AN OVERVIEW OF BEAM COOLING

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I. Introduction

In this brief presentation I thought it would be of value to do two things: first, give some history of the efforts to develop beam-cooling ideas and, second, present elementary and simplified descriptions of the two successful methods of cooling.

It is my hope that you will find the history of interest, for through it one can see the development of the realization of the limits on devices in the absence of cooling and, then, the various unsuccessful attempts to develop cooling which, in due course, culminated in the effective proposals of electron cooling and stochastic cooling.

It is my further hope that some of you, at least, will find useful an over-simplified description of the two cooling techniques. In particular, I think that two simple formulas, summarizing the primary dependence upon parameters, are quite useful in delineating the conditions required for cooling.

II. History

With the development of fixed-field alternating-gradient accelerators and the concept of rf stacking it became possible,[1] for the first time, to seriously consider colliding-beam devices.[2] (The idea goes back to Wideroe, but earlier there was no means proposed to achieve an interesting luminosity.) It was immediately pointed out by E. Wigner that Liouville's theorem imposed a limit upon performance, but the MURA Group concluded that other limits were, in practice, more severe.

In the very first paper proposing a storage ring, by G. K. O'Neill, a crucial part of the proposal was the use of tapered foils to provide a non-Liouvillian injection mechanism.[3] (As a historical footnote, the storage ring idea--without the use of foils--was independently conceived by W. Brobeck and by D. Lichtenberg, R. Newton, and M. Ross.)

Immediately, the MURA Group set to work to study the effect of foils.[4-6] They were able to show that the non-Liouvillian character of even an "ideal" foil is small; that is, the relative reduction in phase volume is just twice the relative reduction in longitudinal momentum.[6] Thus a foil which would significantly reduce phase volume must significantly change particle energy (which could, of course, be resupplied by an rf cavity). However, because of the small phase volume reduction by an "ideal" foil, scattering by a real foil would more than cancel the reduction in phase volume. (As a second historical footnote, it is interesting that the formulas for foil damping are the same as those for radiation damping of electrons. See H. G. Hereward, Brookhaven Symposium 1961, p. 222; with a comment by K. R. Symon.)

The work on foils showed that although a foil is non-Liouvillian, it is almost Liouvillian. However, a wedged foil was effective at interchanging radial and longitudinal phase space (which is why it superficially looked attractive). This is an interesting possibility in its own right. It had been proposed by L. Smith that cavities with electric fields which varied with radius might also have this capability, but N. Francis at MURA proved this not to be the case.[7]

Stimulated by these considerations and more particularly by the desire to develop a mathematical technique for handling self-field phenomena, R. L. Mills and I examined, in 1958, the limits of applicability of Liouville's theorem to particle beams. In this discussion we identified the need for neglect of small-angle collisions (which is employed in electron cooling) and the neglect of fluctuation phenomena (which is employed in stochastic cooling). Neither of us, however, had the slightest idea of how to circumvent the theorem.

In 1966, Budker introduced the idea of electron cooling, giving G. K. O'Neill credit for independent discovery of the concept.[9] In 1968, S. vander Meer conceived of stochastic cooling, although he did not write the work up until 1972.[10]

Now, in this brief review I have neglected to mention very much more work which provided the background against which were discovered the two successful concepts for cooling. In particular, mention must be made of the studies of radiation damping which provided much insight into cooling.[11]

III. Electron Cooling

The idea is, simply, to couple the proton beam, in a frame moving with the average longitudinal speed of the protons, to something with less transverse and less longitudinal energy (or roughly speaking temperature). Then by simple thermodynamics, one must have a cooling of the protons.

If the coupling is to a system which interchanges energy in the various degrees of freedom then one runs the risk that the "infinite" reservoir of longitudinal energy is coupled into the transverse motion or the longitudinal spread in motion. An example of this is the resistive wall instability where the wall resistance takes energy out of the average longitudinal motion while, at the same time, it transfers energy from the longitudinal to the transverse motion at an even higher rate and hence there is a net loss (in terms of cooling).

