BEAM TRANSPORT DESIGN USING A THIN LENS APPROXIMATION

by

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Abstract

The design of a beam transport system which will deliver a beam with a specified emittance profile is frequently troubled by lack of convergence of the beam transport calculation. A method is described which uses a thin lens approximation to find all solutions in the region of interest; these can then be used as approximate solutions for the more accurate thick lens calculation.

A beam matching problem is equivalent to solving 4 non-linear equations in 4 unknowns. If the thin lens approximation is used these equations are polynomials and the problem can be reduced to finding the intersections of two 8th order equations.

Introduction

The task of designing a beam transport system to deliver a beam with a specified emittance profile arises frequently with accelerator beam lines. Elaborate computer programs have been written to follow beam through a proposed transport system; a subroutine to modify the transport system to improve the fit to some requirement can be included. The SLAC program TRANSPORT as described in SLAC-91 is a good example (1). This technique has its limitations, for unless the trial system is sufficiently close to a solution the iterative process may not converge. In many cases no solution can be found, and in others only impractical solutions are found (quadrupole magnet strengths too large or component spacing too large or small). The existence of other solutions is uncertain and it is not apparent how to modify the system so that useful solutions will occur. This paper describes a technique for obtaining a thin lens solution which can be used as a starting approximation in a thick lens program like TRANSPORT.

Waist to Waist Matching

A problem that arises frequently is that of matching from a double waist to double waist using quadrupole lenses. Four variables are needed to permit the required matching and the following discussion treats four lenses in fixed positions with variable strength; a similar calculation could be done using for example two lenses with variable strength and position. Four equations in 4 unknowns may be written down and these can be reduced to 2 equations in 2 unknowns containing polynomials up to the 8th power. These equations can be plotted and solutions will correspond to intersections. In cases where the curves do not intersect in a desired region it may be possible to adjust them so that they do by changing the lens spacing.

The beam transformation matrices R_x, R_y are 2 x 2 matrices which change trajectory coordinates from X_i, X_i' to X_f, X_f' and Y_i, Y_i' to Y_f, Y_f'

$$\begin{pmatrix} X_f \\ X_f' \end{pmatrix} = \begin{pmatrix} R_{x11} & R_{x12} \\ R_{x21} & R_{x22} \end{pmatrix} \begin{pmatrix} X_i \\ X_i' \end{pmatrix}$$

$$\begin{pmatrix} Y_f \\ Y_f' \end{pmatrix} = \begin{pmatrix} R_{y11} & R_{y12} \\ R_{y21} & R_{y22} \end{pmatrix} \begin{pmatrix} Y_i \\ Y_i' \end{pmatrix}$$

If this transforms a beam with a double waist $1/\sqrt{\gamma_{xi}}$, $1/\sqrt{\gamma_{yi}}$ to a double waist $1/\sqrt{\gamma_{xf}}$, $1/\sqrt{\gamma_{yf}}$ where γ is the Twiss beam ellipse parameter, R is of the form

$$R_x = \begin{pmatrix} \sqrt{\gamma_{xi}/\gamma_{xf}}\cos\theta & (1/\sqrt{\gamma_{xf}\gamma_{xi}})\sin\theta \\ -\sqrt{\gamma_{xf}\gamma_{xi}}\sin\theta & \sqrt{\gamma_{xf}/\gamma_{xi}}\cos\theta \end{pmatrix}$$

$$R_y = \begin{pmatrix} \sqrt{\gamma_{yi}/\gamma_{yf}}\cos\varphi & (1/\sqrt{\gamma_{yf}\gamma_{yi}})\sin\varphi \\ -\sqrt{\gamma_{yi}\gamma_{yf}}\sin\varphi & \sqrt{\gamma_{yf}/\gamma_{yi}}\cos\varphi \end{pmatrix}$$

where the angles θ, φ are arbitrary. This is equivalent to saying that the matrix elements must satisfy the relations

