV, $E = 1$ MJ, $\tau_f = 6$ nsec, $A = 200$. Then $Q = 50q$ μCoul so that $\mathcal{J} = 50/\tau(\mu\mathrm{sec})$; the constraints become

$$(\Delta v_z/v_z)_s <\sim 2.4 \times 10^{-4} F/[\tau_s(\mu\mathrm{sec})T_s(\mathrm{GeV})] \ ,$$

$$\Theta_s <\sim 58F^2/[\tau_s^2(\mu\mathrm{sec})T_s(\mathrm{GeV})] \ \ \mathrm{eV}.$$

To proceed with the illustration we take a specific design[4] developed at the Lawrence Berkeley Laboratory, for which $q = 4$. The pulse duration and current shown there are moderately well reproduced by

$$\tau(\mathrm{nsec}) \sim 1380\ [T(\mathrm{GeV})]^{-0.8}, \quad I(\mathrm{Amp}) \sim 145[\ T(\mathrm{GeV})\ ]^{0.8} \ ,$$

which lead to the following values:

T(GeV)	β	τ (μsec)	L(m)	I(Amp)	$\mathcal{J}$(p.A)	$\Delta v_z/v_z$	Δv_z(m/sec)	Θ(eV)
0.2	0.045	5	68	40	10	$2.4F \times 10^{-4}$	$3.2F \times 10^3$	$11F^2$
1	0.1	1.4	41	144	36	$1.7F \times 10^{-4}$	$5.1F \times 10^3$	$30F^2$
4	0.2	0.46	27	440	110	$1.3F \times 10^{-4}$	$7.9F \times 10^3$	$70F^2$
10	0.32	0.22	21	920	230	$1.1F \times 10^{-4}$	$10F \times 10^3$	$120F^2$
20	0.45	0.125	17	1600	400	$1.0F \times 10^{-4}$	$13F \times 10^3$	$184F^2$

These velocities and temperatures are very small indeed.

2. Relative Distance of Travel

The distance of travel, relative to the bunch center, by an ion having maximum velocity defect depends on the designed variations of both mean bunch velocity and bunch length along the linac;

$$\Delta z(s) \;=\; \int_0^s (\Delta v_z/v_z)_s ds.$$

The integrand is proportional to $(v_z L)^{-1}$; thus

$$\Delta z_f \;=\; (\Delta v_z/v_z)_i \; s_f \int_0^1 \left[L_i/L(\sigma)\right]\left[v_i/v(\sigma)\right] d\sigma \;=\; (\Delta v_z)_f t_f \int_0^1 \left[L_i/L(\mu)\right] d\mu$$

with i and f initial and final values and $\sigma = s/s_f$, $\mu = t/t_f$. The product $v_z L$ varies by only a small factor; it is proportional to $T\tau(T)$, which is proportional to $T^{0.2}$ in the example design approximated above. If the mean accelerating field is assumed uniform along the linac, $T - T_i$ is proportional to s so that the first dimensionless integral above is an elementary one, closely approximated by $[(T_i/T_f)^{0.2}]/0.8$, which equals 0.5 for this design. Numerical experiments have shown that if $\mathcal{E}$ is linear in s with space-average $\bar{\mathcal{E}}$ this integral is only larger by $\leqslant \sim 6\%$ if $\mathcal{E}$ varies by a factor $\leqslant 3$. For $\mathcal{E}$ linear in s, $s_f = (T_f - T_i)/qe\bar{\mathcal{E}}$. For the example design

$$\Delta z_f \;\sim\; (\Delta v_z/v_z)_i (T_f - T_i)/2qe\bar{\mathcal{E}} \;=\; 0.57F/\bar{\mathcal{E}} \text{ (MV/m) meter.}$$

For the reasonable assumptions $F \leqslant 3$, $\bar{\mathcal{E}} \geqslant 1$ MV/m this travel is less than 10% of the final bunch length of 17 m in this design.

Estimates based on simple assumptions about the low-beta section (from ~ 1 MeV to 0.2 GeV) and the final compression system after acceleration in this example indicate that comparable or lesser fractions of a bunch length may be traversed in these sections by an ion having maximum velocity defect. Thus it appears that for systems with parameters in this range it is not merely desirable, but required, that individual ions shall not move along a bunch by more than a fraction of its length during acceleration and compression. This might tend to reduce concern about nonlinear couplings between longitudinal and transverse motions of ions during multiple reflections from steep longitudinal potential "walls" near the bunch ends during most of the acceleration; however, the tolerances required to create and maintain such low longitudinal temperatures are extremely stringent.

