FLAVOR DYNAMICS

Makoto KOBAYASHI

National Laboratory for High Energy Physics

Cho-machi, Tsukuba-gun, Ibaraki-ken, 305, Japan
§1 Introduction

The Weinberg-Salam theory of unified weak and electromagnetic interactions has achieved a remarkable success in the neutral current phenomena, and no one can deny its excellent features as a fundamental theory. In contrast to the simplicity of gauge coupling, however, this theory contains too many parameters in the interactions with the Higgs boson, which generate the masses of quarks and leptons, the Cabibbo mixing angle etc, through the spontaneous breaking of the gauge symmetry. Furthermore we do not know how many quark or lepton flavors exist. Thus, it can be said that our understanding of the nature still lacks some fundamental physical principle which regulates this complication.

Recent discovery of the $T$ particle, which indicates the existence of the $b$-quark, has raised the importance of this problem.

The purpose of this lecture is to present an introduction to overcome the above problem. We concentrate ourselves to quark masses and mixing angles. How to treat them in a certain extended version of the Weinberg-Salam (W-S) model will be explained in detail. A basic knowledge about the W-S theory is assumed, but not necessarily required. All discussions will be made in the tree approximation, and no renormalization problem will be considered since it does not change the essential features of the following discussion except finite corrections.

In §2, the construction of quark mass terms is discussed and the number of mixing parameters in the sequential scheme is given. §3 is devoted to some phenomenological analyses in the six-quark case. In §4, an attempt to obtain relations between masses and mixing angles is explained.

§2 Quark masses and the generalized Cabibbo mixing

2.1 Gauge fields and Higgs fields

First, we recall the basic quantities of the W-S model. The model is the SU(2) x U(1) gauge theory, which requires four gauge fields:

\[
\begin{align*}
A_a & \quad \text{SU}(2), \\
B & \quad \text{U}(1)
\end{align*}
\]

where $Y$ is the weak hypercharge which describes the transformation property under U(1). Spontaneous symmetry breaking is realized by introducing a Higgs doublet:

\[
\Phi = \begin{pmatrix} 
\phi^+ \\
\phi
\end{pmatrix},
\]

\[(2.1)\]

It is useful to notice that

\[
\Phi = \begin{pmatrix} 
\phi^+ \\
\phi
\end{pmatrix},
\]

\[(2.2)\]

is also transformed as a doublet. The Lagrangian of this partial system is given by

\[
L = -\frac{1}{4} (F^a_{\mu\nu})^2 - \frac{1}{4} F_{\mu\nu}^2 + \left| (\partial_\mu \phi^a_{\nu} - ig_a \frac{1}{2} A^a_{\mu} - ig' A^a_{\mu} - ig B_{\mu}) \phi \right|^2 - V(\Phi)
\]

\[(2.4)\]

where

\[
\begin{align*}
F_{\mu\nu}^a &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu B^c_\nu \\
F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu
\end{align*}
\]

\[(2.5)\]
In the tree approximation, the vacuum expectation value of $\phi$ is given by

$$
\langle \phi \rangle = \begin{bmatrix} 0 \\ \phi/\sqrt{2} \end{bmatrix},
$$

(2.7)

where

$$
\phi = \sqrt{\lambda}/(\sqrt{2} g),
$$

(2.8)

The last equality follows from the equivalence with the Fermi's theory at the low energy. Then, the shifted fields are expressed as follows;

$$
\phi = \begin{bmatrix} 0 \\ \phi/\sqrt{2} + i \frac{1}{\sqrt{2}} \left( \frac{x^1 + x^2}{\sqrt{2}} \right) \end{bmatrix},
$$

(2.9)

where $x^i$ are the Goldstone modes to be absorbed into the weak bosons and $\phi$ describes the physical Higgs boson. The physical gauge fields are given by

$$
\begin{align*}
W_+ &= \frac{1}{\sqrt{2}} \left( a^3 - i a^2 \right), \\
Z_+ &= \frac{1}{\sqrt{2}} \left( ga^3 - g a^2 \right), \\
A_0 &= \frac{1}{\sqrt{2}} \left( g a^3 + g a^2 \right).
\end{align*}
$$

(2.10)

where $W_+$, $Z_+$, and $A_0$ are the charged, neutral weak bosons and the photon, respectively.

2.2 Quarks --- four-quark case.

In this subsection and the next one, we describe the way of constructing the quark parts of the Lagrangian, especially their mass terms. We first illustrate it in a simple four-quark case, so called G.I.M. model$^{(2)}$, so that notational complexity of the general case which is discussed in the next subsection may not prevent understanding of physical implications.