$$R_{x11} = \frac{\gamma_{xi}}{\gamma_{xf}} R_{x22}$$

$$R_{x12} = -\frac{1}{\gamma_{xf}\gamma_{xi}} R_{x21}$$

$$R_{y11} = \frac{\gamma_{yi}}{\gamma_{yf}} R_{y22}$$

$$R_{y12} = -\frac{1}{\gamma_{yf}\gamma_{yi}} R_{y21}.$$

If we write $M = \gamma i/\gamma f \quad P = -1/\gamma i\gamma f$

the relations become

$$\begin{aligned} R_{x11} - M_x R_{x22} &= 0 \\ R_{x12} - P_x R_{x21} &= 0 \\ R_{y11} - M_y R_{y22} &= 0 \qquad (1) \\ R_{y12} - P_y R_{y21} &= 0 \end{aligned}$$

Matching With 4 Thin Lenses

Consider a system of four thin quadrupole lenses in fixed positions

d_1 $\quad$ d_2 $\quad$ d_3 $\quad$ d_4 $\quad$ d_5

A ———————————————————————— B

F_1 $\quad$ F_2 $\quad$ F_3 $\quad$ F_4

where d_i are distances

F_i are thin lenses with focal lengths $-1/F$.

The transformation matrices from A to B are

$$R_x = \begin{pmatrix}1 & d_5\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ F_4 & 1\end{pmatrix}\begin{pmatrix}1 & d_4\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ F_3 & 1\end{pmatrix}\begin{pmatrix}1 & d_3\\ 0 & 1\end{pmatrix}$$
$$\times \begin{pmatrix}1 & 0\\ F_2 & 1\end{pmatrix}\begin{pmatrix}1 & d_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ F_1 & 1\end{pmatrix}\begin{pmatrix}1 & d_1\\ 0 & 1\end{pmatrix}$$

$$R_y = \begin{pmatrix}1 & d_5\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -F_4 & 1\end{pmatrix}\begin{pmatrix}1 & d_4\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ -F_3 & 1\end{pmatrix}\begin{pmatrix}1 & d_3\\ 0 & 1\end{pmatrix}$$
$$\times \begin{pmatrix}1 & 0\\ -F_2 & 1\end{pmatrix}\begin{pmatrix}1 & d_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -F_1 & 1\end{pmatrix}\begin{pmatrix}1 & d_1\\ 0 & 1\end{pmatrix}$$

Multiplying out, these become

$$\begin{aligned}
R_{x11} = \ & 1\\
&+ F_4(d_5)\\
&+ F_3(d_4 + d_5)\\
&+ F_3F_4(d_4)(d_5)\\
&+ F_2(d_3 + d_4 + d_5)\\
&+ F_2F_4(d_3 + d_4)(d_5)\\
&+ F_2F_3(d_3)(d_4 + d_5)\\
&+ F_2F_3F_4(d_3)(d_4)(d_5)\\
&+ F_1(d_2 + d_3 + d_4 + d_5)\\
&+ F_1F_4(d_2 + d_3 + d_4)(d_5)\\
&+ F_1F_3(d_2 + d_3)(d_4 + d_5)\\
&+ F_1F_3F_4(d_2 + d_3)(d_4)(d_5)\\
&+ F_1F_2(d_2)(d_3 + d_4 + d_5)\\
&+ F_1F_2F_4(d_2)(d_3 + d_4)(d_5)\\
&+ F_1F_2F_3(d_2)(d_3)(d_4 + d_5)\\
&+ F_1F_2F_3F_4(d_2)(d_3)(d_4)(d_5)
\end{aligned}$$

$$\begin{aligned}
R_{x12} = \ & d_1 + d_2 + d_3 + d_4 + d_5\\
&+ F_4(d_1 + d_2 + d_3 + d_4)(d_5)\\
&+ F_3(d_1 + d_2 + d_3)(d_4 + d_5)\\
&+ F_3F_4(d_1 + d_2 + d_3)(d_4)(d_5)\\
&+ F_2(d_1 + d_2)(d_3 + d_4 + d_5)\\
&+ F_2F_4(d_1 + d_2)(d_3 + d_4)(d_5)\\
&+ F_2F_3(d_1 + d_2)(d_3)(d_4 + d_5)\\
&+ F_2F_3F_4(d_1 + d_2)(d_3)(d_4)(d_5)\\
&+ F_1(d_1)(d_2 + d_3 + d_4 + d_5)\\
&+ F_1F_4(d_1)(d_2 + d_3 + d_4)(d_5)\\
&+ F_1F_3(d_1)(d_2 + d_3)(d_4 + d_5)\\
&+ F_1F_3F_4(d_1)(d_2 + d_3)(d_4)(d_5)\\
&+ F_1F_2(d_1)(d_2)(d_3 + d_4 + d_5)\\
&+ F_1F_2F_4(d_1)(d_2)(d_3 + d_4)(d_5)\\
&+ F_1F_2F_3(d_1)(d_2)(d_3)(d_4 + d_5)\\
&+ F_1F_2F_3F_4(d_1)(d_2)(d_3)(d_4)(d_5)
\end{aligned}$$

$$\begin{aligned}
R_{x21} = \ & 0\\
&+ F_4\\
&+ F_3\\
&+ F_3F_4(d_4)\\
&+ F_2\\
&+ F_2F_4(d_3 + d_4)\\
&+ F_2F_3(d_3)\\
&+ F_2F_3F_4(d_3)(d_4)\\
&+ F_1\\
&+ F_1F_4(d_2 + d_3 + d_4)\\
&+ F_1F_3(d_2 + d_3)\\
&+ F_1F_3F_4(d_2 + d_3)(d_4)\\
&+ F_1F_2(d_2)\\
&+ F_1F_2F_4(d_2)(d_3 + d_4)\\
&+ F_1F_2F_3(d_2)(d_3)\\
&+ F_1F_2F_3F_4(d_2)(d_3)(d_4)
\end{aligned}$$

$$\begin{aligned}
R_{x22} = \ & 1\\
&+ F_4(d_1 + d_2 + d_3 + d_4)\\
&+ F_3(d_1 + d_2 + d_3)\\
&+ F_3F_4(d_1 + d_2 + d_3)(d_4)\\
&+ F_2(d_1 + d_2)\\
&+ F_2F_4(d_1 + d_2)(d_3 + d_4)\\
&+ F_3F_3(d_1 + d_2)(d_3)\\
&+ F_2F_3F_4(d_1 + d_2)(d_3)(d_4)\\
&+ F_1(d_1)\\
&+ F_1F_4(d_1)(d_2 + d_3 + d_4)\\
&+ F_1F_3(d_1)(d_2 + d_3)
\end{aligned}$$

$+ F_1F_3F_4(d_1)(d_2 + d_3)(d_4)$

$+ F_1F_2(d_1)(d_2)$

$+ F_1F_2F_4(d_1)(d_2)(d_3 + d_4)$

$+ F_1F_2F_3(d_1)(d_2)(d_3)$

$+ F_1F_2F_3F_4(d_1)(d_2)(d_3)(d_4)$

The R_y terms are similar with $-F$ for F. In practice the rule for writing down these terms is obvious so it is not necessary to do the actual matrix multiplication.

Analytical Solution

Equations (1) are 4 simultaneous equations in 4 unknowns F_1, F_2, F_3, F_4. A complete analytical solution is not feasible but they can be reduced to 2 equations in 2 unknowns F_3, F_4 which can be presented graphically. A solution to eqn (1) must lie on both graphs (i.e. at an intersection) but all intersections need not be solutions. These superfluous intersections can easily be identified and rejected.

Write eqn (1) as

$$\begin{aligned} a_1F_1 + a_2 &= 0 \\ a_3F_1 + a_4 &= 0 \\ a_5F_1 + a_6 &= 0 \\ a_7F_1 + a_8 &= 0 \end{aligned} \qquad (2)$$

where $a_1, a_2 \ldots a_8$ are functions of F_2, F_3, F_4.

Eliminate F_1.

$$-F_1 = \frac{a_2}{a_1} = \frac{a_4}{a_3} = \frac{a_6}{a_5} = \frac{a_8}{a_7}$$

The resulting equations are

$$\begin{aligned} a_2a_3 - a_1a_4 &= 0 \\ a_4a_5 - a_3a_6 &= 0 \\ a_6a_7 - a_5a_8 &= 0. \end{aligned}$$

The a's are linear in F_2 so the above equations can be written as

$$\begin{aligned} b_1F_2^2 + b_2F_2 + b_3 &= 0 \quad \text{(i)} \\ b_4F_2^2 + b_5F_2 + b_6 &= 0 \quad \text{(ii)} \\ b_7F_2^2 + b_8F_2 + b_9 &= 0 \quad \text{(iii)} \end{aligned} \qquad (3)$$

Write y for F_2^2, x for F_2 and the condition which must be met for solutions to exist is

$$\begin{vmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{vmatrix} = 0 \qquad (4a)$$

A second condition needed to ensure that $y = x^2$ is

$$(b_3b_4 - b_1b_6)^2 - (b_1b_5 - b_2b_4)(b_2b_6 - b_3b_5) = 0. \qquad (4b)$$

Write
$$\begin{aligned} B_7 &= b_2b_6 - b_3b_5 \\ B_8 &= b_3b_4 - b_1b_6 \\ B_9 &= b_1b_5 - b_2b_4 \end{aligned}$$

and eqns 4 become

$$\begin{aligned} b_7B_7 + b_8B_8 + b_9B_9 &= 0 \\ B_8^2 - B_7B_9 &= 0. \end{aligned} \qquad (5)$$

These equations can be expressed as polynomials in F_3 each coefficient being a polynomial in F_4. Eqn (4a) contains powers up to the 6th and is of the form

$$\begin{aligned} &(c_{11} + c_{12}F_4 + \ldots + c_{17}F_4^6) + \\ &(c_{21} + c_{22}F_4 + \ldots + c_{27}F_4^6)F_3 + \\ &\ldots \\ &(c_{71} + c_{72}F_4 + \ldots + c_{77}F_4^6)F_3^6 = 0 \end{aligned}$$

Eqn (4b) is similar, containing powers up to the 8th.

The individual c_{ij} are functions of the lens spacings d_1 to d_5 and in principal all 130 of them could be written out.

Eqns (4) cannot be solved analytically but can be handled graphically. Graphs of values of F_3, F_4 which satisfy each equation are drawn and points common to both curves are solutions. F_1, F_2 are found from

$$F_2 = B_8/B_9$$
$$F_1 = -a_2/a_1$$

These values can then be tested in eqn (1) to reject extraneous solutions.

Computer Evaluation

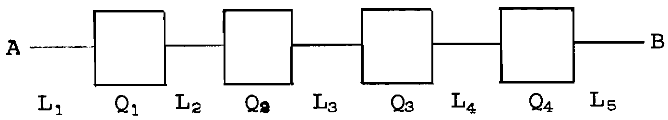
The expressions which arise in the solutions contain a large number of terms and are most easily evaluated numerically by a computer. Subroutines have been written which build up 9 x 9 matrices representing the coefficients of F_3, F_4 in eqn 4 in terms of the transformation matrix R.

The use of F as a lens strength rather than using focal length allow the region of interest to be written

$$-F_{max} < F_4 < F_{max}, \qquad -F_{max} < F_3 < F_{max}$$

which can be scanned easily. Typically $F_{max} = 0.2$ corresponding to a focal length greater than 5 cm. F_4 is scanned in 100 steps and at each step all values of F_3 in the range which gives zeros for each equation (5) are determined. They are printed on the line printer as a graph with 100 x 100 positions. Intersections are noted and the solutions are refined to the required accuracy.

Sample Problem



Consider the waist to waist matching problem with the beam at A satisfying

$$x_{max} = y_{max} = .9 \text{ cm}$$
$$x'_{max} = y'_{max} = 12.5 \text{ m radians,}$$

and the beam at B required to satisfy

$$x_{max} = y_{max} = .4 \text{ cm.}$$

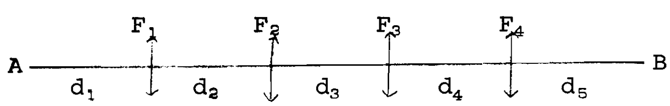
The drift spaces are

L_1 = 50 cm
L_2 = 20.32 cm
L_3 = 25 cm
L_4 = 12.7 cm
L_5 = 10 cm

and magnet lengths are

Q_1 = 30.48 cm
Q_2 = 30.48 cm
Q_3 = 12.7 cm
Q_4 = 12.7 cm

TRANSPORT failed to find a solution using as a starting point, values which gave a beam at B with $x_{max} = y_{max} = .5$ cm. Consider the corresponding thin lens problem



with d_1 = 65.24 cm
d_2 = 50.8 cm
d_3 = 46.59 cm
d_4 = 25.4 cm
d_5 = 16.35 cm

Graphs of eqns (4) are normally plotted together to determine intersections but for this presentation they are shown separately.

Fig. 1 is $\left| b \right| = 0$

and Fig. 2 is $B_8{}^2 - B_7 B_9 = 0$.

Note that the curves are unchanged in this problem if all F_i are replaced by $-F_i$ because $x_{max} = y_{max}$ at both A and B, but this symmetry is not a general property.

The curves on figs. 1 & 2 intersect in 13 pairs of points, only 4 pairs of which satisfy eqn (1). The complete thin lens solution is

F_1 ±.0122, ±.0314, ±.0244, ±.0158 cm^{-1}
F_2 ∓.0278, ∓.0313, ∓.0281, ∓.0265 cm^{-1}
F_3 ±.0416, ±.0477, ±.0421, ±.0387 cm^{-1}
F_4 ∓.0721, ∓.0946, ∓.0287, ∓.0372 cm^{-1}

A thick lens solution can be obtained by using TRANSPORT to follow the solutions to successively thicker lenses. This yields the values

Q_1 ±.059, ±.183, ±.105, ±.085, kgauss/cm (30.48 cm)
Q_2 ∓.138, ∓.168, ∓.142, ∓.138, kgauss/cm (30.48 cm)
Q_3 ±.492, ±.581, ±.465, ±.453, kgauss/cm (12.7 cm)
Q_4 ∓.805, ∓1.396, ∓.332, ∓.401, kgauss/cm (12.7 cm)

General Applicability of the Method

The thin lens approximation was studied to solve a particular problem but appears to be generally useful and can be modified to include a range of related matching problems. I found it a bit disappointing that the solutions corresponded to regions on the graph where the curves coincided for some distance, rather than to a sharp intersection but that is not fundamental and could probably be altered by the appropriate scale change.

The advantage of this method is that an approximate solution is not required so all the solutions in the region of interest can be found. This may give alternative values for a known beam line, possibly with a superior feature. Conversely the non-existence of a thin lens solution, while not guaranteeing that a thick lens solution does not exist, suggests that a search would be futile and a modified configuration should be sought.

References

(1) K.L. Brown, B.K. Kear, S.K. Howry, TRANSPORT/360, SLAC-91

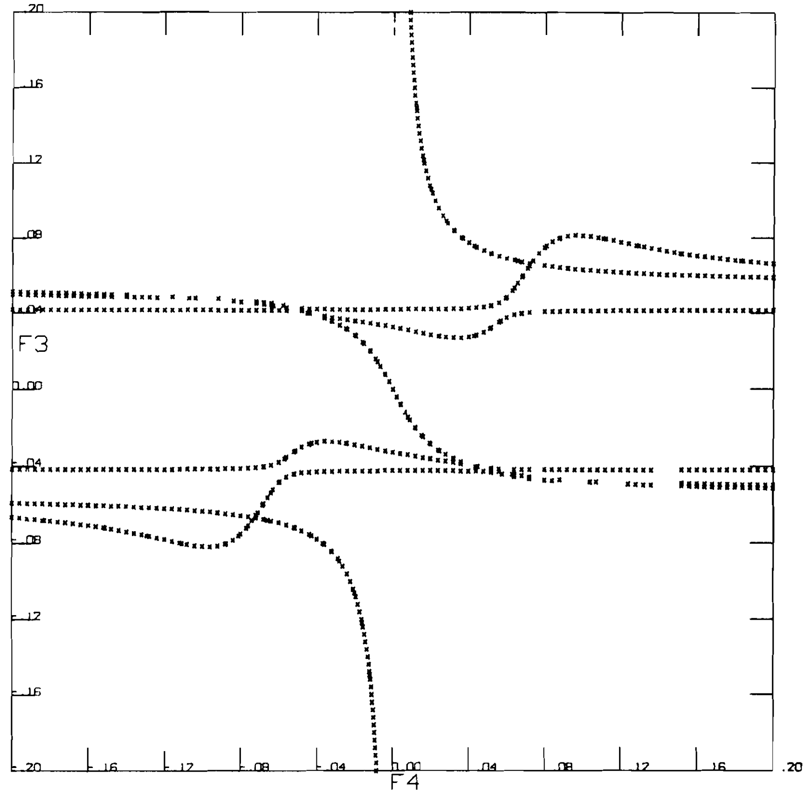


Figure 1

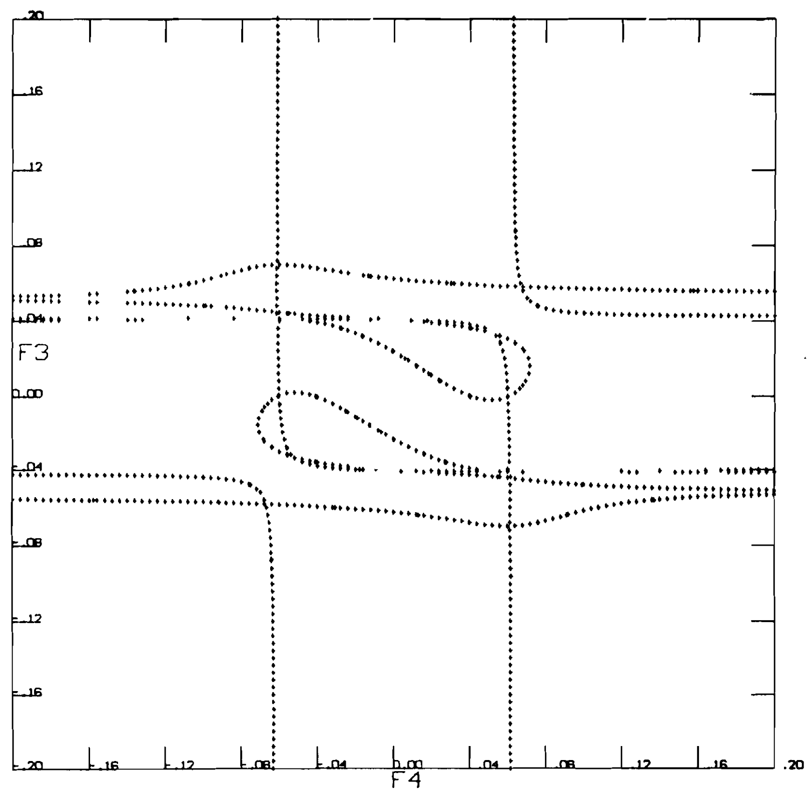


Figure 2