III. BEHAVIOR OF SMALL DISTURBANCES ON A BUNCH VARYING IN SPACE AND TIME

We employ the commonly used assumption[5] that the longitudinal space-charge field of a beam of charged particles moving inside a conducting pipe is given (non-relativistically) by*

$$\mathcal{E}_{z_{sc}} \simeq - gqe\ \partial\lambda/\partial z$$

with λ the number of ions per unit length, g a geometrical factor of order unity, and z distance along the bunch measured from its center, for ions of charge qe and mass m. Just as a plasma frequency is defined by ion charge, mass, and number per unit

* Introduction of a more general assumption would allow consideration of resistive-wall and related effects which are not considered here.

volume, so is a charateristic velocity V defined here by charge, mass, and number per unit length. It is given by

$$V^2 = g(qe)^2/m \text{ (cgs)}.$$

To show this consider a uniform bunch (constant λ_o) at rest. For a small density perturbation $\delta\lambda(z,t) = \lambda(z,t) - \lambda_o$ and correlated small velocity perturbation $\delta u(z,t)$, the equation of continuity is

$$\frac{\partial\lambda}{\partial t} + \frac{\partial}{\partial z}(\lambda u) \simeq \frac{\partial\lambda}{\partial t} + \lambda_o \frac{\partial u}{\partial z} \doteq 0$$

and the equation of motion is

$$a \simeq \frac{\partial u}{\partial t} = - \frac{g(qe)^2}{m} \frac{\partial\lambda}{\partial z} .$$

Evaluating $\partial^2\lambda/\partial z\partial t$ from each equation and equating, we have

$$\frac{\partial^2 u}{\partial z^2} - \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} = 0 ,$$

the wave equation, with V^2 as given above.

For $g \sim 2$, ions of mass Am_p, and a uniform bunch of length L meters containing a total charge Q microcoulomb, one finds

$$V \sim 1.3\ (qQ/AL)^{\frac{1}{2}} \times 10^6 \text{ m/sec} .$$

For example, if $q = 4$, $Q = 200$ μcoul, $A = 200$

$$V \sim 2.7\ L^{-\frac{1}{2}} \times 10^6 \text{ m/sec} .$$

This velocity is to be compared with that of a bunch end toward its center during bunch compression in an induction linac, which we denote by u_{max}. If a bunch is compressed from $L \sim 75$ m to $L \sim 15$ m during an acceleration time ~ 60 μsec, the average value of u_{max} is $\sim 0.5 \times 10^6$ m/sec, so that V and u_{max} are comparable, being equal in this example at $L \sim 30$ m.

As shown earlier, target requirements and final lens parameters constrain the maximum allowable longitudinal thermal (random) velocity. It is a general property of system designs presently under consideration that this velocity is smaller by more than an order of magnitude than the characteristic signal velocity V and the bunch end velocity u_{max}. Therefore it is reasonable to neglect thermal spread and to regard the medium as being at zero temperature when discussing the propagation of small disturbances along a bunch in which density λ and unperturbed ion velocity u (relative to the bunch center) vary in both space and time. We now derive the linearized wave equations for disturbances on such bunches.

We assume that in the absence of perturbations the density and velocity $\lambda_o = \lambda_o(z,t)$ and $u_o = u_o(z,t)$ satisfy the equation of continuity

$$\partial\lambda/\partial t + \partial(\lambda u)/\partial z = 0$$

and are consistent* with the unperturbed externally applied and

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* See Section IV below.

space-charge fields. With perturbations present,

$$\lambda = \lambda_o + \phi(z,t), \quad u = u_o + \psi(z,t)$$

with ϕ, ψ small of first order. Neglecting the second-order term, the equation of continuity is

$$\partial\phi/\partial t + \partial(u_o\phi + \lambda_o\psi)/\partial z = 0. \tag{1}$$

The equation of motion is

$$du/dt = \left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial z}\right] u = \partial(u_o + \psi)/\partial t + (u_o + \psi)\ \partial(u_o + \psi)/\partial z$$

$$= qe\mathcal{E}/m;$$

again neglecting the second-order term and cancelling those present in the absence of perturbations,

$$\partial\psi/\partial t + \partial(u_o\psi)/\partial z = -\left[g(qe)^2/m\right]\partial\phi/\partial z + \delta a_{ext}(z,t) \tag{2}$$

in which δa_{ext} is the acceleration due to departures of the external field from the form required for consistency in the absence of perturbations.