In plasma physics there are well-known formulas for exchange of energy amongst plasma components. From any standard text one may obtain for the non-relativistic regime, in the frame of average proton speed,

$$\tau^*_{eq} = \frac{3M_p m_e}{8(2\pi)^{\frac{1}{2}} n^*_e e^4 \log\Lambda}\left(\frac{kT_p^*}{M_p} + \frac{kT_e^*}{m_e}\right)^{3/2} , \quad (1),$$

where all quantities are in the moving frame and

T^*_p = proton temperature ($kT_p = 1/2 M_p V^{*\,2}_\perp$),

T^*_e = electron temperature,

n^*_e = density of electrons,

$\log\Lambda$ involves the Debye length,

τ^*_{eq} = equilibration time.

Assuming T^*_p dominates T^*_e (very cold electron beam), one obtains for τ_{eq} in the laboratory:

$$\tau_{eq} \approx \frac{\beta^{*3}\gamma^2}{200\, r_e r_p n_e c}\,, \qquad (2)$$

where r_e abd r_p are the classic electron and proton radii, n_e is the electron density in the laboratory, and β^* is the proton velocity (in units of c) in the moving frame. The factor of γ^2 is from transformation of n^*_e and τ^*_{eq}. For transverse temperature $\beta^* = \gamma\theta_{lab}$, where θ_{lab} is the angle of deviation of a proton from the average (or electron) direction. For a proton with a longitudinal deviation in momentum $\beta^* \equiv \Delta p/p$.

Thus one can see, especially for the cooling of betatron amplitudes, the strong, dependence upon γ, namely as γ^5. For damping of longitudinal phase space the γ dependence is weaker. Also, of course, one wants a high electron current so as to decrease τ_{eq}.

An alternative--and very simple--derivation of Eq. (2) may be obtained by considering elementary collisions between electrons and protons, but that is, of course, the basis for the quoted Eq. (1).

The inclusion of many complicating features, such as a strong longitudinal magnetic field, have been considered in recent years. Also, careful theoretical treatments have been given employing distribution functions. You will be hearing about this later in the Workshop. Finally, I want to emphasize the very extensive experimental work, leading to a demonstration of the practicality of this concept, which has been carried out at Novosibirsk.

IV. Stochastic Cooling

This technique employs the fluctuations in a beam of a finite number of particles to provide the cooling. The mean lateral position position, $\bar{x}$, of a section of the beam is sensed by a system with gain, g, and bandwidth, W. At some other point of the ring a correcting signal is applied to the beam so as to reduce $\bar{x}$. The correcting section is placed close enough to the pick-up station so that most of the particles detected by the pick-up are in the sample that is affected by the correcting element. The process can be repeated, effectively, if on subsequent passages through the pick-up, different particles are in the sample. In short, one wants little spread in particle transit time between pick-up and corrector (compared to 1/W) and a large spread in transit times between different encounters of the pick-up (compared to 1/W). In this case,

$$\tau_{damping} = \frac{2N}{W}\,(2g - g^2)^{-1}, \qquad (3)$$

where N is the total number of beam
A finite signal-to-noise ratio, $1/\eta$, modifies the factor of g^2 to $g^2(1+\eta)$.

We may readily derive a formula for stochastic damping, by employing a simplistic model, which is very close to Eq. (3). Suppose the pick-up is of length ℓ, in a ring of radius R, containing N particles. Then there will be

$$n = \frac{N\ell}{2\pi R}$$

particles in the sample, and the centroid of this group will be

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i\,,$$ ad

Suppose the corrector changes each particle position x_i to $x_i - g\bar{x}$ where g is the gain of the system.

A measure of the beam spread is

$$\overline{x^2} = \frac{1}{n}\sum_{i=1}^{n} x_i^{\,2},$$

since although the sample of n-particles has a non-zero centroid $\bar{x}$, the centroid of all N particles is zero. It is easy to compute that after the corrector, $\overline{x^2}$ changes to

$$\overline{x^2}\left[1 - (2g - g^2)\,\frac{\bar{x}^2}{\overline{x^2}}\right].$$

Thus, if the correction is made once per revolution a characteristic time is

$$\tau_{damping} \approx \frac{N\ell}{\beta c}\,(2g - g^2). \qquad (4)$$

Clearly, $\ell/\beta c$ is close to the bandwidth W, while the numerical factor comes from a more careful definition of the characteristic time.

As in the case of electron cooling, much theoretical and experimental work has been done beyond that leading to Eq. (3). In particular, a great deal of experimental work, demonstrating the practicality of the concept, has been done on the ISR at CERN.