Interactions between the gauge bosons and the quarks are determined by the transformation property of the quarks under $SU(2) \times U(1)$. Since we are considering parity violating interactions, there is no need for the left and right components of the quark fields to have the same transformation properties. The left and right components are defined by

$$
\begin{align*}
\bar{u}_L &= \frac{1}{2} (1 + \gamma_5) \bar{u}, \\
\bar{u}_R &= \frac{1}{2} (1 - \gamma_5) \bar{u}.
\end{align*}
$$

(2.11)

Note that we have simple but useful relations;

$$
\begin{align*}
\bar{u}_L \gamma^\mu L &= \bar{u}_R \gamma^\mu R = \bar{u}_L \gamma^\mu R = \bar{u}_R \gamma^\mu L = 0.
\end{align*}
$$

(2.12)

In the G.I.M. model, the assignments of transformation properties are as follows

$$
\begin{align*}
\begin{bmatrix} u \\ d' \end{bmatrix} &= Y = 1/3, \\
\begin{bmatrix} c' \\ s' \end{bmatrix} &= Y = 1/3,
\end{align*}
$$

where

$$
\begin{align*}
\begin{bmatrix} u \\ d' \end{bmatrix} L &= Y = 1/3, \\
\begin{bmatrix} c' \\ s' \end{bmatrix} L &= Y = 1/3,
\end{align*}
$$

$$
\begin{align*}
\begin{bmatrix} u_x \\ s_x \end{bmatrix} R &= Y = 1/3,
\end{align*}
$$

$$
\begin{align*}
\begin{bmatrix} u_x \\ s_x \end{bmatrix} R &= Y = 1/3,
\end{align*}
$$

- 46 -
where \( u, d, s, c \) are the usual quark fields which diagonalize the quark mass terms. For later convenience we note that we can represent two left-handed doublets equivalently as

\[
\begin{align*}
|u|_L, & \quad Y = 1/3, \quad u' = \cos \theta_u - \sin \theta_u c, \\
|s|_L, & \quad Y = 1/3, \quad c' = \sin \theta_u + \cos \theta_u c.
\end{align*}
\]

(2.14)

Writing down the kinetic parts of the lagrangian, which include the couplings to the gauge fields, is straightforward from the above assignment;

\[
L = i \left[ \bar{u} D \gamma \gamma (\partial \bar{u} - ig \frac{2}{3} \alpha^B \bar{u} - ig \frac{1}{6} B_0 \bar{u}) | u \right] + [u + \bar{c}] L + \left(u + \bar{c}ight) + (u + c) + (d + s).
\]

(2.15)

Since, as is seen in (2.13), the left and the right components have different transformation properties, any scalar density of quark bilinear form cannot be gauge invariant. Therefore, quark masses arise only from the yukawa coupling to the higgs field, through the spontaneous breaking of the symmetry. Then, we find that the proper form of the lagrangian is

\[
L = \bar{u} D \gamma \gamma (\partial \bar{u} - ig \frac{2}{3} \alpha^B \bar{u} - ig \frac{1}{6} B_0 \bar{u}) | u \right] + [u + \bar{c}] L + \left(u + \bar{c}ight) + (d + s).
\]

(2.16)

Replacing the higgs field by its vacuum expectation value, we can easily check the adequacy of (2.16). Note that other gauge invariant forms, such as

\[
\bar{u} \gamma \gamma \gamma (\bar{t} + \bar{c}) L, \quad etc.
\]

(2.18)

do not appear, since they violate the assumption that \( u, d, s, c \) diagonalize the mass terms.

An important fact is that, in this scheme, the requirement of gauge invariance does not impose any constraints on the values of masses and mixing angles.

2.3 Quarks --- generalized sequential scheme.

We have already had experimental indications of the existence of more than four quarks. The \( \bar{t} \) and \( \bar{t}' \) are interpreted as \( \bar{t}_0 \) states of a heavy quark, \( b \). A simple and attractive assignment of \( b \)-quark is such that \( b \) forms another left-handed doublet with \( t \)-quark which is not yet discovered. In view of this proliferation of the quark flavour, we give here a general framework for the sequential left-handed doublet scheme.

Let us assume that we have \( N \) sequences of doublet and therefore 2N quark flavours. Then, they are denoted as follows;

\[
L^1 = \left[ \begin{array}{c}
\bar{u} \\
\bar{d} \\
\bar{s} \\
\bar{b} \\
\bar{u}' \bar{d}' \\
\bar{s}' \bar{b}' \\
\bar{u}'' \bar{d}'' \\
\bar{s}'' \bar{b}'' \\
\cdots \\
\bar{u}^{N} \bar{d}^{N} \\
\bar{s}^{N} \bar{b}^{N}
\end{array} \right], \quad Y = 1/3,
\]

(2.17)
where $\bar{\psi}_i$ and $\bar{\psi}_i$ stand for the physical quark fields and $i$ runs over $u,c,t \ldots$ and $j$ runs over $d,s,b \ldots$. Note that the position of sufficies of $L$ and $R$ indicates which set of the flavour indices they should take.

The generalized Cabibbo-like mixing is expressed by an $N \times N$ unitary matrix $U$, whose $ii$-component is denoted as $U_{ii}$. The mixing matrix $U$ must be unitary in order that the kinetic parts of the quark fields have a properly normalized diagonal form (See below). Corresponding to (2.14), we introduce the following:

\[
E_{\mu i} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \\ \end{bmatrix}, \quad \mu = \{u,c,t \ldots \}  
\]

In this notation, the Lagrangian has a simple form:

\[
L = \frac{1}{\lambda^2} \sum_{\mu} (\partial_{\mu} \bar{\psi}_i \gamma^\mu \psi_i - ig \bar{\psi}_i \gamma^\mu \gamma^5 W_k \psi_i + ig' \bar{\psi}_i \gamma^\mu \gamma^5 Z \psi_i)^2  
\]

\[
+ \sum_{\mu} \gamma^\mu \gamma^5 (\bar{\psi}_i \gamma^\mu \psi_i + \bar{\psi}_i \gamma^\mu \gamma^5 W_k \psi_i) \bar{\psi}_i \gamma^\mu \gamma^5 W_k \psi_i  
\]

\[
- \bar{\psi}_i \bar{\psi}_i \mu^2 \gamma^\mu \gamma^5 W_k \psi_i + \bar{\psi}_i \bar{\psi}_i \gamma^\mu \gamma^5 W_k \psi_i  
\]

\[
- \bar{\psi}_i \bar{\psi}_i \gamma^\mu \gamma^5 W_k \psi_i + \bar{\psi}_i \bar{\psi}_i \gamma^\mu \gamma^5 W_k \psi_i  
\]

(2.23)

where

\[
\bar{Q}_k = \sqrt{2} m_k / r.  
\]

(2.22)

The range of summation will be self evident from the above notational convention.

To make a further investigation, we rewrite the Lagrangian in terms of the physical fields:

\[
L = \sum_{\mu} \left( \bar{U}_{\mu} \bar{\psi}_i \gamma^\mu (1 - \gamma^5) W_k \psi_i - m_k \bar{\psi}_i \psi_i \right)  
\]

\[
+ \sum_{\mu} \left( \bar{U}_{\mu} \bar{\psi}_i \gamma^\mu (1 - \gamma^5) W_k \psi_i + \bar{U}_{\mu} \bar{\psi}_i \gamma^\mu \gamma^5 W_k \psi_i \right)  
\]

(2.23)

where $Q_k$ is the electric charge of $\bar{\psi}_k$ and

\[
\bar{Q}_k = \left[ \frac{1}{2} - 2 \frac{Q_k}{r} + \frac{Q_k^2}{r^2} - \frac{1}{2} \gamma_5 \right], \quad k = u,c \ldots  
\]

\[
\bar{Q}_k = \left[ \frac{1}{2} - 2 \frac{Q_k}{r} + \frac{Q_k^2}{r^2} + \frac{1}{2} \gamma_5 \right], \quad k = d,s \ldots  
\]

(2.24)

We have taken the unitary gauge, so that no Goldstone modes appear in (2.23).

An important fact in this scheme is that flavour changing interactions arise only from the last line in (2.23). In such a case, some of the phases of $U_{ii}$ can be absorbed into the phase convention of the quark. - 51 -
fields and do not have physical significance. As an example, let us consider the case of $N = 2$. The most general form of $2 \times 2$ unitary matrix is given by

$$U = \begin{pmatrix}
\cos \theta & e^{i\phi} \\
\sin \theta & e^{i\phi}
\end{pmatrix}$$

Then, we change the phase convention of quark fields as

$$U_k \rightarrow e^{i\delta_k} \bar{U}_k$$

with

$$\delta_u = \delta_{\bar{u}} = \alpha + \beta, \quad \delta_d = \delta_{\bar{d}} = \alpha + \gamma, \quad \delta_s = \delta_{\bar{s}} = -\beta - \gamma. \quad (2.27)$$

In the new convention, $U$ looks like

$$U = \begin{pmatrix}
d & s \\
\cos \theta & e^{i\phi} \\
\sin \theta & e^{i\phi}
\end{pmatrix}$$

This is nothing but the G.I.M. model. So, we find that the G.I.M. model does not lack any generality in the four-quark (two-doublet) case.

Now, let us turn to an arbitrary $N$. An $N \times N$ unitary matrix has $N^2$ parameters. Among $2N$ phase parameters of $2N$ quark fields, $2N-1$ are responsible for the reduction of number of parameters of $U$. Note that the overall phase of quark fields does not affect to $U$. As a result, we find that the number of parameters of the Cabibbo-like mixing is given by

$$N^2 - (2N - 1) = (N - 1)^2 \quad (2.29)$$

2.4 CP violation.

Here, we investigate the CP properties of the Lagrangian (2.23). Let us consider the following CP-transformation:

$$\bar{Q}_{L(R)} \rightarrow -\gamma C \bar{Q}_{L(R)}, \quad \bar{\nu}_{L(R)} \rightarrow -\gamma C \bar{\nu}_{L(R)}$$

where $C$ is the usual charge conjugation matrix with

$$C_T = -C, \quad C_T \gamma = \gamma C, \quad \eta = \begin{cases}
1 & \nu = 0 \\
N & \nu = 1, 2, 3.
\end{cases}$$

and

$$\eta \mu = \begin{cases}
0 & \nu = 0 \\
1 & \nu = 1, 2, 3.
\end{cases}$$

It is easy to check that the first bracket in (2.23) is invariant under the transformation (2.30). Our main concern is the charged current interaction part, which is
Under (2.30), this is transformed into

\[
\begin{align*}
\sum_{i=\{u, c \ldots \}} \sum_{j=\{d, s \ldots \}} \left( U_{ij} \gamma_5 \right) & (1 - \gamma_3) s_{ij} + U_{ij} \gamma_5 (1 - \gamma_3) s_{ij} \\
& \quad \left( U_{ij} \gamma_5 \right) (1 - \gamma_3) s_{ij} + U_{ij} \gamma_5 (1 - \gamma_3) s_{ij}
\end{align*}
\]

(2.33)

Therefore, if all \( U_{ij} \) are real, (2.32) is CP-invariant. Here, however, we should remind the problem of the phase convention. In general, we can multiply an arbitrary phase factor to the right hand side of the transformation of each quark field in (2.30). But this phase factor can be removed by changing the phase convention of quark fields. This implies conversely that we can assume CP-transformation of the form (2.30) for the field with phase convention whatever we like. Therefore, if we can make all \( U_{ij} \) to be real by suitably adjusting the phases of quark fields, then, defining the CP-transformation as (2.30) for the new fields which make \( U_{ij} \) real, we can see that the system is CP-invariant.

As we have seen, in the case of \( N = 2 \), \( U_{ij} \) can be made real. Therefore we have no CP-violation in this case. For a general \( N \), a unitary matrix whose elements are real is nothing but an orthogonal matrix. The most general \( N \times N \) orthogonal matrix has \( N(N - 1)/2 \) parameters. This number should be compared with \( (N - 1)^2 \) in (2.29). Thus the number of the parameter responsible for CP-violation is given by

\[
(N - 1)^2 - N(N - 1)/2 = (N - 1)(N - 2)/2
\]

(2.34)

From this we can conclude that for \( N \geq 3 \) we have a possibilities of CP-violation through the Cabibbo-like mixing. 3)

In the above arguments, our assumption that the Higgs field is only one doublet has played an essential role. If we have many Higgs fields, we have possibilities of CP-violation through the Higgs boson exchange, even for \( N = 2 \). (For example, see ref. 4) To determine what mechanism is true, detailed phenomenological analyses are necessary.

§3 Phenomenology in the six-quark scheme

3.1 Parametrization

As seen from (2.29), we have four parameters to describe the Cabibbo-like mixing for \( N = 3 \). Here we adopt the following explicit parametrization:

\[
U = \begin{pmatrix}
\frac{c_1 c_2}{\sqrt{2}}, & \frac{s_1 c_2}{\sqrt{2}}, & \frac{-s_1}{\sqrt{2}} \\
\frac{s_1 c_3}{\sqrt{2}}, & \frac{c_1 c_3}{\sqrt{2}} - s_1 s_2, & \frac{-s_1 c_2}{\sqrt{2}} \\
\frac{-s_1 c_2}{\sqrt{2}}, & \frac{s_1 c_3}{\sqrt{2}} + s_2 c_1, & \frac{c_1 c_2}{\sqrt{2}}
\end{pmatrix}
\]

(3.1)

where

\[ c_1 = \cos \theta_1, \quad s_1 = \sin \theta_1 \]

\[ i = 1, 2, 3. \]

The phase convention is such that the first row and the first column of \( U \) are made real. If \( \delta \) vanishes, then \( U \) is an orthogonal matrix and we have no CP-violation.

In the following we investigate to what extent presently available experimental informations can determine the parameters. Systematical analysis has been done first in ref. 5).

3.2 Nuclear \( \beta \)-decay and \( |\Delta s| = 1 \) semileptonic decays.

Recently, Shrock and Wang 6) made renalysis of the above processes and obtained
\[ |U_{ud}| = |\cos \theta_2| = 0.9737 \pm 0.0025 \]
\[ |U_{us}| = |\sin \theta_2 \cos \delta| = 0.219 \pm 0.011 \]

(3.2)

from which they concluded
\[ |\sin \theta_2| = 0.28 \pm 0.28. \]

(3.3)

We note that \( \theta_2 \) is essentially the Cabibbo angle, and
\[ \theta_2 \gtrsim 13.2^\circ \]

(3.4)

The magnitude of u-b transition is determined by
\[ |U_{ub}| = \sqrt{1 - |U_{ud}|^2 - |U_{us}|^2} \]
\[ \gtrsim 0.06 \pm 0.06 \]

(3.5)

3.3 \( K_L-K_S \) mass difference

Direct measurement of the other components of \( U_{ij} \) useful to determine the rest of parameters are not yet available. Therefore, we must resort to some model dependent arguments. The neutral K-meson system has been well investigated for this purpose.