The two coupled first-order partial differential equations (1) and (2) for ϕ and ψ contain the unspecified functions u_o and λ_o which are constrained as indicated above. In order to proceed analytically it has been found desirable to further constrain these functions so that they represent <u>scaling</u> bunch shapes as described in the introduction. Such bunches have linear densities of the form

$$\lambda_o(x,t) = \text{const. } S(x)/z_o(t)$$

in which $x = z/z_o(t)$ is distance from the bunch center measured in units of the (arbitrarily) varying bunch half-length. The equation of continuity then requires that the form of u_o be

$$u_o(z,t) = z(dz_o/dt)/z_o = x\, dz_o/dt.$$

Thus scaling bunches are characterized by two independent arbitrary functions, the shape function $S(x)$ and the half-length $z_o(t)$, which we express in terms of its initial value z_{oo} as a dimensionless reciprocal compression factor $R(t) = z_o(t)/z_{oo}$ having initial value unity. Similarly we normalize $S(x)$ so that $S(0) = 1$, leading to

$$\lambda_o(z,t) = \lambda_{oo} S(x)/R(t)$$

with λ_{oo} the initial value of λ at the bunch center. We change variables from (z,t) to (x,t) using the relations

$$\partial/\partial z|_t = z_o(t)^{-1}\partial/\partial x|_t ,$$

$$\begin{aligned}\partial/\partial t|_z &= \partial/\partial t|_x + \partial x/\partial t|_z\, \partial/\partial x|_t \\ &= \partial/\partial t|_x - (z/z_o^2)\ (dz_o/dt)\partial/\partial x|_t .\end{aligned}$$

After some manipulation eqs. (1) and (2) become

$$[R(t)\ \partial/\partial t]\ (R\phi) + (\lambda_{oo}/z_{oo})\partial(S\psi)/\partial x = 0,$$

$$\partial(R\phi)/\partial x + (\lambda_{oo}z_{oo}/V_{oo}^{2})\{[R\partial/\partial t]\ (R\psi) - R^{2}\delta a\} = 0\ ,$$

in which $V_{oo} = qe(g\lambda_{oo}/m)^{\frac{1}{2}}$ is the initial mid-bunch value of the characteristic velocity. From this form it is evident that the appropriate time variable is

$$\tau(t) = \int_{0}^{t} R(t')^{-1}dt'\ .$$

In terms of τ the equations are

$$\partial(R\phi)/\partial\tau + (\lambda_{oo}/z_{oo})\ \partial(S\psi)/\partial x = 0,$$

$$\partial(R\psi)/\partial\tau + (V_{oo}^{2}/\lambda_{oo}z_{oo})\ \partial(R\phi)/\partial x = R^{2}\delta a\ .$$

The dependent variable $R\phi$ may now be eliminated, yielding the desired wave equation for ψ ;

$$\partial^{2}(R\psi)/\partial\tau^{2} = (V_{oo}/z_{oo})^{2}\partial^{2}(S\psi)/\partial x^{2} + \partial(R^{2}\delta a)/\partial\tau\ .$$

We now confine our attention to the normal-mode (standing-wave) solutions of the homogeneous equation in the absence of external perturbations δa. Separating the variables by setting $\psi(x,\tau) = X(x)T(\tau)$, we have

$$T^{-1}d^2(RT)/d\tau^2 = (V_{oo}/z_{oo})^2 \; X^{-1}d^2(SX)/dx^2$$

$$= \text{constant} = -\,\omega_o^2 ,$$

yielding th two ordinary differential equations

$$d^2(RT)/d\tau^2 + (\omega_o^2/R)(RT) = 0 ,$$

$$d^2(SX)/dx^2 + (\kappa_o^2/S)(SX) = 0 ,$$

with $\kappa_o = \omega_o z_{oo}/V_{oo}$ the radian wave-number at the bunch center (dimensionless, in units of x^{-1}) corresponding to angular frequency ω_o of a mode at $t = 0$.

From the form of these equations we may gain a picture of the scaling in space and time of small perturbations in the idealized system to which these equations apply. Consider, for example, the temporal behavior of modes with frequencies high enough that the variation of R may be regarded as adiabatic. [This requires that $|\Delta R/R| << 1$ in one period $\Delta t = R\Delta\tau \sim R^{3/2}/\omega_o$.] Then from the WKB approximation RT is proportional to the real part of

$$R^{\frac{1}{4}} \exp[i\omega_o \int^{\tau} R^{-\frac{1}{2}}(\tau')d\tau']$$

so that the real-time variation of the velocity perturbation in such a mode is

$$T(t) \propto R(t)^{-3/4}\, R\ell\, \exp[i\omega_o \int^{\tau} R^{-3/2}(t')dt'] .$$

Similarly, some wavelengths are short enough that the variation of S may be regarded as adiabatic in space. [This requires that $|\Delta S/S| << 1$ over the spatial region $\Delta z = z_o \Delta x = z_o S^{\frac{1}{2}}/\kappa_o$.] In the same way the spatial variation of the velocity perturbation is

$$X(x) \propto S(x)^{-3/4}\, R\ell\, \exp[i\kappa_o \int^{z} S^{-\frac{1}{2}}(x')dx'] .$$

Thus the amplitude of such a mode's velocity perturbation is $\psi_{oo}[R(t)S(x)]^{-3/4}$, with local frequency $\omega_o/[R(t)^{3/2}]$ and radian wave-number $\kappa_o/[S^{\frac{1}{2}}(x)z_o(t)] = \kappa_o/[z_{oo}S^{\frac{1}{2}}(x)R(t)]$ in physical units (length^{-1}).

The density perturbation $\emptyset$ is found from

$$\emptyset(x,\tau) = -R^{-1}(\tau)(\lambda_{oo}/z_{oo})\int^{\tau} \frac{\partial(S\psi)}{\partial x}\, d\tau$$

$$= -R^{-1}(\tau)(\lambda_{oo}/z_{oo})\, \frac{d}{dx}(SX)\int^{\tau} T(\tau')d\tau' .$$

In the spirit of the WKB approximation, the operations are applied only to the rapidly varying phases in SX and T;

$$d(SX)/dx \mathrel{\tilde{\propto}} S^{\frac{1}{4}} R\ell\, [(i\kappa_o S^{-\frac{1}{2}})\, \exp(i\kappa_o \int^{x} S^{-\frac{1}{2}} dx'] ,$$

$$\int^{\tau} T(\tau')d\tau' \mathrel{\tilde{\propto}} R^{-3/4} R\ell\, [(i\omega_o R^{-\frac{1}{2}})^{-1} \exp(i\omega_o \int^{\tau} R^{-\frac{1}{2}} d\tau'] .$$

Thus the amplitude of the density perturbation is

$$(\psi_{oo}/R)(\lambda_{oo}/z_{oo})(\kappa_o/\omega_o)(S^{\frac{1}{2}}/R^{3/4})(R/S)^{\frac{1}{2}}$$

$$= \phi_{oo} R(t)^{-5/4} S(x)^{-1/4}$$

with ϕ_{oo}, ψ_{oo} the amplitudes at $z = 0$, $t = 0$, satisfying the relation

$$|\phi_{oo}/\psi_{oo}| = \lambda_{oo}/V_{oo} \quad .$$

For lower frequencies, longer waves, and in regions of the bunch where $S(x)$ is changing rapidly the WKB approximations are not valid. To illustrate what can be done with the equations we give here exact solutions of the equation for T for a one-parameter family of functions $R(t)$ which has some generality, and an exact solution of the equation for X for the parabolic shape $S(x) = 1 - x^2$. We set $R(t) = (1 + \alpha\nu t)^{-1/\alpha}$, in which α is a parameter determining the functional form of the bunch compression; since $dR/dt = -\nu$ for all α, ν is the initial fractional compression rate. The form of $R(\tau)$ is found from

$$\tau(t) = \int_0^t R^{-1}(t')dt' = \left[(1 + \alpha\nu t)^{(\alpha+1)/\alpha} - 1\right]/\nu(\alpha + 1)$$

so that

$$R(\tau) = (1 + \beta\nu\tau)^{-1/\beta} \text{ with } \beta = 1 + \alpha .$$

As examples we list

1. Steady compression ($\alpha = -1, \beta = 0$), $R(t) = 1 - \nu t$, $R(\tau) = e^{-\nu\tau}$
2. Exponentially increasing compression ($\alpha = 0, \beta = 1$), $R(t) = e^{-\nu t}$, $R(\tau) = (1 + \nu\tau)^{-1}$
3. An intermediate case ($\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$), $R(t) = (1 - \frac{1}{2}\nu t)^2$, $R(\tau) = (1 + \frac{1}{2}\nu\tau)^{-2}$

[In the first two examples one must use the limiting form

$$\lim_{n \to 0} (1 + nx)^{1/n} = e^x .]$$

For this family $R(\tau)$ the equation may be written as

$$d^2(RT)/d\xi^2 + (\omega_o/\beta\nu)^2 \xi^{1/\beta}(RT) = 0 ,$$

with $\xi = 1 + \beta\nu\tau$. Its solution is

$$RT = \xi^{\frac{1}{2}}[AJ_p(\zeta) + BY_p(\zeta)]$$

where the order p of the Bessel functions is $\beta/(2\beta + 1)$ $= (\alpha + 1)/(2\alpha + 3)$ and their argument ξ is $(\omega_o/\nu)(\beta + \frac{1}{2})^{-1} \xi^{-(\beta+\frac{1}{2})/\beta}$; A and B are arbitrary constants. It is more useful to express T in terms of R and α since the latter gives the dependence of R on real time t directly;

$$T = R^{-(\alpha+3)/2}[AJ_p(\zeta) + BY_p(\zeta)]$$

with $\zeta = (\omega_o/\nu)(\alpha + 3/2)^{-1} R^{-(\alpha+3/2)}$.

This form of solution fails for the first example above (steady compression) because R is not a power of ξ for $\beta = 0$. For this case it is simpler to express the equation in terms of t though $\xi = 1 - \nu t$;

$$\xi(d/d\xi)[\xi(dT/d\xi)] + (\omega_o/\nu)^2 T = 0.$$

The solution is $T = A \cos[(\omega_o/\nu)\ln(1 - \nu t) + \delta]$ with A, δ constants.

The Bessel form of solution is also invalid for the special case $\alpha = -3/2$, $\beta = -\frac{1}{2}$, for which the order p becomes infinite. In this case the equation is

$$d^2(RT)/d\xi^2 + (2\omega_o/\nu)^2 \xi^{-2}(RT) = 0.$$

The form of solution depends on the value of the parameter $\mu = (2\omega_o/\nu)^2$. For $\mu < \frac{1}{4}$ it is $T(\xi) = \xi^{-3/2}[A\xi^{(\frac{1}{4}-\mu)^{\frac{1}{2}}} + B\xi^{-(\frac{1}{4}-\mu)^{\frac{1}{2}}}]$;

for $\mu > \frac{1}{4}$, $T(\xi) = A\xi^{-3/2} \cos[(\mu - \frac{1}{4})^{\frac{1}{2}} \ln\xi + \delta]$, and for $\mu = \frac{1}{4}$, $T(\xi) = \xi^{-3/2}[A + B\ln\xi]$.

In considering normal modes of small oscillations on a finite bunch the equation for $X(x)$ is to be solved subject to appropriate boundary conditions at its ends, thereby defining spatial eigenfunctions X_n and associated eigenvalues κ_{o_n} leading to eigenfrequencies ω_{o_n} through the relation $\omega_o/\kappa_o = V_{oo}/z_{oo}$. The simplest non-trivial example of a shape function appears to be the parabolic one, $S(x) = 1 - x^2$, for which the equation is

$$d^2[(1 - x^2)X]/dx^2 + \kappa^2 X = 0;$$

it may be written in the form

$$(x^2 - 1)X'' + 4x X' + (2 - \kappa^2) X = 0,$$

which is Gegenbauer's equation[6] with its parameters α and β (not the α and β used above) given by $(\alpha + 1)(\alpha + 2) = \kappa^2$, $\beta = 1$. For this value of β, the general solution is[7]

$$X(x) = (d/dx)[AP_{1+\alpha}(x) + BQ_{1+\alpha}(x)]$$

in which P and Q are Legendre functions.

To proceed further one must establish boundary conditions. Unfortunately, both the physical assumption $\mathscr{E}_{sc} \propto \partial\lambda/\partial z$ and (therefore) the mathematics become inadequate near the bunch ends where $\lambda \to 0$; the bunch ends are singular points of the differential equation, at which only a single specific linear combination of the P and Q terms is non-singular for any given value of α. For such solutions the ratio of slope to value at these points is $(X'/X)|_{x=\pm 1} = \pm\frac{1}{4}\alpha(\alpha+3)$, leaving no freedom to impose a physical condition on the ratio. The problem will not be pursued further here because with our assumptions there is no clear justication for any particular boundary condition. More work might be appropriate should analytic solutions be desired for comparison with a computer simulation, provided the latter embodies a well defined bunch-end condition.

After establishing by some means a set of normal modes, examination of the inhomogeneous equations ($\delta a \neq 0$) could lead to an evaluation of tolerances on allowable departures of the applied electric field from that required for the equilibrium distribution.

Among additional topics remaining for future study are inclusion of the space periodicity of the surrounding structure and its applied electric field, and of its finite resistivity.

IV. GENERALIZATIONS OF NEUFFER'S SELF-CONSISTENT PHASE SPACE DISTRIBUTIONS

Neuffer has derived a "self-consistent" stationary distribution for longitudinal transport of a beam bunch, and a similar nonstationary distribution together with its envelope equation[1]. In subsequent work[2] he has analyzed the stability of these "standard longitudinal distributions" in continuous and periodic transport systems. His distributions are characterized by longitudinal self-fields and external fields proportional to the distance z from the bunch center [based on the usual approximation[5] $\varepsilon_z(\text{sp.chg.}) \simeq -gqe\, d\lambda/dz$] and therefore by parabolic dependence of the number of ions per unit length λ on z. Although these dependences provide analogies with the K-V distribution[8] (which has appeared to be uniquely tractable for analytic stability studies), they are very different from those for the nearly uniform λ expected over most of the bunch length (beginning at injection and continuing over a large part of the acceleration) in an induction linac driver for a heavy-ion inertial-fusion power plant. The generalizations described below were developed in an effort to arrive at forms more relevant to the physical situation in such a linac. Here, as in Neuffer's work, relativistic corrections are ignored.

It is perhaps worth noting that the property of self-consistency is in one sense a matter of definition. If a distribution function is specified there will exist some external force field which, together with the self-field, is required to produce it; the total field can be calculated from conservation of phase space density (Vlasov equation). However, in our application and many others a velocity dependence of the force due to this field is unacceptable; the general problem of finding distributions without velocity dependence of the force is a difficult one. In Neuffer's work the desired forms of λ and the corresponding consistent total field (parabolic and linear, respectively, in z) were established in advance.

Here we explore in turn alternatives to the following assumptions made by Neuffer:

1. $\lambda(z)$ is required to be parabolic.
2. The stationary distribution function $f(z, z') = f(H)$ is a <u>specific</u> function of the single-ion Hamiltonian $H(z, z')$; $f = \text{const.}(H_{max} - H)^{\frac{1}{2}}$.
3. The nonstationary distribution function $\lambda(z, s)$ is required to be parabolic in z.
4. The bunch is coasting; its center has zero acceleration.

In what follows the shape function is denoted by $F(x)$ rather than $S(x)$, and the meanings of the symbols p, q, r, S, R, T, μ, ν, ξ, ζ, ϕ, and ψ are not related to those above.

1. Stationary Distributions

First we explore alternatives to the first assumption, retaining the others. Consider the class of stationary distribution functions for which

$$f(z,\ z') \quad = \quad (3N/2\pi\varepsilon)(\nu/\mu)\ [F(x)\ -\ (\nu z_o/\varepsilon)^2 z'^2]^{\frac{1}{2}}$$

within the area in $z - z'$ space where the square root is real and $f = 0$ outside it. Here $F(x)$ is a dimensionless shape function; $x = z/z_o$, with z_o the bunch half-length, and $F(\pm 1) = 0$; N is the total number of particles; ε is the un-normalized emittance $=$ (occupied area in $z - z'$)$/\pi$; primes are d/ds with s, the independent variable, equal to vt for a coasting beam; μ and ν are constants defined by

$$\mu \quad = \quad (3/4)\int F(x)\ dx, \quad \nu \quad = \quad (2/\pi)\int F^{\frac{1}{2}}(x)\ dx,$$

and unstated limits of integration here and below are those at which the integrand vanishes. The quantity within the square root is proportional to $H_{max} - H(z,\ z')$ with the Hamiltonian H the sum of kinetic and potential energies; the potential energy per particle is

$$V(z) \quad = \quad V_{ext} + V_{sc}$$

and the space charge potential energy is

$$V_{sc} \quad = \quad g(qe)^2\lambda(z)$$

with λ the number of particles per unit length;

$$\lambda(z) = \int f\, dz' = (3N/4z_o)\mu^{-1}F(x) .$$

For a coasting beam the total force on an ion is mv^2z'', and from the Vlasov equation $z'' = -z'(\partial f/\partial z)/(\partial f/\partial z') = \frac{1}{2}(\varepsilon/\nu z_o)^2 dF/dz$; the total force is also given by $-d(V_{ext} + V_{sc})/dz$. Combining the expressions above, the self-consistent potential energy per ion of the external force is found to be

$$V_{ext}(z) = -\left[1 + 2(S/R)(\mu/\nu^2)\right] V_{sc}(z)$$

in which

$$2S/R = (2mv^2\varepsilon^2)/(3gq^2e^1Nz_o) ;$$

S and R are dimensionless parameters I introduced in an earlier work[9] on longitudinal dynamics [in which $F(x) = 1 - x^2$ as in Neuffer's distribution and $\mu = \nu = 1$] as measures of the emittance and space charge terms. The external force must not only cancel that due to space charge but in addition must contain the thermal pressure of the bunch which is proportional to ε^2.

Thus we have constructed a family of stationary distribution functions generalized from that of Neuffer which allow an arbitrary linear density $\lambda(z)$ and have determined the corresponding self-consistent external field; for this family it has the same form of space dependence as $d\lambda/dz$.

Next we explore alternatives to the second assumption, retaining the first and last ones. Consider the class of stationary distribution functions

$$f(z,z') = (3N/2\pi\varepsilon)(\nu/\xi)\ f\left[F(x) - (\nu z_o/\varepsilon)^2 z'^2\right]$$

with the same definitions and conditions as before; the function $f(H_{max} - H)$ is assumed to be such that the integral $\int f\ dz'$ can be performed analytically so as to obtain an analytic expression for $\lambda(z)$. We define a function $G(F(x))$;

$$G(F(x)) = F^{\frac{1}{2}} \int_{-1}^{1} f\left[F(1 - \zeta^2)\right] d\zeta$$

The constants ν and ζ are defined by

$$\nu = (2/\pi)\int F^{\frac{1}{2}}(x)\ dx, \quad \xi = (3/2\pi)\int G\ dx.$$

Then

$$\lambda(z) = (3N/4z_o)\xi^{-1}G$$

Following the same procedure as before, we find

$$V_{ext}(z) = -\left[V_{sc}(z) + 2m(v\varepsilon/\nu z_o)^2\ F(x)\right]$$

Thus we have constructed a more general family of stationary distribution functions which allow an arbitrary density $\lambda(z)$, and have determined the corresponding self-consistent external field; for this wider class the part of the external field which contains the thermal pressure has in general a form of dependence on z which differs from that of $d\lambda/dz$.

As examples of this class we display the results for $f(h) = h^{\eta}$ with η an arbitrary number, restricted to avoid non-integrable singularities. Then

$$\lambda(z) = (N/z_o)\, F^{\eta+\frac{1}{2}} \Big/ \int F^{\eta+\frac{1}{2}}(x)\, dx.$$

For $\eta = \frac{1}{2}$ we reproduce the results for the less general family above. For $\eta < \frac{1}{2}$ the space charge field becomes singular at the bunch ends where $F(x) \to 0$, unless $dF/dx \to 0$ there. For $\eta = 0$ the distribution function is constant within the boundary in $z - z'$ space for any $F(x)$; to avoid a singularity in the space charge field at the bunch ends one must require $F^{-\frac{1}{2}}dF/dx \to 0$ there. As $\eta \to -\frac{1}{2}$, $\lambda(z) \to N/(2z_o) =$ constant within the bunch for any $F(x)$, and the distribution function becomes increasingly singular toward the boundary in $z - z'$ space, approaching a square root singularity. For Neuffer's choice $F(x) = 1 - x^2$, the external field required will always be proportional to z while λ varies as $[1 - (z/z_o)^2]^{\eta}$ for any value of η.

These examples serve to indicate a special feature of Neuffer's distribution $\left[F(x) = 1 - x^2,\ \eta = \frac{1}{2}\right]$; all forces (space charge, external, and thermal) are linear in z. This feature no doubt contributes to the tractability of analytic stability analyses. More importantly, the examples show that the z dependence of the external field required to maintain a bunch in equilibrium will depend on the form of its distribution function's dependence on z', even for a fixed $\lambda(z)$. Should actual distributions have thermal pressures requiring applied containing-field components of magnitude greater that the smallest allowable error fields it will be necessary to determine their z dependences.

2. Nonstationary Distribution Functions

Here we explore alternatives to the third assumption, considering not only the specific functional dependence of the second assumption but also more general ones. The requirement that the forces be velocity-independent $(\partial z''/\partial z' = 0)$ is

$$(\partial/\partial z')[\,(\partial f/\partial s + z'\ f/\partial z)/(\partial f/\partial z')\,] = 0$$

which imposes a constraint on the form of the distribution function $f(z, z', s)$ and leads to the expected conservation equation

$$\partial\lambda/\partial s + \partial(\lambda u)/\partial z = 0,$$

in which $\lambda(z, s) = \int f\, dz'$ and $u(z, s) = \langle z' \rangle = \int z' f\, dz'/\lambda$, obtained by integrating the Vlasov equation over z' and noting that $f = 0$ at the limits of integration.

The severity of this constraint on the form of f may be seen by considering f to be <u>any</u> appropriate function of $H_{max} - H(z,z',s)$

and assuming the somewhat general form

$$H_{max} - H = \phi(z,s) - \psi(s)[z' - u(z,s)]^2 .$$

Then

$$\lambda(z,s) = (\phi/\psi)^{\frac{1}{2}} \int_{-1}^{1} f[\phi(1-\zeta^2)]\,d\zeta = \psi^{-\frac{1}{2}} G(\phi(z,s)).$$

After some calculation one finds that the constraint requires

$$2\psi z'(z' - 2u)\left[(\partial u/\partial z) - \frac{1}{2}\psi'/\psi\right] - [\phi' + u(\partial\phi/\partial z)] = 0$$

for all z', which can be satisfied only if

$$u(z,s) = \frac{1}{2}(\psi'/\psi)z$$

and

$$\phi' + u\partial\phi/\partial z = 0.$$

Using this form for u, the continuity equation becomes

$$\psi^{-\frac{1}{2}}(dG/d\phi)(\phi' + u\partial\phi/\partial z) = 0 ,$$

showing consistency; $\phi(z,s)$ must be of the form $\phi\left[z/\psi^{\frac{1}{2}}\right]$. All such distribution functions are <u>scaling</u>; the rate of dilation (change of length per unit length per unit change of the independent variable s) is independent of z, so that the <u>shape</u> of $\lambda(z,s)$ is preserved.

Should one try a more general form by allowing ψ to depend on z as well as on s, the requirement $\partial z''/\partial z' = 0$ forces the condition $\partial\psi/\partial z = 0$. The problem of developing non-scaling non-stationary distributions appears to be a difficult one, and will not be attempted here.

All of the generalized stationary distributions derived above may be converted into scaling nonstationary ones, with arbitrary shape function and arbitrary dilation dependence on s, in the following way:

$$f(z,z',s) = (3N/2\pi\varepsilon)(\nu/\xi) f\left[F(x) - \{\nu z_o(s)/\varepsilon\}^2 \{z' - (z_o' z/z_o)\}^2\right]$$

in which f is any suitable function of the argument in the square brackets; $z_o(s)$ is any function; $x = z/z_o(s)$;

$$\nu = (2/\pi)\int F^{\frac{1}{2}}(x)dx, \quad G(F(x)) = F^{\frac{1}{2}} \int_{-1}^{1} f\left[F(1 - \zeta^2)\right] d\zeta$$

with f the function selected and $F(1 - \zeta^2)$ its argument, and

$$\xi = (3/2\pi)\int G(F(x))dx.$$

The density is

$$\lambda(z,s) = [3N/4z_o(s)] \; \xi^{-1} G(F(z/z_o)).$$

The potential energy per ion of the external force required by self-consistency is

$$V_{ext}(z,s) = - V_{sc}(z,s) - \frac{1}{2}mv^2 \left[(\varepsilon/\nu z_o)^2 F + (z_o''/z_o) z^2\right]$$

for a coasting beam with constant v. The force due to the second term in the bracket is that required to produce the specified dilation.

3. Effect of Acceleration

Here we consider the correction needed if the fourth assumption is not valid. For an accelerating bunch

$$d^2z/dt^2 = z''v^2 + z' a$$

with $a = dv/dt$. We will estimate the importance of the second term by evaluating the ratio

$$R = z'a/z''v^2$$

on the assumption that the acceleration is constant and that the bunch length varies as a power of the ion kinetic energy T. (In the example design[4] used above the pulse duration τ is approximately proportional to $T^{-0.8}$, so that bunch length $L \propto T^{-0.3}$.) With $L \propto T^{-p}$, and dT/ds constant,

$$R = L'(T'/m)L''(2T/m) = (dL/dt)/\left[2T(d^2L/dt^2)\right] = -\left[2(1+p)\right]^{-1}$$

and if $p = 0.3$ the correction ratio $|R| \sim 40\%$.

The correction ratio R is even larger if the acceleration increases with distance along the accelerator. Retaining the power law assumption for $L(T)$ but assuming the acceleration to increase linearly with distance s from an initial value a_i to a final value a_f, the value of R is

$$R = -\frac{1}{2}\{(1 + p) - d[\ell n(dT/ds)]/d(\ell n\ T)\}^{-1}$$

and the logarithmic derivative is

$$\frac{1}{2}(1 - q)r/[q + (1 - q)r]$$

with $q = (a_i/a_f)^2$ and $r = (T - T_i)/(T_f - T_i)$. This term (and therefore also $|R|$) attains its maximum value $\frac{1}{2}(1 - q)$ at the final energy, where

$$|R|_{max} = (1 + 2p + q)^{-1} ;$$

if the accelerating field increases by a factor ~ 3, as suggested in some designs, $q \sim 0.1$ and $R \sim -0.6$; the ratio $z''v^2/(d^2z/dt^2)$ becomes as large as ~ 2.4, compared with unity for a coasting beam.

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