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APPENDIX

The paper is complete without the following Appendix material which consists of the transparencies shown at the Workshop (the marked parts were read aloud). They are included here to provide the reader with some appreciation for the spirit and character of the Workshop.

Attainment of Very High Energy by Means of Intersecting Beams of Particles

D. W. Kerst,[*] F. T. Cole,[†] H. R. Crane,[‡] L. W. Jones,[‡] L. J. Laslett,[§] T. Ohkawa,[||] A. M. Sessler,[¶] K. R. Symon,[**] K. M. Terwilliger,[‡] and Nils Vogt Nilsen[††]

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In planning accelerators of higher and higher energy, it is well appreciated that the energy which will be available for interactions in the center-of-mass coordinate system will increase only as the square root of the energy of the accelerator. The possibility of producing interactions in stationary coordinates by directing beams against each other has often been considered, but the intensities of beams so far available have made the idea impractical. Fixed-field alternating-gradient accelerators[1] offer the possibility of obtaining sufficiently intense beams so that it may now be reasonable to reconsider directing two beams of approximately equal energy at each other. In this circumstance, two 21.6-Bev accelerators are equivalent to one machine of 1000 Bev.

The two fixed-field alternating-gradient accelerators could be arranged so that their high-energy beams circulate in opposite directions over a common path in a straight section which is common to the two accelerators, as shown in Fig. 1. The reaction yield is proportional to the product of the number of particles which can be accumulated in each machine. As an example, suppose we want 10^7 interactions per second from 10-Bev beams passing through a target volume 100 cm long and 1 cm^2 in cross section. Using 5×10^{-26} cm^2 for the nucleon interaction cross section, we find that we need 5×10^{14} particles circulating in machines of radius 10^4 cm.

There is a background from the residual gas proportional to the number of particles accelerated. With 10^{-6} mm nitrogen gas, we would have 15 times as many encounters with nitrogen nucleons in the target volume as we would have with beam protons. Since the products of the collisions with gas nuclei will be in a moving coordinate system, they will be largely confined to the orbital plane. Many of the desired p-p interaction products would come out at large angles to the orbital plane since their center of mass need not have high speed in the beam direction, thus helping to avoid background effects.

Multiple scattering at 10^{-6} mm pressure is not troublesome above one Bev; but beam life is limited by nuclear interaction with residual gas to $\sim$1300 seconds. Consequently, in about 1000 seconds the high-energy beam of 5×10^{14} particles must be established in each accelerator. The fixed-field nature of the accelerator allows it to contain beams of different energy simultaneously. It may be possible to obtain this high beam current in this time by using $\sim10^3$ successive frequency modulation cycles of radio-frequency acceleration, each cycle bringing up 5×10^{11} particles. It is encouraging to learn that Alvarez and Crawford[2] succeeded in building up a ring of protons by successively bringing up several groups of particles to the same final energy by frequency modulation in the 184-in. Berkeley cyclotron.

The number of particle groups which may be successively accelerated without leading to excessive beam spread can be estimated by means of Liouville's theorem.[3] One can readily convince himself that there is adequate phase space at high energy to accommodate

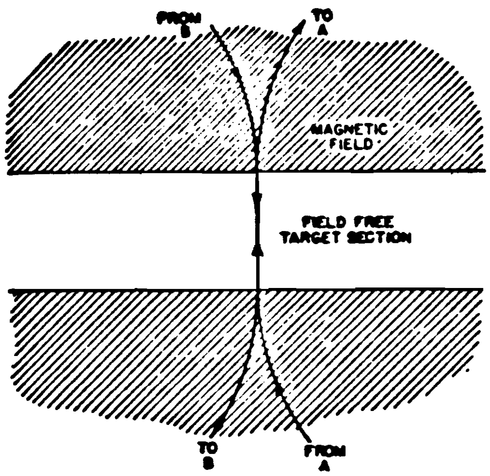

Fig. 1. The target straight section. *B* and *A* can be adjacent or concentric fixed-field alternating-gradient accelerators.