Let us consider the mass matrix \( M \) of \( K^0-K^0 \) system. It can be decomposed into the dispersive and absorptive parts;

\[ M = \Gamma + \frac{i}{2} \Gamma, \]

(3.6)

where \( \Gamma \) and \( \Gamma \) are 2 \times 2 Hermitian matrices. Since we are considering a

\[ -56 - \]
with

\[ x_1 = \frac{2}{3} \frac{m^2}{m_w^2}, \]  

(3.11)

and \( \beta \) is a model dependent factor. The bag model calculation gives \( \beta \approx 0.4 \).

In order to see how this formula works, we consider the limited case of \( \sin \theta \approx 0 \). Assuming \( x_j \ll 1 \), we have

\[ 2G \frac{m^2}{B} f^2 \sim \frac{1}{c^2 s^2} \sim \frac{1}{\beta^2} \sim \frac{1}{0.4^2} \sim 2.5 \]

(3.12)

which yields

\[ c^4 + \frac{2}{c^2} \frac{m^2}{m_w^2} + \frac{2}{c^2} \frac{1}{1 - \frac{m^2}{m_w^2}} \sim 2.5 \]

(3.13)

where we have used \( \Gamma \approx 0.4 \), \( f_\pi \approx 170 \) MeV, and \( m \approx 1.5 \) GeV.

For example, setting \( m = 15 \) GeV, we have

\[ |x_2| \approx 0.33 \]  

(3.14)

The value of \( |x_2| \) decreases with increasing \( m \).

As seen from (3.3), \( \sin \theta \) is not necessarily small. Therefore the required numerical analysis becomes complicated. This, combined with the constraint arising from the CP violation effects, has been done recently in refs. 9) and 10) (See below).

### 3.4 CP violation

As we have seen in §2-3, in the six-quark scheme we have a possibility of CP violation through the Cabibbo-like mixing. In the following we investigate how the observed CP violating phenomena are described in this scheme. Allowed regions of the parameters are also discussed.

It is worth while reminding the following point. Since the effect of CP violation is very small, the observed CP violation may arise from other additional interaction which is very weak and practically does not change the ordinary weak interaction phenomenology. In such a case, the following arguments should be changed substantially.

Now, we consider again the \( K^0 \rightarrow \pi^0 \) system. The mass matrix \( M \) contains informations on the CP violation. However, we can not say anything about CP violation from the dispersive part alone. The relevant quantity is the phase difference between \( \langle K^0 | \pi^0 \rangle \) and \( \langle K^0 | \pi^0 \rangle \). The latter is given by

\[ \langle K^0 | \pi^0 \rangle = 2\pi \Gamma_p \langle \pi^0 | \pi^0 \rangle \]

(3.15)

where \( \Gamma_p \) is the phase volume factor. Therefore we must investigate first the nonleptonic decay amplitudes.

The \( K^0 \rightarrow \pi^0 \) amplitudes can be written as

\[ \langle \pi^0 | \pi^0 \rangle = e^{i\delta_4} A_4, \]

(3.16)

where \( \langle \pi^0 | \pi^0 \rangle \) denotes the final \( \pi^0 \) state with isospin I, and \( \delta_4 \) is the strong interaction phase shift. Since we have already fixed the phase convention, \( A_4 \) is no longer a real quantity in general. It is necessary not to confuse
with the usual Wu Yang convention in which $A_0$ is real.

Let us consider the quark diagrams which contribute to the $K^0 \rightarrow \pi^-$ decay. The contribution of Fig. 2(a) to $A_0$ is real, because no complex mixing parameters appear in this diagram. Diagrams (b) and (c) contribute only to the $\Delta I = 1/2$ part, and give rise to the imaginary part of $A_0$.

Therefore, we can consider that in our phase convention only $A_0$ is complex and $A_2$ is real. It has been argued that the contribution of (b) may be small because of the OZI rule. Then the diagram (c), so called penguin diagram, is important for the phase of $A_0$, as well as for the $\Delta I = 1/2$ rule in nonleptonic decays. Theoretical estimate, however, seems to have some uncertainty. Here, we consider an experimental upper bound of the phase of $A_0$

\[
\frac{\rho_{0}}{\rho_{B}} \lesssim 0.2
\]

Let us consider the ratio

\[
\frac{\rho_{0}}{\rho_{B}} = \left| \frac{\epsilon - \bar{\epsilon} c'/\epsilon + c'}{1 - 2\epsilon} \right|
\]

where

\[
\epsilon = \frac{<0|T|K_0^->}{<0|T|K^-_0>}, \quad c' = \frac{<2|T|K_0^->}{<2|T|K^-_0>}, \quad \epsilon = \frac{<2|T|K_0^->}{<2|T|K^-_0>}
\]

Since $\epsilon' \ll \epsilon$, we have

\[
\frac{\rho_{0}}{\rho_{B}} \lesssim |1 - 3\epsilon' + 3\epsilon|
\]

where $\epsilon$ is defined by

\[
\epsilon = |A_0| e^{i\alpha}
\]

It is known that the contribution of the penguin diagram gives $\epsilon < 0.11$, which implies $|\rho_{0}/\rho_{B}| < 1$.

