BASIC PROPERTIES OF THE SYMMETRIC INSERTION

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1. Introduction

Meads has suggested (1) the use of a symmetric insertion for obtaining a phase advance of 2π in one plane and a phase advance of π in the perpendicular plane.

The purpose of this paper is to write down some analytic relations for this device.

2. The Device

Consider (Fig. 1) two quadrupole multiplets M and $\bar{M}$ and a field-free drift length L lying between them. If we assume that M and $\bar{M}$ are symmetric with respect to each other, their matrices are

$$M = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \qquad (1)$$

and

$$\bar{M} = \begin{Vmatrix} D & B \\ C & A \end{Vmatrix} \qquad (2)$$

Let the beam direction be from M to $\bar{M}$. The total transfer matrix of the device is then

$$M_t = \begin{Vmatrix} A_t & B_t \\ C_t & D_t \end{Vmatrix} = \bar{M} \times \begin{Vmatrix} 1 & L \\ 0 & 1 \end{Vmatrix} \times M \qquad (3)$$

i.e.

$$M_t = \begin{Vmatrix} AD + BC + LCD & 2BD + LD^2 \\ 2AC + LC^2 & AD + BC + LCD \end{Vmatrix} \qquad (4)$$

On account of the relation $A_t = D_t$ the device is self-symmetric.

We want a phase advance of 2π (identity or reproduction matrix) in the x-plane and a phase advance of π (turn-over or minus one matrix) in the y-plane. In either case afocality of the whole system is a first requirement (2) which we write, from Eq (4),

$$C(2A + LC) = 0 \qquad (5)$$

There are two possibilities to satisfy this condition :

a) If we take

$$C = 0 \qquad (6)$$

rendering thus afocal the basic multiplets M and $\bar{M}$, we have

$$AD - BC = AD = 1 \qquad (7)$$

and the total transfer matrix (4) becomes

$$M_t = \begin{Vmatrix} 1 & 2BD + LD^2 \\ 0 & 1 \end{Vmatrix} \qquad (8)$$

Denoting now by p and q (Fig. 1) respectively the distances from the entrance and the exit of the system of the two points between which a phase advance of 2π is wanted, we must have (2)

$$p + q + 2BD + LD^2 = 0 \qquad (9)$$

and

$$D(2B + LD) < 0 \qquad (10)$$

This is the situation in the x-plane.

b) If we take

$$2A + LC = 0 \qquad (11)$$

in Eq (5), the total transfer matrix (4) becomes

$$M_t = \begin{Vmatrix} -1 & -2\frac{D}{C} \\ 0 & -1 \end{Vmatrix} \qquad (12)$$

and in order to achieve a phase advance of π between the two abscissas described by p and q we must have (2)

$$p + q + 2\frac{D}{C} = 0 \qquad (13)$$

with

$$\frac{D}{C} < 0 \qquad (14)$$

This is the situation in the y-plane.

We now introduce indices x and y for the two planes and write the conditions for achieving at the same time a phase advance of 2π in the x-plane and a phase advance of π in the y-plane.

3. Conditions for Achieving Simultaneously Reproduction in the x-plane and Turn-over in the y-plane

From the preceding theory we conclude that in order to achieve at the same time a phase advance of 2π in the x-plane and of π in the y-plane, we must have

$$C_x = 0 \tag{15}$$

$$2A_y + LC_y = 0 \tag{16}$$

$$-(p+q) = 2B_xD_x + LD_x^2 = \frac{2D_y}{C_y} \tag{17}$$

and the inequality

$$C_yD_y < 0 \tag{18}$$

must be satisfied. According to the circumstances the sum p+q appearing in these equations will be a given quantity or an unknown parameter.

We can write the four equations (15)-(17) in the form

$$C_x = 0 \tag{19}$$

$$D_y = D_x(B_xC_y - A_yD_x) \tag{20}$$

$$L = -\frac{2A_y}{C_y} \tag{21}$$

$$p + q = -\frac{2D_y}{C_y} \tag{22}$$

Equations (19) and (20) may be called "internal" relations because they represent the constraints imposed on the multiplet M (or its symmetric counterpart $\bar{M}$). Equations (21) and (22) may be called "external" because they determine the drift length between the multiplets M and $\bar{M}$ and the positions between which the $2\pi - \pi$ transfer is achieved.

On account of the relation

$$A_xD_x = 1 \tag{23}$$

which follows from Eq (7), we can also write Eq (20) in the more symmetric form

$$A_xD_y + A_yD_x = B_xC_y \tag{24}$$

Eqs (21) shows that A_y and C_y cannot be of the same sign so that

$$A_yC_y < 0 \tag{25}$$

From Eqs (18) and (25) we conclude that A_y and D_y must be of the same sign, opposite to the sign of C_y.

Eq. (17) shows that B_x and D_x cannot be of the same sign so that

$$B_xD_x < 0 \tag{26}$$

Moreover from Eq (23) we have

$$A_xD_x > 0 \tag{27}$$

From Eqs (26) and (27) we conclude that A_x and D_x must be of the same sign, opposite to the sign of B_x.

Let us recall that the matrix elements we are using here are those of the entrance multiplet ; those of exit multiplet are obtained by interchanging A and D. Physically, the fact that A and D should be of the same sign in either plane results from the overall symmetry of the structure.

The preceding relations are quite general in the sense that they apply to any mirror-symmetric structure. We now investigate in more detail a few specific structures.