the necessary number, N, of particle groups. Assume for simplicity that synchrotron and betatron phase space are separately conserved, so that for the former

$$(\Delta p)_f(\Delta S)_f = N(\Delta p)_i(\Delta S)_i,$$

where ΔS and Δp are the arc length and momentum spread at injection and final energy. Then, employing the fact that $P \sim R^{k+1}$, where R is the radius and k is the field index, one obtains

$$N = 2(k+1)(\Delta R/R)(p_f/p_i)(\Delta S_f/\Delta S_i)(E_i/\Delta E_i).$$

Using typical numbers such as

$$(p_f/p_i) \sim 100, \quad k \sim 100, \quad R \sim 0.5 \text{ cm},$$
$$R \sim 10^4 \text{ cm}, \quad (\Delta E_i/E_i) \sim 10^{-3},$$

one finds that there is room for $N \sim 10^3$ frequency-modulation cycles.

The betatron phase space available is so large that it cannot be filled in one turn by the type of injectors used in the past which can inject 10^{11} particles. Thus there is the possibility of attaining and exceeding the yield used for this example by improving injection.

The more difficult problem of whether one can, in fact, use all of the synchrotron and betatron phase space depends in detail upon the dynamics of the proposed scheme and this is presently under study.

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[1] Keith R. Symon, Phys. Rev. 98, 1152(A) (1955); L. W. Jones *et al.*, Phys. Rev. 98, 1153(A) (1955); K. M. Terwilliger *et al.*, Phys. Rev. 98, 1153(A) (1955); D. W. Kerst *et al.*, Phys. Rev. 98, 1153(A) (1955).

[2] L. Alvarez and F. S. Crawford, private communication.

[3] We are indebted to Professor E. Wigner who pointed out to us the importance of this consideration.

Storage-Ring Synchrotron: Device for High-Energy Physics Research*

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AS accelerators of higher and higher energy are built, their usefulness is limited by the fact that the energy available for creating new particles is that measured in the center-of-mass system of the target nucleon and the bombarding particle. In the relativistic limit, this energy rises only as the square root of the accelerator energy. However, if two particles of equal energy traveling in opposite directions could be made to collide, the available energy would be twice the whole energy of one particle. Kerst, among others, has emphasized the advantages to be gained from such an arrangement, and in particular of building two fixed-field alternating gradient (FFAG) accelerators with beams interacting in a common straight section.

It is the purpose of this note to point out that it may be possible to obtain the same advantages with any accelerator having a strong, well-focused external beam. Techniques for beam extraction have been developed by Piccioni and Ridgway for the Cosmotron, and by Crewe and LeCouteur for lower energy cyclotrons.

In the scheme proposed here (see Fig. 1), two "storage rings," focusing magnets containing straight sections one of which is common to both rings, are built near the accelerator. These magnets are of solid iron and simple shape, operating at a high fixed field, and so can be much smaller than that of the accelerator at which they are used.[1] The full-energy beam of the accelerator is brought out at the peak of each magnet cycle, focused, and bent so that beams from alternate magnet cycles enter inflector sections on each of the storage rings. In order to prevent the beams striking the inflectors on subsequent turns, each ring contains

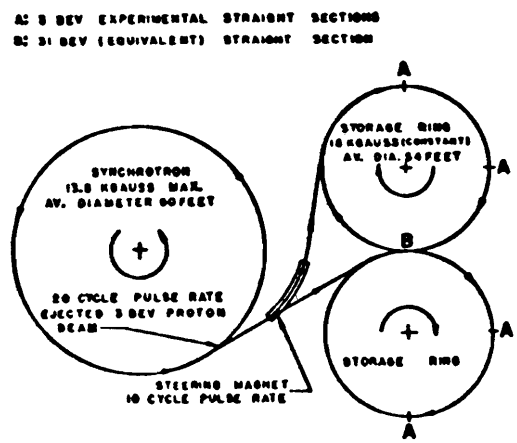

FIG. 1. Plan view of particle orbits in a hypothetical arrangement of storage rings at a 3-Bev proton synchrotron.

a set of foils, thick at the outer radius but thinning to zero about one inch inside the inflector radius. The injected beam particles lose a few Mev in ionization in the foils; so their equilibrium orbit radii shrink enough to clear the inflectors after the first turn. After several turns, the beam particles have equilibrium orbits at radii at or less than the inside edge of the foils.