From the present experimental value of

\[
\frac{\rho_{0}}{\rho_{B}} = 1.02 \pm 0.04,
\]

we have

\[
|\epsilon| < 0.2
\]
It is critical whether this bound is compatible with the contribution of the penguin diagram. Improvement of data on $|V_{ub}|/|V_{cb}|$ will provide more stringent test of this scheme.  

Now we turn to the estimation of the mixing parameters. In terms of the mass matrix elements, $c$ is given by

$$
\begin{align*}
    c &= \frac{\text{Im}\langle K^0|\Gamma|K^0\rangle - \frac{1}{2}\text{Im}\langle K^0|\pi|K^0\rangle}{\text{Im}\langle K^0|\pi|K^0\rangle - \frac{1}{2}\text{Im}\langle K^0|\pi|K^0\rangle + 1} \\
    &= \frac{\text{Im}\langle K^0|\Gamma|K^0\rangle}{\text{Im}\langle K^0|\pi|K^0\rangle} \\
    &\geq m_\Delta
\end{align*}
$$

(3.23)

where we have used the approximate relation

$$
\text{Im}\langle K^0|\Gamma|K^0\rangle \geq m_\Delta
$$

(3.24)

which holds because the main contribution to $\Gamma$ comes from the $I = 0 \pi \pi$ state. From (3.22), $a$ in (3.23) may be negligible. In this case, $c$ is given by in terms of the mixing parameters by using (3.9), and experimental constraint on them can be obtained. References 9) and 10) made detailed numerical analyses. Results are seen in Fig. 3, which show regions allowed by $m_\Delta$ and $c$. According to the sign of $\cos \theta$, $c \equiv \cos \theta / |\cos \theta|$, we have two solutions for each $m_\Delta$ and $c_3$.

Although the constraints obtained above are rather loose, we can extract interesting consequences on the heavy quark decays. For example, we can say, that the branching ratio of $b \to c$ decay will be larger than $b \to u$. Further discussion is seen in ref. 9) and 10).
§4 Relations between masses and mixing angles.

4.1 Diagonalization of the mass matrix

In order to understand the close relation between quark masses and mixing angles, we investigate again the origin of quark mass terms. In §2 we started with the gauge multiplets which are written in terms of the physical quark fields. Here we go back to a more fundamental step.

Let us consider $N$ left-handed doublets and $2N$ right-handed singlets:

$$
\begin{align*}
\mathbf{P}_1^T, & \quad \mathbf{P}_{1R}, \quad \mathbf{n}_1^R, \quad i = 1, \ldots, N \\
Y = 1/3, & \quad Y = 4/3, \quad Y = -2/3
\end{align*}
$$

(4.1)

where $\mathbf{P}_i$ and $\mathbf{n}_i$ are not yet related to the physical quark fields. Relations should be determined by diagonalizing the mass terms. The most general Yukawa interaction with the Higgs field invariant under the $SU(2)_L \times U(1)_Y$ transformation is given by

$$
\begin{align*}
L = & \sum_{i,j=1}^{N} \left( \frac{1}{2} g_{ij} \mathbf{P}_i \mathbf{P}_j + \frac{1}{2} g_{ij} \mathbf{n}_i \mathbf{n}_j \right) \\
+ & \sum_{i,j=1}^{N} \left( h_{ij} \mathbf{P}_i \mathbf{n}_j + h_{ij} \mathbf{n}_i \mathbf{P}_j \right)
\end{align*}
$$

(4.2)

Replacing $\phi$ with its vacuum expectation value, we have

$$
\begin{align*}
L = & \sum_{i,j=1}^{N} \left( \frac{1}{2} g_{ij} \mathbf{P}_i \mathbf{P}_j + \frac{1}{2} g_{ij} \mathbf{n}_i \mathbf{n}_j \right) \\
+ & \sum_{i,j=1}^{N} \left( h_{ij} \mathbf{P}_i \mathbf{n}_j + h_{ij} \mathbf{n}_i \mathbf{P}_j \right)
\end{align*}
$$

(4.3)

where we have used a matrix notation with $N \times N$ matrices $M_p$ and $M_n$ defined by

$$
\begin{align*}
(M_p)_{ij} = \frac{g}{\sqrt{2}} \delta_{ij}, & \quad (M_n)_{ij} = \frac{g}{\sqrt{2}} \delta_{ij}
\end{align*}
$$

(4.4)

Note that $\mathbf{P}_L$ and $\mathbf{n}_L$, as well as $\mathbf{P}_R$ and $\mathbf{n}_R$, are independent quantities and therefore $M_p$ and $M_n$ are not necessarily Hermitian. Diagonalisation is performed by the unitary transformations

$$
\begin{align*}
\mathbf{P}_i^L = U_P \mathbf{P}_i^L, & \quad \mathbf{P}_i^R = U_P \mathbf{P}_i^R \\
\mathbf{n}_i^L = U_n \mathbf{n}_i^L, & \quad \mathbf{n}_i^R = U_n \mathbf{n}_i^R
\end{align*}
$$

(4.5)

The unitary matrices $U_P$, $V_P$, $U_n$ and $V_n$ are determined so that $M_p$ and $M_n$ become diagonal;

$$
\begin{align*}
U_P M_P U_P^T = & \begin{bmatrix}
\mathbf{n}_0 & 0 \\
0 & \mathbf{n}_0
\end{bmatrix} \\
V_P M_P V_P^T = & \begin{bmatrix}
\mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{bmatrix}
\end{align*}
$$

(4.6)

This is possible, since any complex matrices can be diagonalized by two unitary matrices as shown below.

Let us consider an arbitrary matrix $M$. Since $M^* M$ is Hermitian, usual diagonalization can be performed by a unitary matrix $U$ as

$$
U M U^T = \begin{bmatrix}
\mathbf{u}_0 & 0 \\
0 & \mathbf{u}_0
\end{bmatrix}
$$

(4.7)

It is easy to see that $\mathbf{u}_0 \geq 0$. For the purpose of simplicity, let us assume here that no zero eigenvalues exist. Then, we can define
Since we have

\[ V^+ = \sqrt{2/\beta} U M^+ \frac{1}{\beta} \]

(4.9)

\[ V \text{ is unitary. Equation (4.8) can be rewritten as} \]

\[ V M^+ U^T = \sqrt{2/\beta} \]

(4.10)

which completes the proof.

Now the original left-handed doublets are expressed in terms of the physical quark fields as

\[ p_{\ell} = \begin{cases} \frac{1}{\sqrt{2}} \left( \begin{array}{c} \bar{u_c} \\ \bar{d} \end{array} \right) & \ell = 1, \ldots, N \end{cases} \]

(4.11)

By taking linear combinations of them, we can rewrite them in the following way,

\[ \tilde{p}_\ell = \sum_{j=1}^{n} \frac{1}{\sqrt{2}} \left( \begin{array}{c} \bar{u} \\ \bar{c} \end{array} \right)_{\ell,j} \]

(4.12)

This is nothing but (2.19) with

\[ U = \sqrt{2/\beta} p_{\ell} \]

(4.13)

From the above arguments, we can see that the original Higgs coupling (4.2) completely determines both the quark masses and mixing angles.

As seen in (4.13), the Cabibbo-like mixing arises from the difference between \( U_p \) and \( U_n \).

In the present scheme, however, no constraints are imposed on the mass matrices \( M_p \) and \( M_n \) by the SU(2) \( \times \) U(1) gauge invariance alone. Accordingly it has no predictive powers on the quark masses and the mixing angles. It is hard to think that a fundamental theory has so many free parameters. We believe that there exists a missing principle which governs Higgs interactions.

An attempt, which may provide a hint to this problem, is to derive some relations between masses and mixing angles by imposing higher symmetries. This will be discussed in the next subsection.

4.2 Discrete symmetry and Cabibbo angle

Gauge symmetry larger than SU(2) \( \times \) U(1) have been considered from many motivations. Another possibility of a larger symmetry is a discrete symmetry, which is fully utilized in the following discussions. The basic idea is to obtain constraints on the mass matrices as a result of certain discrete symmetries. Below we illustrate it in a simple model.\(^{12}\)

Let us consider the SU(2)_L \( \times \) SU(2)_R \( \times \) U(1) gauge theory instead of \( SU(2) \times U(1) \).\(^{13}\) To make a discussion simple, we consider the four-quark case. Then, we have

\( ^{12}\) We have interesting models within the SU(2) \( \times \) U(1) gauge theory.\(^{13}\)

They, however, require rather involved calculations.
\[ L_i = \begin{pmatrix} p_i \\ n_i \end{pmatrix}, \quad R_i = \begin{pmatrix} p_i \\ \bar{n}_i \end{pmatrix}, \quad i = 1, 2 \]

(2, 2) \quad (1, 2) \quad (3, 2)

(4.15)

where \((2, 1)\), for example, implies SU(2)_L doublet and SU(2)_R singlet.

Higgs fields are two complex 2 \times 2 matrices:

\[ \phi, \quad \chi \]

(2, 2) \quad (2, 2)

(4.16)

Note that \(\phi \equiv \phi^{+}\phi\) and \(\chi \equiv \chi^{+}\chi\) are also transformed as \((2, 2)\) representation. In this system, the most general Higgs-quark coupling invariant under SU(2)_L \times SU(2)_R \times U(1) is

\[ L \frac{1}{2} \sum_{i,j=1}^{2} \left[ g_{i,j} \bar{R}_i \phi \bar{L}_j + g_{i,j} \bar{R}_i \chi \bar{L}_j \right] + h.c. \]

(4.17)

At this stage we have no constraints on the mass matrices for general vacuum expectation values of \(\phi\) and \(\chi\). First we assume a discrete symmetry under the following transformation:

\[ L_j \rightarrow iL_j, \quad \phi \rightarrow -i\phi, \quad \chi \rightarrow -i\chi \]

(4.18)

From this invariance we have

\[ g_{i,j} = h_{i,j} = 0. \]  

(4.19)

Next, we require a further discrete symmetry under the transformation,

\[ L_1 \rightarrow iL_1, \quad R_1 \rightarrow -iR_1, \]
\[ L_2 \rightarrow L_2, \quad R_2 \rightarrow R_2, \]
\[ \phi \rightarrow \phi, \quad \chi \rightarrow -i\chi, \]  

(4.20)

from which we obtain

\[ g_{11} = g_{12} = g_{21} = h_{11} = h_{22} = 0. \]  

(4.21)

Finally we assume the invariance under parity transformation:

\[ L_j \rightarrow R_j, \quad \phi \rightarrow \phi, \quad \chi \rightarrow \chi \]  

(4.22)

Then (4.17) becomes

\[ L = -g_{i,j} \bar{R}_i \phi \bar{R}_j - h_{i,j} \bar{R}_i \chi \bar{R}_j + h.c. \]

(4.23)

where \(g\) is a real constant.

General vacuum expectation values of \(\phi\) and \(\chi\) are given by

\[ \langle \phi \rangle = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad \langle \chi \rangle = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \]

(4.24)

where \(v_1\) and \(x_2\) are complex numbers. Replacing \(\phi\) and \(\chi\) in (4.23) by (4.24), we have quark mass terms of the following form:
\[ L = - \sum_{i=1}^{2} \left( \begin{array}{c} a^i \hline b \\ c \end{array} \right) \begin{pmatrix} P_2 \\ \bar{P}_2 \end{pmatrix} - \frac{1}{n} \left( \begin{array}{c} a^i \hline b \\ c \end{array} \right) \begin{pmatrix} n^2 \\ \bar{n}^2 \end{pmatrix} + \text{h.c.} \]

\[ - \sum_{i=1}^{2} \left( \begin{array}{c} a^i \hline b \\ c \end{array} \right) \begin{pmatrix} P_2 \\ \bar{P}_2 \end{pmatrix} = \frac{1}{n} \left( \begin{array}{c} a^i \hline b \\ c \end{array} \right) \begin{pmatrix} n^2 \\ \bar{n}^2 \end{pmatrix} + \text{h.c.} \]  

(4.25)

where \( a, a', b, c, c' \) and d are complex numbers with \( |a| = |a'|, \quad |c| = |c'| \).

(4.26)

Special forms of mass matrices should be noted.

Now we diagonalize these mass matrices in the following way:

\[ U = U' \begin{pmatrix} \cos \theta_m & -\sin \theta_m \\ \sin \theta_m & \cos \theta_m \end{pmatrix} \]

(4.31)

with

\[ \tan \theta_m = \frac{a_b}{a_c} \]  

(4.32)

Similarly we have

\[ U_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i \phi_n} \end{pmatrix} \]

(4.33)

with

\[ \tan \theta_n = \frac{b_n}{c_n} \]

(4.34)

Then, the Cabibbo mixing matrix is given by

\[ U = U_m U_n \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \]

(4.35)

where

\[ \delta = \phi - \phi_n, \quad \xi_n = \cos \theta_n, \quad \eta_n = \sin \theta_n, \]  

etc. \hfill (4.36)
Reminding the arguments in 2-3, we have
\[
\frac{\tan \theta_c}{\tan \phi} = \frac{\tan \theta_c - \frac{a_4}{a_5} \tan \phi}{1 + \frac{a_4}{a_5} \tan \phi \tan \theta_c}.
\]
(4.37)

Since \( \frac{a_4}{a_5} \ll 1 \), we can write this as
\[
|\tan \theta_c| \approx \frac{a_4}{a_5} |\tan \phi|.
\]

It is well known that this relation is phenomenologically well-satisfied.

From the arguments on the chiral symmetry breaking we have an estimation
\[
\frac{\eta_4}{\eta_5} \approx \frac{1}{20},
\]
(4.38)

which should be compared with
\[
|\tan \theta_c|^2 \approx 0.055.
\]
(4.39)

In this way, we have obtained a remarkable relation by requiring
discrete symmetries. Similar discussions on the six-quark case are seen
in ref. 14). Although these arguments are very suggestive, the discrete
symmetries which we have considered are rather artificial and their
physical meanings are not clear. Further studies are required to reveal a
new physical principle.

References
   A. Salam; in "Elementary Particle Theory" edited by N. Svartholm
   (Almqvist and Wiksell, Stockholm, 1968) P.367.
   213.
    (1979) 1589.
12) S. Weinberg: The problem of mass, Harvard preprint (1979);
14) H. Fritzsch; Phys. Letters 72B (1978) 317;
   76B (1978) 602;