4. Self-Symmetric Multiplets

If the two multiplets used in the structure are self-symmetric we have $A_x = D_x$ and $A_y = D_y$ so that the two mirror multiplets are identical. Eq (23) can then be satisfied only if we have either $A_x = D_x = 1$ or if we have $A_x = D_x = -1$. In the first case Eqs (19)-(22) reduce to

$$C_x = 0 \tag{28}$$

$$A_x = D_x = 1 \tag{29}$$

$$L = p + q = -B_x = -\frac{2A_y}{C_y} \tag{30}$$

In the second case Eqs (19)-(22) become

$$C_x = 0 \tag{31}$$

$$A_x = D_x = -1 \tag{32}$$

$$L = p + q = B_x = -\frac{2A_y}{C_y} \tag{33}$$

Eqs (28)-(30) represent the solution of the problem for $B_x < 0$ while Eqs (31)-(33) should be taken for $B_x > 0$. Although solutions can therefore be found, at least in principle, they may not be very interesting in practice, especially in the case where one wants L as large as possible and p + q as small as possible ; with self-symmetric multiplets one can only achieve L = p + q as the preceding relations show. Eqs (19)-(22) point the way to more practical solutions for the case considered : take A_y/C_y as large as possible and D_y/C_y as small as possible. Physically this means that the multiplets should display a pronounced degree of asymmetry (in the reverse order when going from the first to the second multiplet). How far can one go this way ? The answer to this question is revealed by considering the cases of degeneracy.

5. Cases of Degeneracy

Eq (10) shows that there is a limit for the drift length, given by

$$L_{max} = -\frac{2B_x}{D_x} \quad ; \tag{34}$$

beyond this limit it is not possible to achieve a 2π phase advance. Comparison of Eqs (21) and (34) yields for this limiting case

$$A_y D_x = B_x C_y \tag{35}$$

so that Eq (24) becomes

$$A_x D_y = 0 \tag{36}$$

Now A_x cannot be zero [on account of Eq (23), this would require an infinite value for D_x], consequently we can satisfy Eq (36) only by putting

$$D_y = 0 \tag{37}$$

This case of degeneracy corresponds therefore to the maximum value of L given by Eq (34) and to the minimum value of p + q for which we find from Eq (22) $(p + q)_{min} = 0$. Practically, housing of the coils and other problems may prevent the rigourous achievement of this situation.

Another, probably less interesting case of degeneracy, is represented by

$$A_y = 0 \tag{38}$$

In this case we have from Eq (21) L = 0 and from Eq (20)

$$\frac{D_y}{C_y} = B_x D_x \tag{39}$$

Eq (22) yields then for the maximum value of p + q

$$(p + q)_{max} = -2B_x D_x \tag{40}$$

That this is indeed a maximum can easily be seen by using Eq (17). We notice the symmetry played in the theory by the quantity L on one hand and the quantity p + q on the other. We also note that it is not possible to achieve a phase advance of 2π if the points between which one wants this phase shift are farther apart than the positions given by Eq (40).

6. Use of Simple Multiplets

It is easy to show that a structure composed of mirror doublets cannot be used to achieve a $2\pi - \pi$ phase advance. Indeed, if one takes the focusing-defocusing plane of the entrance doublet as the x - plane, one has $B_x > 0$, $D_x > 0$ so that B_x and D_x cannot be of opposite signs ; if one takes the defocusing-focusing plane of the entrance doublet as the x-plane, one has $A_x > 0$, $B_x > 0$ so that, again, A_x and B_x cannot be of opposite signs.

In the case of mirror triplets solutions can be found but a more detailed investigation shows that a large number of unphysical cases have to be eliminated.

References

1. P.F. Meads, Jr, An Invisible Long Straight Section, Nuclear Instruments and Methods 96 (1971), 351-354.
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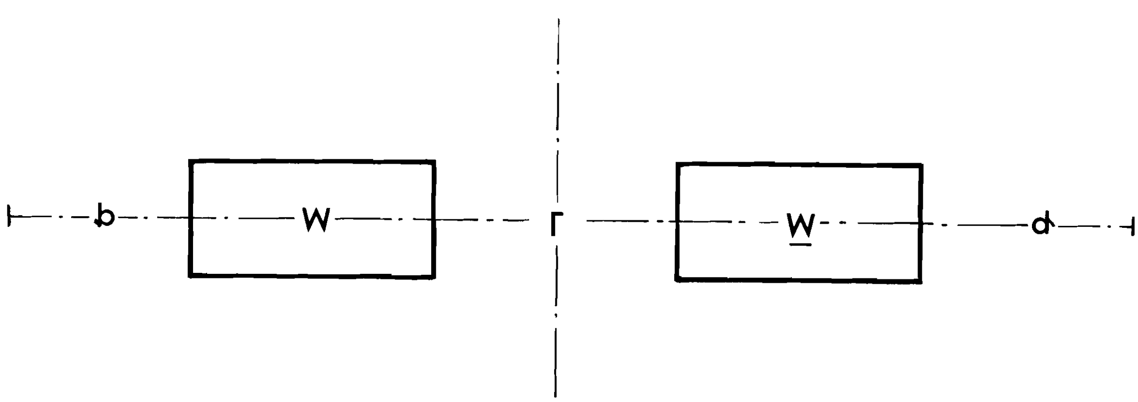

Figure 1