The possibility exists of storing a number of beam pulses in these storage rings, since space charge and gas scattering effects are small at high energies. Preliminary calculations have been carried out on a hypothetical set of storage rings for the 3-Bev, 20 cycle per second Princeton-Pennsylvania proton synchrotron. Since the storage rings would be simple and almost entirely passive devices, their cost would be small compared with that of the accelerator itself. It was estimated that a pair of storage rings operating at 18 000 gauss with a 2 in.×6 in. good-n region would weigh a total of 170 tons. The magnet of the synchrotron itself would weigh 350 tons, and would be of much more complicated laminated transformer iron. In the event that one could obtain an average current of 1 microampere from the synchrotron, and an average particle lifetime of a few seconds for the storage rings, there would be about 1000 strange-particle-producing reactions per second at each of two beam crossover points, for an estimated 1.5-millibarn total cross section. The center-of-mass energy, 7.8 Bev, would be equivalent to that of a 31-Bev conventional accelerator. If storage rings could be added to the 25-Bev machines now being built at Brookhaven and Geneva, these machines would have equivalent energies of 1300 Bev, or 1.3 Tev.

If only one storage ring were used, tangential to the accelerator itself, the interaction rate would be reduced by a factor S/D, where S is the average number of beam pulses stored in each ring, and D is the fraction of time the accelerator beam is at full energy. The interaction rate would be proportional to S^2 if two storage rings were used.

The advantage of systems involving energy-loss foils is that they provide an element of irreversibility; with foils, the area in phase space available to a particle can be made to decrease with time. This makes it possible to insure that particles once injected will never subsequently strike the injector, no matter how long they may circulate in the storage ring. Preliminary work with a stabilized electronic analog computer indicates that foils may also allow the stable and irreversible capture of roughly half of the circulating particles by a fixed-frequency rf system, which in turn may allow the storage of a large number of beam pulses in each storage ring. It appears that a thin hydrogen jet inside the equilibrium orbit of a conventional synchrotron would, in some energy ranges, reduce radial betatron oscillations even when scattering is taken into account.

The major difficulties in the use of storage rings with foils may result from the amplification of radial betatron oscillations by the foils. Quantitative calculations of this effect have been carried out on the analog computer. It was found that the effect would be serious unless the initial injection to the storage rings could be very precise. However, calculations were also made on a system involving a second foil placed at the inner limit of the good-n region. This foil would move the particle orbits inward as soon as betatron oscillation became serious, and would then continue reducing the betatron oscillation amplitude until the foil itself was rotated out of the median plane. During the long interval (about 0.1 second, or 600 000 turns) before the next beam pulse, the betatron oscillations would continue to be reduced by a thin hydrogen "target" jet also at the radius of the second foil. The process of orbit shrinkage would stop when the particles were captured in stable synchrotron phase by a low-power fixed-frequency rf system; the reduction in betatron oscillations due to the hydrogen would continue. The rf system would define an equlibrium orbit just outside the radius of the hydrogen jet, so that particles whose betatron oscillation amplitudes had been reduced to low values would circulate in a high-vacuum region, where the mean lifetime for nuclear interactions would be long. When the moving foil returned to assist in the acceptance of the next beam pulse, all particles that had been captured by the rf in previous pulses would have small oscillation amplitudes, and so would miss the foil. In this way particles from many beam bursts could be concentrated in a small region, with very little deviation in energy or position.

The author takes pleasure in acknowledging very helpful discussions on this subject with Dr. M. G. White and Dr. F. C. Shoemaker. The assistance of Dr. I. Pyne in setting up problems for the GEDA computer of the Princeton engineering school is also very gratefully acknowledged.

* This work was supported by The Higgins Scientific Trust Fund.

[1] Between the dates of submitting this letter and its publication, it has come to the author's attention that the basic idea of a storage-ring synchrotron has also occurred, at about the same time, to W. M. Brobeck of the Berkeley accelerator group, and to D. Lichtenberg, R. Newton, and M. Ross of the MURA group.

16 May 1956

Memorandum to : Jim Snyder

From: A. M. Sessler

Topic: Proposed Digital Computer Program to Study the Coupling of Radial Betatron Oscillations; and Synchrotron Oscillations; in the Presence of Foils; and Non-slaunched, Non-Radially Terminating, Leakage Flux and Magnetic Effects Absent, R. F. Gaps.*

Motivation: 1) It appears important that we study the effects of coupling between orbital motion and synchrotron oscillations in order to be able to understand completely such things as R. F. knockout.

