STATISTICAL BEAM TRANSPORT

FOR

HIGH INTENSITY ION CURRENTS*

by

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ABSTRACT

One of the problems associated with the operation of a high-intensity 14-MeV neutron facility is that of transporting an ampere beam of tritium ions from the ion source to a small target area (1 cm^2). The intensity profile history of such an intense beam is difficult to predict correctly as it depends upon detailed knowledge of the nonlinearities in the system and a complete description of the individual particle trajectories, neither of which is sufficiently known.

This paper describes one way in which the beam characteristics may be described in sufficient detail to design completely a transport system to follow an intense beam through a nonrelativistic accelerator structure. The statistical beam transport is a rms average description in which the detailed charge distribution or particle velocity distribution need not be known. The size and growth of the beam is related to the rms values of both the positions and velocities of the individual particles.

I. INTRODUCTION

The limited success in using the Kapchinskij-Vladimirskij[1] (K-V) equations for describing the envelope of an assemblage of charged particles has prompted a different approach. To characterize properly the assemblage using the K-V equation is often most difficult as the edge of the beam is quite sensitive to the slightest change in the charge distribution. For this reason, the K-V theory is quite restrictive in that the particle distribution is constrained to lie on an ellipsoidal surface in the four-dimensional phase space x, $\dot{x}$, y, $\dot{y}$.

In the following presentation, the beam is characterized statistically by its rms position and velocity. It thus becomes insensitive to small changes of density distribution in phase space and the description is valid for the practical profiles usually observed. The need for a realistic description was recognized many years ago and came to the attention of the author early in the spring of 1970.[2,3] Although this paper presents the theory as applied to beam transport in a low-energy pre-injector area[4] (accelerating column, beam transport), it can obviously be

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extended to many areas of accelerator physics (accelerating gaps, beam focusing, and experimental area beam transport).

II. BEAM GEOMETRY

The distribution of particles in the ion beam is assumed to be symmetric in each component direction and all the particles in a particular group have a longitudinal velocity very close to $v_z(z)$. That is, $\dot{x}_j(z)$, $\dot{y}_j(z)$, and $\dot{z}_j(z) << v_z(z)$, where x_j, y_j, and z_j are the component positions of the j^{th} particle with respect to the centroid, z, of the group of particles under consideration. The initial restriction that the particles be nonrelativistic can be removed with some difficulty and will be the subject of a later paper.

III. RMS EQUATION

The rms value of the position of all the particles in a given assemblage at a given time (t) [or place (z)] is defined as

$$R_x{}^2(t) = \frac{1}{N_v} \sum_{j=1}^{N_v} 4\, x_j{}^2\,(t), \tag{1}$$

and the rms value of the velocity is defined as

$$V_x{}^2\,(t) = \frac{1}{N_v} \sum_{j=1}^{N_v} 4\dot{x}_j{}^2\,(t)\ . \tag{2}$$

The factor 4 is arbitrarily introduced so that the rms radius of a uniform density beam will match the real radius. N_v is the total number of particles in the assemblage under consideration. Similar equations exist for the y and z components.

Consider only the x-component, Eq. (1), and perform two time derivatives. After the first time derivative, the x-component becomes

$$2R_x\ \dot{R}_x = \frac{4}{N_v} \sum_j 2x_j\dot{x}_j\ . \tag{3}$$

For simplification, the time dependence will be implied and the summation will be over all particles in the group under consideration. After the second time derivative, the x-component equation becomes

$$\ddot{R}_x - \frac{R_x{}^2V_x{}^2 - R_x{}^2\,\dot{R}_x{}^2}{R_x{}^3} - \frac{4}{N_vR_x} \sum_j x_j\ddot{x}_j = 0\ . \tag{4}$$

The various terms in this equation have special meanings relating to the energy and forces involved in the motion of the group of particles.

IV. INTERNAL FORCES - PULSE BEAM

The forces involved in determining the beam characteristics are contained in the term

$$\frac{4}{N_vR_x} \sum_j x_j\ddot{x}_j \tag{5}$$

of Eq. (4). The forces can be classed into internal forces due to the space charge distribution and to external forces caused by accelerating electrodes and focusing devices.

Considering only nonrelativistic velocities the space charge internal forces are entirely electrostatic. The reaction on each individual particle is

$$\ddot{x}_j = F_x\ (x_j\text{'s},\ y_j\text{'s},\ z_j\text{'s})\ /m_j \tag{6}$$

where F_x is the x-component of the space charge force and m_j is the mass of the j^{th} particle.

For a given assemblage of particles, the potential at any point $(x_j,\ y_j,\ z_j)$ due to the electrostatic field of all the rest of the particles can be written

$$\phi_j = + \frac{\eta q}{4\pi\varepsilon} \sum_i \left[(x_j-x_i)^2 + (y_j-y_i)^2 + (z_j-z_i)^2\right]^{-\frac{1}{2}} \tag{7}$$

where ηq is the charge on the individual particle in coulombs, ε is the capacitivity of vacuum in Farads per cm, η is the charge state of the ion and ϕ is the potential in volts. All dimensions are in cm.

The force term, Eq. (5), can be written in terms of the potential

$$-\frac{4\eta q}{N_vMR_x} \sum_j x_j\ \frac{\partial}{\partial x_j}\ \phi_j$$

$$= -\frac{\eta^2q^2}{\pi\varepsilon MN_vR_x} \sum_j \lim_{a\to1} \frac{\partial}{\partial a} \sum_i \Big[(ax_j-x_i)^2 + (y_j-y_i)^2 + (z_j-z_i)^2\Big]^{-1/2} \tag{8}$$

Here the free variable, a, has been introduced to simplify taking the indicated summations and all particle masses, m_j, are the same, M. Considering only those parameters which are involved in the double-summation and re-arranging the terms, the bracketed expression becomes

$$\sum_j \sum_i \frac{1}{R}\left[1 + \frac{(ax_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2 - R^2}{R^2}\right]^{-1/2} . \tag{9}$$

R is introduced as a constant over the summations and is chosen to justify an expansion into a power series,

$$\frac{1}{R}\sum_j \sum_i \left[1 - \frac{(ax_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2 - R^2}{2R^2} + \ldots\right] \tag{10}$$

and further to make the second term zero over the double-summation. The succeeding terms then become negligible. In making the double-summation, symmetry of particle distribution must be invoked, then

$$R^2 = \left(\frac{a^2 + 1}{4}\right) R_x^2 + \frac{1}{2} R_y^2 + \frac{1}{2} R_z^2 . \tag{11}$$

Carrying out the indicated operations on the free variable, a, in Eq. (8), the internal force term becomes

$$\frac{\eta^2 q^2 N_v}{4\pi\varepsilon M} \cdot \frac{2\sqrt{2}\, R_x}{\left(R_x^2 + R_y^2 + R_z^2\right)^{3/2}} . \tag{12}$$

It becomes quite clear that the ratio

$$\frac{\eta \cdot q N_v}{(4\pi/3)} \cdot \frac{1}{\left(R_x^2 + R_y^2 + R_z^2\right)^{3/2}} \tag{13}$$

can be defined as the rms charge density for the assemblage and that the total charge,

$$\eta q N_v = Q(z), \tag{14}$$

can be a function of t and/or z to account for such variations in charge density distribution due to beam neutralization, charge stripping, or charge exchange. Beam losses due to collimation can also be accounted for; however, a corresponding step change in other parameters will have to be applied.

The term $(R_x^2 + R_y^2 + R_z^2)^{3/2}$ occurring in the denominator of the force term represents a coupling between the otherwise separate dimensional equations.

V. INTERNAL FORCES - d.c. BEAM

It is far more difficult to derive the rms force term for a d.c. beam as it is dependent on the particle distribution both far upstream and far downstream. However, some rather reasonable approximations reduce the problem to a relatively simple one. Consider a very narrow disk whose thickness is dS, and whose volume contains $N_L \cdot dS$ particles. The total force on a particle located within this disk at x_j, y_j will be due to the coulomb forces of all the other particles in that disk plus those particles contained in similar neighboring disks. In a manner similar to that of calculating the force term for the pulse beam, the force term, Eq. (5), can be expressed as

$$\frac{4\eta q}{N_L M R_x} \sum_j x_j \frac{\partial}{\partial x_j} \phi_j$$

$$= -\frac{2\eta^2 q^2}{\pi\varepsilon M N_L R_x} \int_s \sum_j \sum_i \lim_{a\to 1} \frac{\partial}{\partial a} \left[(ax_j - x_i)^2 + (y_j - y_i)^2 + S^2\right]^{-1/2} dS \tag{15}$$

where N_L is the number of particles per unit length (cm^{-1}). It must be assumed that all disks, dS, will have the same particle distribution in both x and y dimensions and that for every upstream disk there is an identical downstream disk -- that is, the beam distribution is essentially symmetrical in z. This is usually a good assumption as in most practical beam transport the beam does not change appreciably in a distance of the order of a few beam diameters. Carrying out the indicated operations, Eq. (15) reduces to

$$\frac{\eta^2 q^2 N_L}{2\pi\varepsilon M} \cdot \frac{2R_x}{R_x^2 + R_y^2} . \tag{16}$$

A similar equation exists for the y dimension; however, the z-dimension equation cancels to zero over the S integration, as it should.

VI. EXTERNAL FORCES

The external forces involved in beam transport are stationary in position but not necessarily in time and are usually applied for the purpose of focusing and accelerating beams. For the most part, such forces are considered linear (terms in x_j^2 will cancel out over the double summation),

$$F_x(z,t) = K_x(z,t) \cdot x_j \cdot \eta \qquad (17)$$

so that the force term, Eq. (5), for external linear forces can be represented by

$$\eta . K_x(z,t) \cdot R_x/M \ , \ K_x(z) = \frac{4M}{\eta R_x^2} \frac{1}{N_L} \sum_j x_j \ddot{x}_j \quad . \qquad (18)$$

Other than linear forces, for example, those applied by an accelerating column of Pierce design, are special cases and easily handled by these equations by appropriate modifications (see Appendix).

VII. EMITTANCE

The second term of Eq. (4) is the emittance term of the beam transport. The rms approach has brought to light a physical interpretation of this term which has to do with the chaotic energy contained in the beam.

It is first desirable to establish that the quantity

$$R_x^2 V_x^2 - R_x^2 \dot{R}_x^2 \qquad (19)$$

is related with an area in phase-space occupied by the beam. Consider Fig. 1, which is an x-dimensional phase plot of the individual particles of the beam. The line

$$\dot{x}_j = x_j \dot{R}_x / R_x \qquad (20)$$

is constructed for reference.

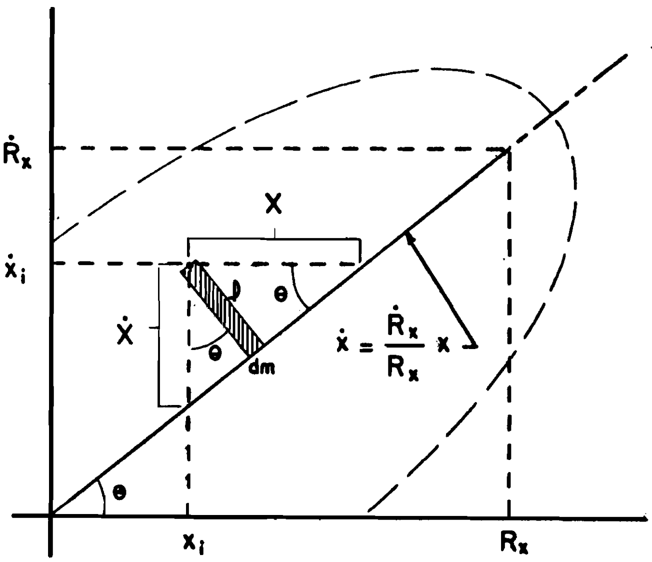

Fig. 1. Phase plot of beam characteristics.

An area, which is associated with each particle, is <u>defined</u> by a perpendicular length ℓ from the reference line and a width, dm, which is the length of the reference line, $(R_x^2 + \dot{R}_x^2)^{1/2}$, associated with the rms beam phase space, divided by the number of particles N_v. The length, ℓ, can be obtained by comparing the similar triangles designated by the angle θ,

$$\ell^2 = X\dot{X}R_x\dot{R}_x \ / \ (\dot{R}_x^2 + R_x^2), \qquad (21)$$

where X and $\dot{X}$ are the distances from the reference line as indicated in the diagram. The area associated with each particle dA, is then

$$dA = (X\dot{X}R_x\dot{R}_x)^{1/2} \ / \ N_v \ . \qquad (22)$$

Again, from the diagram

$$X = \dot{x}_i \left(\frac{R_x}{\dot{R}_x}\right) - x_i, \text{ and } \dot{X} = \dot{x}_i - \left(\frac{\dot{R}_x}{R_x}\right) x_i \quad . \qquad (23)$$

Making these substitutions, the area is

$$dA = \frac{1}{N_v}\left[R_x^2\dot{x}_i^2 - 2x_i\dot{x}_iR_x\dot{R}_x + \dot{R}_x^2 x_i^2\right]^{1/2} . \tag{24}$$

By differentiating dA with respect to time, one can show that this expression for area is conserved if the forces involved are linear, that is, $\ddot{x}_i$ can be represented by $K \cdot x_i$ everywhere. As dA and its reciprocal, the particle density in phase space, is conserved, this representation is a graphical confirmation of Liouville's Theorem.

The rms total area, $A^2 = 4N \sum (dA)^2$, defined as the rms emittance squared, is given by

$$E_x^2 = R_x^2 V_x^2 - R_x^2 \dot{R}_x^2 . \tag{25}$$

This emittance term also can be shown to be constant for linear forces and certain other forces in the following manner. Expressing the emittance in basic terms according to our definitions,

$$E_x^2 = \frac{16}{N_v^2}\left[\sum_j x_j^2 \cdot \sum_j \dot{x}_j^2 - \left(\sum_j x_j\dot{x}_j\right)^2\right] . \tag{26}$$

Differentiating the emittance with respect to time

$$2E_x\dot{E}_x = \frac{16}{N_v^2}\left[\sum_j 2x_j\dot{x}_j \cdot \sum_j \dot{x}_j^2 + \sum_j x_j^2 \sum_j 2\dot{x}_j\ddot{x}_j - 2\sum_j \dot{x}_jx_j \cdot \sum_j x_j\ddot{x}_j - 2\sum_j \dot{x}_jx_j \cdot \sum_j \dot{x}_j^2\right]$$

$$= \frac{32}{N_v^2}\left[\sum_j x_j^2 \cdot \sum_j \dot{x}_j\ddot{x}_j - \sum_j \dot{x}_jx_j \cdot \sum_j x_j\ddot{x}_j\right] \tag{27}$$

It is quite evident that for linear forces, $\ddot{x}_j = K \cdot x_j$, expression (27) reduces to zero, both surviving terms being identical. Consider further the effect on these two terms due to internal space charge forces. From the preceding sections, the second term is equal to

$$\frac{\sqrt{2}\, N_v \cdot \eta^2 q^2}{\pi\varepsilon M} \cdot \frac{R_x^2}{(R_x^2 + R_y^2 + R_z^2)^{3/2}} \cdot R_x\dot{R}_x . \tag{28}$$

A similar type of calculation can be made for the first term. That is, it can be written

$$\frac{2R_x^2\eta^2q^2}{\pi\varepsilon MN_y}\sum_j\sum_i \lim_{a\to o}\frac{\partial}{\partial a}\frac{1}{R}\left[1 - \frac{(a\dot{x}_j + x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2 - R^2}{2R^2} + \ldots\right] . \tag{29}$$

By expanding the various squares and summing over both summations, the first term can be made equal to zero if R is chosen so that

$$R^2 = \left[\frac{a^2V_x^2}{4} + \frac{aR_x\dot{R}_x}{2} + \frac{R_x^2}{2} + \frac{R_y^2}{2} + \frac{R_z^2}{2}\right] . \tag{30}$$

Carrying out the indicated operations on the free variable, a, the first term becomes

$$\frac{\sqrt{2}\, N_v\eta^2q^2}{\pi\varepsilon M} \cdot R_x^2 \frac{R_x\dot{R}_x}{(R_x^2 + R_y^2 + R_z^2)^{3/2}} . \tag{31}$$

Thus, the surviving two terms of Eq. (27) are identical to a second order approximation and opposite in sign. This rather remarkable result indicates that the rms emittance is essentially constant under space charge forces regardless of the x, y, or z particle density profiles. The only constraint placed upon the assemblage is that of symmetry in each component direction.

It is interesting to derive the rms emittance from a different approach to appreciate its physical significance. Consider a beam of zero emittance, whose rms position is R_x and whose rms velocity V_x is $\dot{R}_x$, and whose trajectories are laminar (for example, the reference line in Fig. 1). The added kinetic energy, K.E., needed to form a beam of finite emittance having the same rms values of R_x and $\dot{R}_x$ is given by

$$\text{K.E.} = \sum \frac{m_i}{2} \left\{ \dot{x}_i^2 - \left(\frac{\dot{R}_x}{R_x} x_i \right)^2 \right\} \quad . \tag{32}$$

From the definitions of R_x^2 and V_x^2, the above equation reduces to

$$\text{K.E.} = \frac{M N_v}{8} \left(V_x^2 - \dot{R}_x^2 \right) \quad . \tag{33}$$

Thus, the rms emittance is directly related to the added kinetic energy associated with the departure of the beam from ideal (straight-line configuration in phase space). This K.E. is related, at the extraction surface of a plasma, where $\dot{R}_x = 0$, to an elevated beam temperature,(1/2)kT. It should be noted that the K.E. associated with an increase in emittance from a rms laminar flow is not conserved but that the quantity $R_x^2(V_x^2 - \dot{R}_x^2)$ is conserved for linear forces and essentially conserved for space charge forces. This is not too surprising as the R_x, $\dot{R}_x$ reference line is not a physically meaningful quantity.

VIII. SUMMARY

The rms description of an intense beam of ions can be expressed by the equation

$$\ddot{R}_x - \frac{E_x^2}{R_x^3} - \frac{\eta K_x \cdot R_x}{M} - \frac{q^2\eta^2 N_v}{4\pi\epsilon M} \cdot \frac{2\sqrt{2}\ R_x}{(R_x^2 + R_y^2 + R_z^2)^{3/2}} = 0 \tag{34}$$

for a single pulse of particles.

Of more practical concern for the 14-MeV Neutron Facility, is the rms description for the history of the beam cross section. From the preceding sections, the equation is

$$\ddot{R}_y - \frac{E_x^2}{R_x^3} - \frac{\eta K_x \cdot R_x}{M} - \frac{q^2\eta^2 N_L}{2\pi\epsilon M} \cdot \frac{2 R_x}{R_x^2 + R_y^2} = 0 \quad . \tag{35}$$

A similar equation exists for the y-dimension. In these expressions, the independent variable is time, t. However, custom has preferred the use of the longitudinal direction, z, as the independent variable. A transformation in variables can be made by dividing the above equation by the longitudinal velocity squared, $v_z^2 = dz^2 / dt^2$. This allows a recasting of several of the parameters into more commonly used ones. For the steady current

$$R_x'' - \frac{H_x(z) \cdot R_x^2}{\phi(z) R_x^3 \eta} - \frac{K_x(z) \cdot R_x}{2\phi(z)} - K\left(\frac{M}{\eta M_p}\right)^{1/2} \frac{I(z)}{\phi^{3/2}(z)} \cdot \frac{2 R_x}{R_x^2 + R_y^2} = 0 , \tag{36}$$

where $H_x(z) = (1/2)M(V_x^2 - \dot{R}_x^2)$ units of electron volts.

$\phi(z)$ = average longitudinal energy per unit charge, units of electron volts.

$K_x(z)$ = external force constant per unit charge in units of electron volts/cm^2

$I(z)$ = beam current in units of amperes.

M, M_p = mass of ion particle, proton particle in same units.

R_x, R_y = the rms dimensions in units of cm.

η = charge state of the ion.

(K is a constant and has the value 6.487×10^5 (volts)$^{3/2}$/ampere.)

Because of the complexity of this expression, it is best solved on a computer. The term $H_x(z) \cdot R_x^2$ is the rms emittance squared and is normally conserved during beam transport. However, its value can change should the beam be subjected to collimation or heating as in the transmission through a gas target, for example. The beam current I(z) is normally conserved, however its value can also change due to collimation, stripping, charge exchange or to beam neutralization from the residual gases in the beam transport vacuum system. The average longitudinal energy, $\phi(z)$, is normally conserved except in the accelerating column and in the gas target.

Application of this theory to the beam transport problems of the 14-MeV neutron facility will be the subject of subsequent reports.

IX. APPENDIX

A number of corrections can be made to the rms equations to account for beam transport through various types of focusing devices and accelerating structures. In many cases, such fields are constrained to a small linear region and their effects can be accurately approximated by a step function in one of the transport equation parameters at the appropriate position. In other cases, the fields may be such that the effect is uniform over long distances and again a step function in one or more of the parameters is appropriate. Examples of these latter cases would be the bending magnet and the accelerating column.

Homogeneous Field Bending Magnet

In this type of magnet, the horizontal focusing in the plane of the beam trajectory is continuously applied over the entire effective length of the bending magnet. This focusing results from the different path lengths of the various particles in the beam. The effective focusing strength can be easily calculated in the following manner.

The linear trajectory equation for the horizontal transverse displacement, x_j, referenced to the magnetic radius of curvature ρ_m is given by Steffen,[5]

$$x_j(z) = {}_ox_j \cos\left(\frac{z}{\rho_m}\right) + {}_ox_j' \, \rho_m \sin\left(\frac{z}{\rho_m}\right) . \quad (37)$$

After taking two time derivatives of equation (37) and eliminating the initial condition parameters, ${}_ox_j$ and ${}_ox_j'$, the x-dimension acceleration becomes

$$\ddot{x}_j = - x_j \, \dot{z}^2/\rho_m^2 \quad . \quad (38)$$

The individual particle acceleration can now be substituted into the rms equation force term equations (4) and (5), with the result

$$\frac{4}{N_v R_x} \sum x_j \ddot{x}_j = \eta \, . \, K_x \, . \, R_x/M \, , \quad (39)$$

where the focusing strength per unit charge, K_x, in electron volts/cm^2 is given by

$$K_x = - 2\phi/\rho_m^2 \quad . \quad (40)$$

As this development applies to an orthogonal system, the above focusing force is for a sector-shaped magnet. In a similar manner, the vertical focusing strength $K_y = 0$.

In magnets which do not have entrance and exit apertures normal to the beam, there is an additional amount of focusing. As this occurs in a very small linear dimension of the order of the beam diameter or less, a step change in convergence can be made. Again using formulae given by Steffen and ignoring second order effects due to the fringe fields, the step change in the individual trajectories are

$$x_j' = - x_j \tan\Theta/\rho_m$$

and

$$y_j' = y_j \tan\Theta/\rho_m \, , \quad (41)$$

where Θ is the angle of incidence of the beam and is positive if the outermost trajectory in the bending magnet is lengthened. The equations are valid for both entrance and exit effects. Using equation (3), the rms convergence step change can be calculated

$$R_x R_x' = - \frac{1}{N_L R_x} \sum_j 2x_j \, . \, \frac{x_j}{\rho_m} \tan\Theta$$

or

$$R_x' = - (R_x/\rho_m) \tan\Theta \quad (42)$$

and similarly

$$R_y' = (R_y/\rho_m) \tan\Theta \quad .$$

These equations are valid as long as $\rho_m \gg R_x, R_y$. If this criteria is not valid, higher order effects can be included as well as a possible step change in R_x and R_y. In the practical cases, a bending magnet can be accurately included in the rms equations by the following step changes (see TABLE I).

TABLE I

RMS Transport Equation Step Functions
For A Homogeneous Field Sector Bending Magnet

z	R_x'	R_y'	K_x	K_y
z_{in} (entrance)	$-(R_x/\rho_m)\tan\theta_{in}$	$(R_y/\rho_m)\tan\theta_{in}$	-	-
$z > z_{in}$, $z < z_{out}$	-	-	$-2\phi/\rho_m^2$	-
z_{out} (exit)	$-(R_x/\rho_m)\tan\theta_{out}$	$(R_y/\rho_m)\tan\theta_{out}$	-	-

'Pierce' Accelerating Column

In the 'Pierce' column, the main objective is to accelerate an ion beam from a plasma surface into a near parallel flow. This has the effect of stabilizing the plasma extraction surface into a flat boundary and so minimizes the beam emittance. The usual procedure for designing the column electrodes is to solve equation (36) with the first two terms equated to zero.

$$\frac{K_x(z)\cdot R_x}{2} = -K\left(\frac{M}{\eta M_p}\right)^{1/2}\frac{I}{\phi^{1/2}(z)}\frac{1}{R_x}\quad . \tag{43}$$

Only one equation need be considered as the beam is axially symmetric. The force constant, $K_x(z)$, can be derived in the following manner. The potential at the position x_j, y_j, z_j can be expressed in terms of the on-axis potential in the usual manner

$$\phi(x_j, y_j, z_j) = (z_j) + \frac{\phi''(z)}{4}(x_j^2 + y_j^2) + \ldots \tag{44}$$

and the force on the i^{th} particle at the j^{th} position will be

$$F_{x_j} = -\frac{\partial}{\partial x_j}(x_j, y_j, z_j) \simeq -\phi''(z)\cdot x_j/2 \tag{45}$$

or the acceleration on the i^{th} particle will be

$$\ddot{x}_j = F_{x_j}/M = -\phi''(z)\cdot x_j\cdot \eta/2M\quad . \tag{46}$$

Making this substitution into Eq. (18), the free constant becomes

$$K_x = -\phi''(z)/2$$

and equation (43) becomes, for a circularly symmetrical charged beam,

$$\frac{\phi''(z)\cdot R}{4} = +K\left(\frac{M}{\eta M_p}\right)^{1/2}\frac{I}{R}\frac{1}{\phi^{1/2}(z)} \tag{47}$$

where $R_x = R_y = R$. The solution to equation (47) is straight forward and gives the Child-Langmuir space charge law.

$$\phi(z) = \left(9K\right)^{2/3}\left(\frac{M}{\eta M_p}\right)^{1/3}\left(\frac{I}{R^2}\right)^{2/3} z^{4/3} \tag{48}$$

Thus, when the emittance is zero and the potential along the beam is given by Eq. (48), the particle trajectories will effectively be parallel ($\dot{R}_x = \ddot{R}_x = 0$). The electrode configuration needed to insure such a potential distribution can be calculated.[6]

When the emittance is not zero, the beam will expand slightly as given by the second term. As the beam expands, the last two terms become unequal and an additional focusing takes place. By considering only the first two terms and that R_x is very nearly constant, the beam growth can be estimated from

$$R_x'' \sim \frac{\left\{H_x(z)\cdot R_x^2\right\}}{\eta\phi(z)\cdot R_x^3}\quad , \tag{49}$$

where $\phi(z)$ is given by equation (48). Because a singularity exists at z=0, the equation is best solved in terms of the independent variable t, in place of z. The solution gives

$$R_x(z) \sim R_x(0) + \frac{\left\{H_x(z)\cdot R_x^2\right\}\cdot c^2\ t^2}{\left\{Mc^2\right\}\ R_x^3(0)}\quad . \tag{50}$$

Usually, for practical examples, the beam growth during the transient time, t, down an accelerating column is less than one percent.

Quadrupole Focusing Magnet

The quadrupole focusing magnet can be considered as a field distribution which is uniform in the z dimension and has an effective length, L, which in the usual manner includes the effect of the fringe fields. In the rms equations, the focusing of the quadrupole magnet is accurately approximated by a step function in K_x and K_y, which lasts over the effective length of the quadrupole. From Eq. (17), the focusing strength is defined as

$$K_x(z) = \frac{4M}{\eta R_x^2} \frac{1}{N_L} \sum_j x_j \ddot{x}_j .$$

For non-relativistic energies, the acceleration, $\ddot{x}_j$, in the quadrupole field is given by

$$\ddot{x}_j = \frac{2(\eta\phi) \cdot 300 K_g \cdot x_j}{M(2Mc^2 \cdot \eta\phi)^{1/2}} ,$$

where K_g is the quadrupole field gradient in gauss/cm. Making this substitution into equation (51), the focusing constant per unit charge becomes

$$K_x = 600\left(\frac{\eta\phi}{2Mc^2}\right)^{1/2} K_g ,$$

where the sign of K_x is negative for a focusing field and is positive for a defocusing field. In the normal quadrupole magnet $K_x = - K_y$.

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DISCUSSION

A. Sessler, Berkeley: What approximations have you made to get the last term, the self-force term, in your derivation? I presume that it is not exact and that some approximation has been made to put it in that form.

Emigh: That is correct. You have to sum over all the forces by all the particle distributions; the details are presented in my paper. The term presented is a first-order approximation. If you were to make higher-order approximations, you would find that the R_x in the numerator of this expression would have additional terms in higher powers of R_x giving the higher moments.