LONGITUDINAL DYNAMICS OF BUNCHED BEAM IN A MODEL LINAC

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In this note, I will report on some analytical efforts to understand the longitudinal bunched beam dynamics in an induction linac as currently stipulated for the HIF program. The analysis is carried out within the framework of a simple model. It is found that a bunched beam tends to be stable. Further work is necessary to extend the results to a more realistic case.

I. THE MODEL

Since the subject is analytically quite involved, some mathematical idealization is inevitable. The model linac to be considered here has the following features:

a) The external force is described by a rectangular-well potential shown in Fig. (1).

b) The self-force F_s is given by

$$\boxed{F_s = - e^2 \frac{Z_R}{L} p_o \lambda(x) - e^2 Z_C \frac{\partial}{\partial x} p_o \lambda(x)} \qquad (1)$$

Here, e = the proton charge, p_o = the velocity of the bunch center, L = the bunch length, $\lambda(x)$ = the line density, and Z_R and Z_C are the resistive and the capacitive parts of the impedance, respectively. They are taken to be real and positive.

The main motivation for introducing the model is the calculational simplicity. However, it should be noted that the external potential in induction linacs resembles more or less the rectangular well. Also, Eq. (1) represents the simplest possible form of the force incorporating both the space charge repulsion and the deceleration due to the cavity impedance.

The relevant magnitudes for Z_R and Z_C are

$$Z_R \sim Z_C \sim 10^3 \text{ ohms}. \tag{2}$$

The motion in phase space under the influence of the rectangular-well potential is shown in Fig. (1.b). Here and in the following, p is the velocity of the particle relative to the bunch center. A particle is reflected instantaneously when it reaches the edge of the potential well. Therefore, one should identify the points B and C, and also the points A and D. In terms of the distribution function $\psi_T(t,x,p)$ (T for the total, unperturbed plus perturbed), one obtains

$$\psi_T(t,o,p) = \psi_T(t,o,-p), \psi_T(t,L,p) = \psi_T(t,L,-p). \tag{3}$$

Eq. (3) supplies the relevant boundary conditions for the longitudinal motion of the bunched beam for our idealized linac.

The linearized Vlasov's equation for the present problem is

$$\frac{\partial\psi}{\partial t} + p\,\frac{\partial\psi}{\partial x} - f(p)\,[\alpha\,\frac{\partial}{\partial x} + \beta]\,\lambda(x) = 0, \tag{4}$$

where

$$f(p) = \frac{\partial\psi_o(p)}{\partial p}, \quad \lambda(t,x) = \int dp\,\psi(t,x,p), \tag{5}$$

$$\alpha = \frac{e^2 p_o Z_C}{m\gamma^3}, \quad \beta = \frac{e^2 p_o Z_R}{m\gamma^3 L}. \tag{6}$$

In the above, ψ_o and ψ are the unperturbed and perturbed parts of ψ_T, m is the rest mass of particle and γ is the ratio relativistic mass/rest mass. The boundary conditions (3) become the following statement:

$\psi_o(p)$, $\psi(t,o,p)$ and $\psi(t,L,p)$ are even functions of p. (7).

II. METHOD OF ANALYSIS

The boundary value problem specified in the above can be solved in the following steps: First, one seeks a solution in the following form:

$$\psi(t,x,p) = e^{i\omega t} \sum_k e^{-ik.x} A(k,p) \; . \tag{8}$$

From Eq. (4), one obtains

$$A(k,p) + \frac{k\alpha + i\beta}{\omega - kp} f(p) \int A(k,p)\, dp = 0, \tag{9}$$

$$1 + (k\alpha + i\beta) \int dp\, f(p) \frac{1}{\omega - kp} = 0. \tag{10}$$

Next, one solves the dispersion relation (10) to obtain k as a function of ω. In general, there will be many branches $k_\ell(\omega)$, $\ell = 1, 2, \ldots$. In view of Eqs. (8) and (9), the solution has the following structure:

$$\psi(t,x,p) = e^{i\omega t} \sum_\ell e^{-ik_\ell(\omega) \cdot x} \frac{k_\ell(\omega) \cdot \alpha + i\beta}{\omega - k_\ell(\omega) p} f(p)\, A_\ell, \tag{11}$$

where $\left\{A_\ell\right\}$ is a set of constants. The requirement that the function ψ must satisfy the boundary condition (7) results in a discrete set of eigenvalues ω_n. Going back to Eq. (11), one determines the eigenfunction corresponding to ω_n in the following form:

$$\psi_n(t,x,p) = e^{i\omega_n t}\; U_n(t,p). \tag{12}$$

This completes the sketch of the general procedure to obtain the eigenvalues and the eigenfunctions.

It is instructive to compare the situation to the case of a coasting beam circulating in a ring. In this case, Eq. (4) remains the same while the boundary condition (7) is replaced by

$$\psi(t,x,p) = \psi(t,x + C,p), \tag{13}$$

where C is the circumference of the ring. Eq. (13) determines immediately that $k = k_n = 2\pi n/C$, $n = 1, 2, .$ The eigenvalue ω_n is then obtained from Eq. (11).

To compare the theory with numerical simulation, one has to consider the initial value problem. This is easily solved if one could determine the coefficients C_n in the expansion of the initial distribution $\psi(o,x,p)$;

$$\psi(o,x,p) = \Sigma \, C_n \, \psi_n(x,p). \tag{14}$$

For this purpose, it is necessary to consider the following adjoint equation á la Van Kampen[1]:

$$\frac{\partial\phi}{\partial t} + p \frac{\partial\phi}{\partial x} - (\alpha \frac{\partial}{\partial x} - \beta) \int dp \; f(p) \; \phi(t,x,p) = 0. \tag{15}$$

The function $\phi(t,x,p)$ is subject to the same boundary condition as $\psi(t,x,p)$. Following similar steps as in the above, one obtains the eigenvalues $\bar{\omega}_n$ and and the eigenfunctions $V_n(x,p)$ for the adjoint system. The following orthogonality theorem is easily derived:

$$(V_m,U_n) \equiv \int_o^L dx \int dp \; V_m{}^*(x,p) \; U_n(x,p) = 0 \text{ if } \omega_n \neq \bar{\omega}_n{}^*. \tag{16}$$

The coefficients C_n can now be determined by making use of (16).

III. A SIMPLE EXAMPLE

In this section, the general procedure described in the previous section will be illustrated for a special case in which the unperturbed distribution $\psi_o(p)$ is a simple step function as shown in Fig. (2). One has

$$f(p) = \frac{\partial\psi_o(p)}{\partial p} = \frac{N}{2L\Delta}\left[-\delta(p-\Delta) + \delta(p+\Delta)\right], \tag{17}$$

where N is the total number of the particles in the bunch. From the structure of Eq. (4), it is convenient to write

$$\psi(t,x,p) = \psi_I(t,x,p) + \delta(p-\Delta)\, A(t,x) + \delta(p+\Delta)\, B(t,x). \tag{18}$$

The functions ψ_I, A and B satisfy the following equations:

$$\frac{\partial\psi_I(t,x,p)}{\partial t} + p\,\frac{\partial\psi_I(t,x,p)}{\partial x} = 0 \tag{19}$$

$$\left[\frac{\partial}{\partial t} + \Delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x} + \frac{N}{2L\Delta}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\left(\alpha\,\frac{\partial}{\partial x} + \beta\right)\right]\begin{pmatrix} A \\ B \end{pmatrix} + \frac{N}{2L\Delta}\left(\alpha\,\frac{\partial}{\partial x} + \beta\right)\begin{pmatrix} \lambda_I \\ -\lambda_I \end{pmatrix} = 0. \tag{20}$$

In the above, $\lambda_I(t,x)$ is the charge density associated with ψ_I;

$$\lambda_I(t,x) = \int dp\; \psi_I(t,x,p) \tag{21}$$

The function ψ_I satisfies the same boundary condition as ψ, Eq. (7), while the boundary condition for A and B becomes

$$A(t,o) = B(t,o),\quad A(t,L) = B(t,L). \tag{22}$$

The boundary value problem for ψ_I is easily solved. One gets

$$\psi_I(t,x) = g_n(p) \cos K_n(pt \pm x), \text{ or } \; h_n(p) \sin K_n(pt - x). \tag{23}$$

In the above, $g_n(p)$ ($h_n(p)$) is an arbitrary even (odd) function of p, and

$$K_n = \pi n/L \tag{24}$$

Eq. (23) describes waves that move with phase velocity p.

Now consider the "edge wave" described by Eqs. (20) and (22). If ψ_I does not vanish, it drives the edge wave through the last term in Eq. (20). Let us first consider the case $\psi_I = 0$. The resulting system can be solved following the general outline described in the previous section. The details can be found in Reference (2), and the result is

$$\begin{pmatrix} A \\ B \end{pmatrix} = e^{i\omega_n t} \, e^{-Qx} \left[e^{-iK_n x} \begin{pmatrix} \omega_n + k_n^{+}\Delta \\ -\omega_n + K_n^{+}\Delta \end{pmatrix} + e^{iK_n x} \begin{pmatrix} -\omega_n - k_n^{-}\Delta \\ \omega_n - k_n^{-}\Delta \end{pmatrix} \right] \tag{25}$$

Here

$$Q = \frac{\beta'}{2L(1+\alpha')}, \quad \alpha' = \frac{N\alpha}{\Delta^2 L}, \quad \beta' = \frac{N\beta}{\Delta^2}, \tag{26}$$

$$\omega_n^2 = (1+\alpha')\,\Delta^2\,(K_n^2 + Q^2), \; k_n^{\pm} = iQ \pm K_n \; . \tag{27}$$

The factor e^{-Qx} in Eq. (25) means that the particles tend to pile up in the rear part of the bunch (The bunch moves to the positive x-direction). This is due to the fact that the resistive part Z_R of the impedance causes a decelerating force through the first term in Eq. (1). For a machine with a peak current $I = ep_0 N/L \sim 10^4$ Amp, velocity spread $\Delta/p_0 \sim .1\%$, energy $mp_0^2 \sim 10$ GeV and $\gamma \sim 1$, the dimensionless constants α' and β' are numerically equal to the impedances Z_C and Z_R expressed in ohms. From Eqs. (2) and (26), the attenuation factor for such a machine is

$$e^{-QL} \sim e^{-.5} \sim .6 \ . \tag{28}$$

It is perhaps surprising that the eigenfrequency given in Eq. (27) is real so that no instabilities develop for the bunched beam case. This is in sharp contrast to the coasting beam case where the beam is unstable as long as $Z_R \neq 0$. The usual interpretation is that, while the wave grows along the line C → D in Fig. (1.b), it damps on travelling along the line A → B, the net result being stable. Notice that the growth along C → D (or damping along A → B) is consistent with the explanation given in the previous paragraph.

To complete the solution, one has to consider the effect of the last term in Eq. (20), which is analogous to the driving term in an oscillator problem. It is then necessary to expand the last term in terms of the eigenfunctions (25). This is done easily with the help of the adjoint eigenfunctions obtained below. The behavior of the complete solution can only be analyzed numerically, and will not be discussed further in this paper.

The adjoint equation defined by Eq. (15) is, if written in the matrix notation as in Eq. (20), as follows:

$$\left[\frac{\partial}{\partial t} + \Delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{N}{2L\Delta} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \left(\alpha \frac{\partial}{\partial x} - \beta\right)\right] \begin{pmatrix} C \\ D \end{pmatrix} = 0 \ . \tag{29}$$

The functions C and D satisfy the same boundary condition as in Eq. (22). The solution of the adjoint system is quite analogous to the original system. One finds that is has the same set of eigenvalues ω_n, and the corresponding eigenfunctions are as follows:

$$\begin{pmatrix} C \\ D \end{pmatrix} = e^{i\omega_n t} e^{Qx} \left[e^{-iK_n \cdot x} \begin{pmatrix} \omega_n/k_n^{+} + \Delta \\ \omega_n/k_n^{+} - \Delta \end{pmatrix}^{*} - e^{iK_n \cdot x} \begin{pmatrix} \omega_n/k_n^{-} + \Delta \\ \omega_n/k_n^{-} - \Delta \end{pmatrix}^{*} \right] . \tag{30}$$

In the above, * denotes the complex conjugate. After a lengthy algebraic computation, one confirms that the functions in Eq. (25) are orthogonal to those in Eq. (30) for $n \neq m$.

IV. EXTENSION

For an arbitrary shaped unperturbed distribution $\psi_0(p)$, it is convenient to introduce a new set of canonical variables (ρ,θ) as follows:

$$\begin{cases} \rho = |p| \\ \theta = x\, E(p) \end{cases} \quad \text{or equivalently} \quad \begin{cases} p = \rho\, E(\theta) \\ x = |\theta| \end{cases} \tag{31}$$

Here $E(z)$ is a step function defined to be +1 (-1) when $z > 0$ ($z < 0$). The variables (ρ,θ) are analogous to the polar variables in phase plane.

The Vlasov's equation becomes

$$\frac{\partial}{\partial t}\psi(t,\theta,\rho) + \rho\frac{\partial\psi(t,\theta,\rho)}{\partial\theta} + f(\rho)\left(\alpha\frac{\partial}{\partial\theta} + \beta\, E(\theta)\right)\lambda(t,\theta) = 0,$$

$$\lambda(t,\theta) = \int_0^\infty dp[\psi(t,\theta,\rho) + \psi(t,-\theta,\rho)]. \tag{32}$$

The boundary condition (7) can be translated as follows:

$$\boxed{\psi(t,\theta,\rho) \text{ is continuous at } \theta = 0 \text{ and } \psi(t,L,\rho) = \psi(t,-L,\rho).} \tag{33}$$

It follows that $\psi(t,\theta,\rho)$ can be regarded as a continuous periodic function of period 2L in θ. Therefore, an eigenfunction with frequency ω must have the following expansion:

$$\psi = e^{i\omega t}\sum_n e^{ik_n\cdot\theta} A_n(\rho). \tag{34}$$

Inserting Eq. (34) into Eq. (32), one obtains a recursion relation (of infinite order) to determine $A_n(\rho)$. The eigenvalue condition involves an infinite determinant as follows[3]:

$$\begin{vmatrix} \lambda_1(\omega), & \frac{1}{3}, & 0, & \frac{1}{15}, & 0, & \dots \\ -\frac{1}{3}, & \lambda_2(\omega), & \frac{1}{5}, & 0, & \frac{1}{21}, & \dots \\ 0, & -\frac{1}{5}, & \lambda_3(\omega), & \frac{1}{7}, & 0, & \dots \\ -\frac{1}{15}, & 0, & -\frac{1}{7}, & \lambda_4(\omega), & \frac{1}{9}, & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0 \qquad (35)$$

Here

$$\lambda_n(\omega) = \pi \cdot \frac{1 + \alpha K_n D_n(\omega)}{4\beta n D_n(\omega)}, \quad D_n(\omega) = \int_{-\infty}^{\infty} dp\, f(p) \frac{1}{K_n p - \omega}. \qquad (36)$$

Eq. (35) could probably be analyzed numerically.[4]

For the case where the unperturbed distribution consists of two steps as shown in Fig. (3), the procedure described in Section II is still tractable. The dispersion relation (10) becomes a quartic equation in k, which could in principle be solved to obtain $k_\ell(\omega)$, $\ell = 1, 2, 3, 4$. The eigenvalue equation is a complicated transcendental equation involving $k_\ell(\omega)$, which is presently being studied numerically. In some limiting cases, the equation can even be studied analytically. For example, suppose that the shaded area in Fig. (3) is very small compared to the unshaded one. One can then set up a perturbation series $\omega_n = \omega_n^\circ + \omega_n^1 + \omega_n^2 + \dots$. Here ω_n° is the frequency for the simple step function discussed in Section III. The lowest order frequency shift ω_n^1 was computed. Although the formula is too lengthy to be recorded here, ω_n^1 was found to be real for all values of Z_R and Z_C. Therefore, the motion is again stable.

With the examples treated so far, one may get the impression that a bunched beam in our model linac is always stable. However, that is not generally so. A simple example is the case in which $\beta = 0$ and $\psi_o(p)$ has a dip as in Fig. (4). It is not hard to show that when $\beta = 0$, the wave vector k_n is real and given by

$$k_n = K_n \tag{37}$$

Since k_n is known, ω_n is determined from the dispersion relation (10). The situation here is quite similar to the coasting beam case. Now, it is well-known that a coasting beam can develop instability if $\psi_o(p)$ has a dip as shown in Fig. (4) even when the resistive part β vanishes. Thus, one concludes that the longitudinal motion of our model linac can be unstable. However, the instabilities seem to be driven mainly by the geometrical parameters of $\psi_o(p)$, and not by the resistive part of the impedance.

V. DISCUSSIONS AND CONCLUSIONS

In this paper, a linac specified by a) and b) in Sec. (I) is discussed in detail, limiting ourselves to analytical methods. The results obtained here are encouraging in the sense that a sensible theoretical approach to the longitudinal dynamics of bunched beam could be formulated and solved. However, a lot of further work is necessary both within the framework of the present model and beyond. The paper will be concluded by listing some of the immediate problems.

First, within the framework of the model, they are a) the problem of obtaining eigenfunctions and eigenvalues for a general distribution $\psi_o(p)$;

this problem was briefly touched on in Section IV. One might try, for example, to solve the determinant equation (35) by numerical method. b) The initial value problem; given an intitial disturbance at t = 0, how does it develop in time? By analyzing this problem, one would like to understand how the wave disperses and how it reflects at the walls of the potential. The results of Section III should be useful for this analysis.

The model studied in this paper was shown to be stable in most cases (except when $\psi_0(p)$ has a dip), and it does not explain the microwave instabilities observed at CERN and FNAL. Therefore, it is important to consider a more realistic model. In doing so, there are the following problems: c) Replacing the rectangular well potential by a more realistic potential. A realistic potential cannot be rectangular. Also, one would like to understand the differences between the longitudinal dynamics in induction linacs and in storage rings. d) Replacing the self force (Eq. (1)) by a more realistic one. For this purpose, one has to start from a first principle, Maxwell's equations, etc. A possible improvement of Eq. (1) is proposed by L. Smith, who suggested that the factor $p_0\lambda(x)$ in Eq. (1) be replaced by

$$p_0\lambda(x) \rightarrow \int (p_0 + p)\, \psi(t,x,p)\, ds \ . \tag{38}$$

The above replacement is made plausible by arguing that the force should be proportional to the current rather than charge density. With this modification, the Vlasov's equation can again be solved exactly when $\psi_0(p)$ is given by Fig. (2). Following similar steps as in Section III, one finds that the eigenvalues for the edge waves are given by

$$\omega_n = \frac{i\gamma \pm \sqrt{-\gamma^2 + (1+\delta)(1+\alpha')\,\Delta^2(k_n{}^2 + Q^2)}}{1+\delta} \ , \tag{39}$$

where

$$\gamma = \frac{\varepsilon\beta'(1+\alpha')}{4(1+\alpha')} \left(\frac{\Delta}{L}\right) , \quad \delta = \frac{\alpha'\varepsilon}{4(1+\alpha')} , \quad \varepsilon = \frac{\Delta}{p_o} . \tag{40}$$

Notice that ω_n now has a positive imaginary part. Therefore, the perturbation is damped, and again there is no instability. This is easy to understand because the growth rate along C → D in the phase plane is less than the damping along A → B due to the replacement Eq. (38).

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REFERENCES AND FOOTNOTES

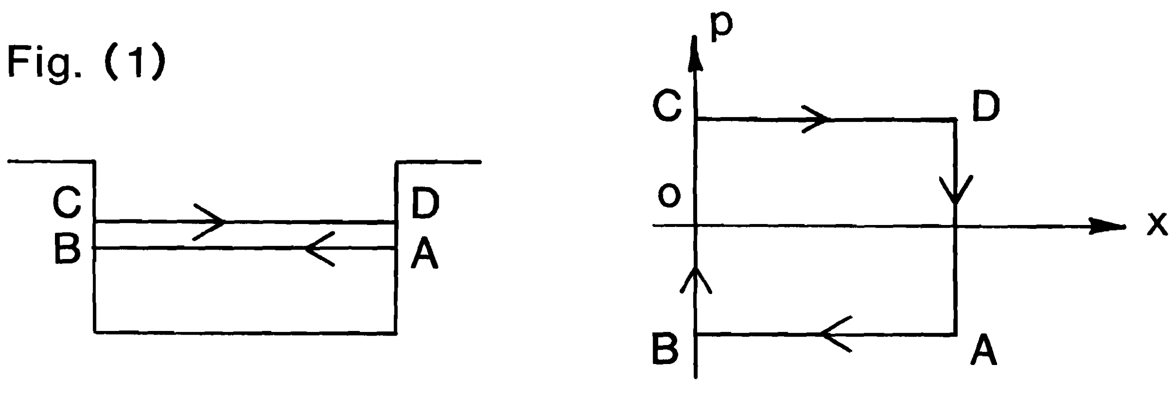
1. N. G. Van Kampen, Physica 21, 949 (1955); See also F. Sacherer, CERN/SI-BR/72-5.

2. K. J. Kim, LBL-9741, presented at 1979 ISABELLE Workshop (to be published).

3. The eigenvalue problem for the Neuffer's model (D. Neuffer, IEEE Trans. NS-26, 3031 (1979)), properly extended to incorporate the resistive part of the impedance, can also be formulated in terms of an infinite determinant, as shown by L. Smith (private communication).

4. Eq. (35) was tested for $\psi_0(p)$ given by Fig. (2). The infinite determinant is approximate by retaining only the first two rows and columns. The approximate eigenvalues so obtained was in good agreement with the results of Section III.

Fig. (1)



(a) (b)

Fig. (2)

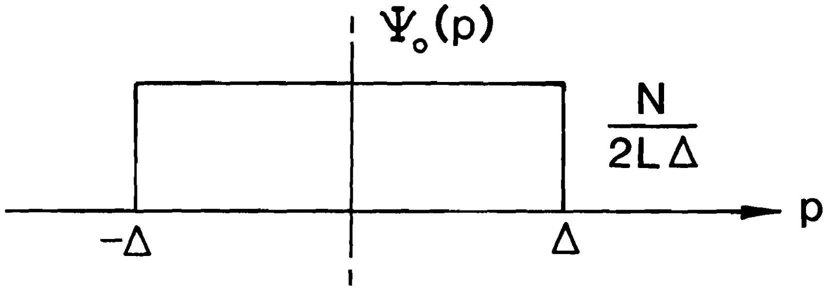


Fig.(3)

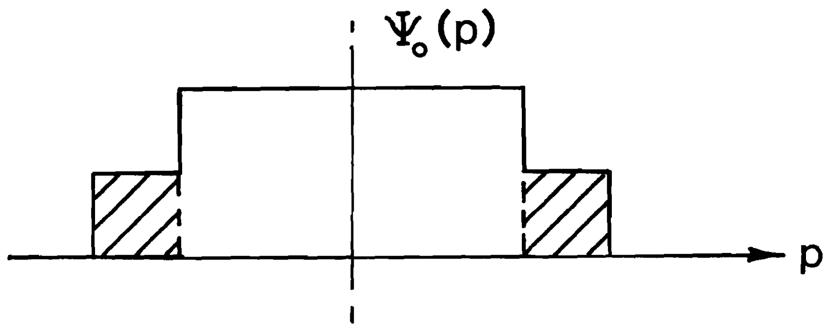


Fig. (4)