2) The Princeton people have made the important observation that it is possible to devise systems which are non-Liouvillian as far as the accelerated particles are concerned. This is readily reconciled with general theorems of dynamics by noting that the proposed schemes introduce other particles (electrons in foils) so that the total phase space is still conserved, or alternatively the accelerated particles are subject to dissipative forces. The possibilities opened up by the observation must be studied, since successful use of foils may allow a storage ring to be substituted for an accelerator--at a considerable saving in cost.

3) The separated sector accelerator has slaunched cavities, and some of the proposed R. F. schemes employ cavities which only extend over part of the radial aperture. It is important to study the effects these cavities may engender, but it was felt that the simpler problem in which these effects were ignored, should at least be formulated first. It may

*I am indebted to Dr. Laslett for constant encouragement and support during the writing of this title.

On the Non-Liouvillian Character of Foils

A. M. Sessler

July 11, 1956

Comments

I was unable to see Lichtenberg on my visit of July 10. Symon says Lichtenberg's results do not agree in detail with those presented here, but it is not clear to me that we are calculating the same quantity. Simply to form a basis for discussion during future visits to Madison I have written this material up. With the technical group in two locations, preliminary drafts with a high probability of included errors, seem unfortunately to be essential.

Lichtenberg has constructed an ingenious proof (1) that thin foils are almost Liovillian in character. He has shown by general arguments that the change in total phase space on passage through a foil is negligibly small. This author felt the need for a specific calculation in order to confirm the general result; as well as to obtain explicit formulas for the change in betatron, synchrotron, and total phase space on traversal of a foil. The results of these calculations have been outlined here.

I. Derivation

The starting point is a mathematical characterization of a foil and its effect on the betatron oscillation coordinates x and p, and on the energy of a particle. The transformation is given in a previous memorandum (2), but will

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(1) Mura lecture of July 2, 1956, soon to be published.
(2) "A Proposed Digital Computer Program to Study Foils, etc." -- Memorandum to J. N. Snyder of May 16, 1956.

MODIFICATION OF LIOUVILLE'S THEOREM REQUIRED BY THE PRESENCE OF DISSIPATIVE FORCES*

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July 12, 1956

It has recently been suggested by O'Neill[1] that high current densities might be achieved in accelerators by the use of foils to reduce the volume in phase space occupied by a beam of particles. It is the purpose of this note to examine under what conditions such a compression of phase space can occur and whether the effect is large enough to be of any practical value in accelerators.

The equations of motion satisfied by a particle can be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

where the Lagrangian L includes all forces derivable from a potential, the Q_i are the forces due to the foil and the q_i are the generalized coordinates of the particle.

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* Supported by the National Science Foundation, Office of Naval Research, and Atomic Energy Commission.

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1. G. K. O'Neill, Phys Rev 102 1418 (1956).

If we define

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

$$H = \sum_i p_i \dot{q}_i - L$$

we get

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} + Q_i \end{aligned}\right\} \tag{1}$$

We now consider a closed region V in phase space. The rate of change of this volume will be equal to the volume integral of its divergence:

$$\frac{dV}{dt} = \int \prod_i dq_i \, dp_i \sum_i \left(\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} \right). \tag{2}$$

Using Eq. (1), Eq. (2) becomes

$$\frac{dV}{dt} = \int \prod_i dq_i \, dp_i \sum_i \frac{\partial Q_i}{\partial p_i}. \tag{3}$$

Therefore to see what happens to a volume in phase space due to a foil, we need merely consider the form of the functions Q_i. Of special interest is the case of an ideal foil, defined as one which produces an energy loss but no scattering. If, furthermore, the energy loss depends on the path length through the foil, but not on the particle velocity, we may write

$$\underset{\sim}{Q} = Q(\underset{\sim}{r}, t) \frac{\dot{\underset{\sim}{r}}}{\dot{r}} = Q(\underset{\sim}{r}, t) \frac{(\underset{\sim}{p} - e\underset{\sim}{A}/c)}{|\underset{\sim}{p} - e\underset{\sim}{A}/c|}$$

where $\underset{\sim}{r}$ is the position of the particle and $\underset{\sim}{A}$ is the vector potential. By writing the time t explicitly, we take into account that the foil need not remain in one position. Note that $Q(\underset{\sim}{r}, t)$, is negative for a foil. With the above choice of Q we obtain

$$\sum_i \frac{\partial Q_i}{\partial p_i} = 2\,\frac{Q(\underset{\sim}{r}, t)}{P} \qquad (4)$$

where $P = |\underset{\sim}{p} - e\underset{\sim}{A}/c|$ is the kinetic momentum of the particle. The factor 2 comes from the fact that we are considering the problem in three dimensions. If the effect of the foil on vertical oscillations is neglected, the factor is unity. Using Eq. (4), Eq. (3) becomes

$$dV/V = 2\,dt\ \overline{Q(\underset{\sim}{r}, t)/P} \qquad (5)$$

where the bar indicates the space average of Q/P.

In most accelerators, the momentum spread of the beam is much smaller than the average momentum P of the particles. Therefore instead of Eq. (5) we may write

$$d\dot{V}/V = (2/P)\,dP \qquad (6)$$

where $dP = \overline{Q}\,dt$ is the average momentum increment of a particle due to the foils in time dt. Integrating, we obtain, if the foils are the only source of momentum increment,

$$\frac{V_f}{V_i} = \left(\frac{P_f}{P_i}\right)^2 \qquad (7)$$

where the subscript i indicates the initial value and f the final value. It is apparent from Eq. (7) that the volume in phase space may be reduced by the use of an ideal foil, but that the average momentum of the particles must be reduced by an amount comparable to the reduction in phase space. To avoid this, an oscillator can be used to supply the energy lost in the foil. Then the average momentum P is kept constant and, on integrating Eq. (6), we get

$$\frac{V_f}{V_i} = e^{-2\Delta P/P_i} \tag{8}$$

where ΔP is the average total momentum loss in the foil.

An actual foil differs from an ideal foil in that the energy loss of a particle depends on the magnitude of the particle momentum. However, in the relativistic region this dependence is small and can be neglected so that Eqs. (7) and (8) still approximately hold.

In the non-relativistic region the energy loss goes approximately as the inverse square of the velocity so that the force becomes

$$\underset{\sim}{Q} = Q(\underset{\sim}{r}, t)\, \underset{\sim}{P} / P^3.$$

Putting this expression for Q in Eq. (3) it turns out that $dV/dt = 0$. From Eqs. (7) and (8) it is apparent that the reduction of the volume in phase space depends only on the energy loss in the foil. Therefore, although foils of odd shapes and those which change with time may twist a volume in phase space, they are no more effective in reducing the volume than are uniform foils which produce the same average energy loss. In order to increase the density of particles in the beam by a factor n, a reduction in phase volume by a factor n is required, which from Eq. (8) implies a loss of momentum to the foil of

$$\Delta P = \frac{P}{2} \log n$$

which is comparable to P itself. An actual foil thickness sufficient to do this would produce more than enough scattering to cancel the compression in phase space obtained above.

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LIOUVILLE'S THEOREM FOR A CONTINUOUS MEDIUM WITH CONSERVATIVE INTERACTIONS**

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ABSTRACT

It is shown that for a continuous medium with conservative interactions the density in six-dimensional phase space is preserved as one follows the motion of the medium.

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I. INTRODUCTION

The study of the motion of particles in an accelerator becomes a many-body problem when the interactions between particles are taken into account. It is thus important to investigate the possibility of establishing the validity--or approximate validity--of general dynamical theorems applicable to the n-body problem. Such a powerful theorem is the theorem proved here to be rigorously valid for continuous media, and asserted to be an extremely good approximate theorem for particles in an accelerator.

Liouville's theorem is a theorem which asserts that in a $2fN$ dimensional space (f is the number of degrees of freedom of one particle) spanned by the coordinates and momenta of all particles (called γ space), the density in phase is a constant as one moves along with any phase point. It is thus a statement about the density of points; each point representing a dynamical system. The systems constitute an ensemble and of course do not interact.

The theorem proven here refers to a system of many interacting particles, and asserts that in the $2f$-dimensional space spanned by a single system of coordinates and momenta (called a μ space), the density in phase is a constant as one moves along with any phase point. It is thus a statement about the behavior of interacting particles, and thus really quite different from Liouville's theorem.

The validity of the theorem, as well as the limits of its validity, may readily be seen by the following intuitive argument:

Consider first a system of many particles, N. Suppose these particles are subject to external forces (which may even be time dependent), but there are no interactions between the particles. Clearly density in phase in μ space is a constant of the motion as one follows the motion of a phase point. This follows then immediately from Liouville's Theorem in γ space, since with no interactions between particles μ space for N particles is simply γ space for a single particle.

Consider now a system of a great many particles N, with interactions between the particles. Imagine that the solution has been obtained so that we know the motion of all the particles as a function of time. Concentrate now on a "small" number of particles n, which initially are localized in μ space. We will define what "small" means shortly. Let all the other particles move along the trajectories appropriate to the solution of the N-body problem. If the interactions between one of the particles and the n particles can be neglected compared to the interactions between the N-n particles and one particle, then these particles are subject to "external forces" and by the first case the density in μ space is a constant as one moves along with the sample group of n particles. This is clearly true for any sample, and hence the theorem is established.

That is, as long as one has sufficient particles N, that a sample can be obtained of sufficiently small number of particles n, that the interactions between these particles and one of their number is negligible compared to the interactions between one of these particles and the N-n particles, while at the same time n is sufficiently large that fluctuation phenomena can be neglected, then the theorem is valid. In the rigorous proof given in the next section, the limit of a continuous medium is taken so that fluctuation phenomena do not exist. For applications to particle accelerators where we consider a number of particles $N \simeq 10^{13}$ this approximation is very valid, corresponding to neglect of particle-particle collisions which throw a particle out of the accelerator, but not neglecting long range electromagnetic interactions which are responsible for space-charge limits, plasma oscillations, beam-beam interactions, and possible two-stream amplification mechanisms.

The practical importance of the theorem can be readily seen by limiting one's attention to systems which initially have a constant density in a restricted region of μ space, and no particles outside this region. (This is determined by the injection mechanism, and is a reasonable approximation to most situations). In this case, the N-body problem is completely characterized by the behavior of the boundary surface as a function of time. This surface satisfies a partial differential integral equation of the first order in at most 2f independent variables, so that the N-body problem((f N) differential equations of the second order) is greatly simplified. In particular, for problems involving one degree of freedom, the equation for the boundary curve as a function of time and one coordinate is quite amenable to analysis.

II. FORMAL PROOF

Let λ_i (i = 1, ---, 2f,) be parameters labelling the particles of the medium (2f dimensional phase space; this is the μ space).

$dn = \sigma\, d\lambda_1 \ldots d\lambda_{2f}$ = number of particles in 'volume' element $d\lambda$.

σ = constant 'density' with respect to λ .

Let $\pi_\alpha(\lambda)$ = momentum density

$= \sigma p_\alpha$

$q_\alpha(\lambda)$ = position of particle λ .

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STOCHASTIC DAMPING OF BETATRON OSCILLATIONS IN THE ISR

by

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Geneva - August, 1972

SUMMARY

In principle, betatron oscillations could be damped by detecting and compensating statistical variations of the average beam position, caused by the finite number of particles present. It is shown that achieving useful damping in the ISR would be difficult with presently available techniques.

1. STOCHASTIC DAMPING

As is well known, Liouville's theorem predicts that betatron oscillations cannot be damped by the use of electromagnetic fields deflecting the particles. However, this theorem is based on statistics and is only strictly valid either for an infinite number of particles, or for a finite number if no information is available about the position in phase plane of the individual particles. Clearly, if each particle could be separately observed and a correction applied to its orbit, the oscillations could be suppressed. It is also well known to be possible to damp coherent betatron oscillations (where the beam behaves like a single particle) by means of pickup-deflector feedback systems. In the same way, the statistical fluctuations of the average beam position, caused by the finite number of particles, can be detected with pickup electrodes and a corresponding correction applied. In other words, the small fraction of the oscillations that happens to be coherent at any time due to the statistical fluctuations, can be damped.

After the beam would have passed through such a damping system (for which the name "stochastic damping" could perhaps be used), it would no longer present any coherent oscillations, and further damping would seem to be impossible. However, there are two effects that reintroduce randomness, and therefore some coherency: