INDUCED EMFS IN TWO-DIMENSIONAL FIELDS*

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Abstract

The magnetic flux linking a coil is composed of external field flux and flux due to current I in the coil itself, $\Psi = \Psi_{ext} + \Psi_I$. Induced emf, e, is proportional to $-\dot{\Psi}$ and hence $\int_1^2 e dt$ is proportional to $\Delta\Psi = \Psi_1 - \Psi_2$. On open circuit, I = 0, and this integral is a measure of the change, $\Delta\Psi_{ext}$, in external flux linking the coil. On closed circuit e causes an induced current, I_{ind}, to flow whose field tends to oppose the flux change which induces e (Lenz's law). If the coil is superconducting $\Delta\Psi = 0$ because $\Delta\Psi_I$ just cancels $\Delta\Psi_{ext}$. Then ΔI_{ind} is also a measure of $\Delta\Psi_{ext}$.

For the case of two-dimensional fields and cylindrical sheet coils detailed calculations can be made because both the fields and the coils can be resolved into multipole or harmonic components. It is shown that a carefully made 2n-pole winding enclosing an aperture can be used·

(1) On open circuit, as a search coil to measure the 2n-pole field component entering the aperture from outside the coil.

(2) On closed circuit, as a passive self-powered shield to oppose the intrusion into the aperture of any 2n-pole field component from outside and, if the coil is superconducting, to exclude that component entirely.

The "two potential" complex formulation for two dimensional fields facilitates the analysis. Expressions are also derived for the self and mutual inductances of cylindrical sheet coils.

I. Flux Linkage and Induced Emf

The magnetic flux linking a coil is given by the line integral of the vector-potential along the conductor of the coil, $\oint \vec{A} \cdot \vec{ds}$.[1] The electromotive force, or emf, induced in the coil by a changing magnetic field is proportional to the negative time derivative of the flux linkage. If the coil circuit is closed an induced current will flow. The magnetic flux due to the induced current will oppose the flux change producing it (Lenz's Law). If the coil is superconducting (zero resistance) the flux due to the induced current will reduce the change in total flux linkage to zero.

Coils which are appropriate for producing or measuring two-dimensional magnetic fields consist of long straight conductors normal to the field plane and connected in coil sequence at the far ends. Thus each straight filament conductor constitutes a <u>half turn</u> of the coil and there are as many negative as positive half turns in the coil sequence. For such coils of indefinite length it is appropriate to deal with the flux linkage <u>per unit length</u> of the coil

$$\psi = \sum s_k A_k \tag{1}$$

where s_k is the signed unit length, + 1 or - 1, of the kth half turn and A_k is the vector potential component along the kth conductor. The induced emf per unit length of the coil is proportional to the negative time derivative of (1), namely $-\dot{\psi}$.

2. Two Dimensional Fields - Multipole Components

The characteristic properties of a two-dimensional magnetic field in empty space are identical with those of an analytic function of a complex variable. This fact emerges from the <u>two-potential complex formulation</u> which may be introduced as follows.[2]

Let X, Y be coordinates in the plane of a two-dimensional magnetic field and let S be the third coordinate of a right-handed rectangular system. In empty space $\mu = B/H = 1$. The field components, H_X and H_Y, depend only on X and Y, but not on S. These components are space derivatives either of the scalar potential, $\Omega(X, Y)$, or of the S-component of the vector potential, $A_S = A(X,Y)$:

$$H_X = -\frac{\partial\Omega}{\partial X} = \frac{\partial A}{\partial Y}$$
$$H_Y = -\frac{\partial\Omega}{\partial Y} = -\frac{\partial A}{\partial X} \tag{2}$$

For a two-dimensional field the two potentials, A and Ω are independent of S, and the remaining vector potential components, A_X and A_Y are constant and of no further interest.

The second parts of equations (2) constitute the Cauchy-Riemann conditions which are necessary and sufficient to make the "complex potential", which we choose to define as

$$W \equiv -(A+i\Omega) = W(Z) \tag{3}$$

an analytic function of the complex variable $Z = X+iY$. It is easily seen from (2) that its derivative gives both field components in the complex combination*

*Note that H_X+iH_Y is <u>not</u> an analytic function of Z.

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$$\frac{dW}{dZ} = H(Z) \equiv H_Y + iH_X \qquad (4)$$

This derivative is again an analytic function of Z and its Cauchy-Riemann equations

$$\frac{\partial H_Y}{\partial X} = \frac{\partial H_X}{\partial Y}$$
$$\frac{\partial H_Y}{\partial Y} = -\frac{\partial H_X}{\partial X} \qquad (5)$$

constitute Maxwell's equations for the field.

Thus we see that every two-dimensional field corresponds to an analytic function of $Z = X+iY$ and every analytic function of Z corresponds to a physically possible two-dimensional Maxwell field.

Around any regular point as origin the analytic potential (3) can always be represented by a power series

$$W(Z) = W_0 + W_1Z + W_2Z^2 + --- = \sum_{n=o}^{\infty} W_nZ^n \qquad (6)$$

whose circle of convergence extends up to the nearest singular point (current or material with permeability $\mu \neq 1$). W_0 is an arbitrary constant and the succeeding W_nZ^n terms correspond to the 2n-pole components of the field. Thus W_1 is the coefficient of the dipole term, W_2 of the quadrupole, W_3 of the sextupole, etc. Several points should be noted:

(a) For a given origin any two dimensional field can be uniquely resolved into a series of multipoles but the resolution is strictly possible only for two-dimensional fields. Any variation of the field with S implies a non-analytic behavior in the X, Y, planes. Equations for this variation have been given.[3]

(b) The resolution into multipoles depends on the choice of origin. Thus, if we transform (6) to a new origin Z_o by substituting $Z = Z_o + Z'$ each multipole coefficient W_n will contribute to all coefficients below it. However, if (6) is a finite series (polynomial) the coefficient of the highest multipole present will not depend on Z_o.

(c) Each multipole depends on <u>two</u> specifying numbers since the coefficients W_n are, in general, complex.

The series (6) leads to general expressions for A and Ω in Fourier form. We set

$$W_n = Q_n e^{-in\varphi_n} \quad \text{and} \quad Z = \rho e^{i\varphi} \qquad (7)$$

Then (6) becomes

$$W = \sum W_nZ^n = \sum Q_n\rho^n e^{in(\varphi - \varphi_n)} = -(A+i\Omega) \qquad (8)$$

and the potentials are

$$A = -\sum Q_n\rho^n \cos n(\varphi - \varphi_n) \qquad (9)$$

$$\Omega = -\sum Q_n\rho^n \sin n(\varphi - \varphi_n) \qquad (10)$$

3. <u>Sheet Coils - Harmonic Components</u>

In designing coils to produce a desired two dimensional field it is often useful to begin with an idealized cylindrical current sheet - a smooth distribution of currents flowing along the elements of a cylindrical surface of zero thickness.

For the problem of calculating flux linkages we introduce the passive equivalent of a current sheet, namely a coil consisting of a practically continuous distribution of filamentary half turns lying along the elements of a cylinder such that a current through the coil (the <u>same</u> series current through all the turns) would imitate a current sheet. This sheet coil idea can be made more precise in the following way.

Let the elements of a circular cylinder of radius a be normal to the $Z = X+iY$ plane at

$$Z = z \equiv ae^{i\varphi}$$

and let dN be the number of coil half turns lying along the cylinder elements in the +S direction in the interval $d\varphi$; dN is a negative quantity for half-turns in the -S direction. Since there must be as many negative as positive half turns in the whole coil the integral of dN taken once around the circumference of the cylinder must be zero, $\oint dN = 0$. It follows that the cumulative number of half turns, $N(\varphi) = \int dN$, must be a periodic function of φ which repeats at intervals of 2π. This turn function can therefore be written as a Fourier series

$$N = N(\varphi) = \sum N_m \sin m(\varphi - \theta_m) \qquad (11)$$

whence

$$dN = \sum mN_m \cos m(\varphi - \theta_m)\, d\varphi \qquad (12)$$

Thus the sheet coil consists of a series of harmonic components and each component depends on two specifying numbers, the amplitude N_m and the reference azimuth θ_m.

An example of a sheet coil is the measuring coil described by G. H. Morgan[4] and shown in his Figure 1. It consists of six half turns placed at 60^o intervals around a circular cylinder with $a = 2.22$ cm. The turn function (11) is alternately + 1/2 and - 1/2 in successive 60^o intervals; if we place a positive half turn at $\varphi = 0$ the turn function (11) is

$$N = \frac{2}{\pi} [\sin 3\varphi + 1/3 \sin 9\varphi + 1/5 \sin 15\varphi + \cdots] \quad (13)$$

Thus the harmonic coefficients are zero except when m is an odd multiple of 3:

$$N_3 = \frac{2}{\pi}, \; N_9 = \frac{2}{3\pi}, \; N_{15} = \frac{2}{5\pi}, \; \cdots$$

The series (13) represents sharp discontinuities at $\varphi = k\pi/3$ corresponding to infinitely thin filaments; for finite size conductors the higher coefficients will fall off faster.

4. Flux Linkage and Induced Emf for Sheet Coils

For a sheet coil normal to a two-dimensional field the flux linkage per unit length (1) becomes

$$\psi = \oint A dN$$

From (9) with $\rho = a$ and (12) we see that ψ consists of a double series of terms

$$\psi_{nm} = - Q_n a^n m N_m \int_0^{2\pi} \cos n(\varphi - \varphi_n) \cdot \cos m(\varphi - \theta_m) d\varphi. \quad (14)$$

The integrals vanish for $m \neq n$, leaving only the terms ψ_{nn} for which, after reduction, we can write

$$\psi_n = - \pi n a^n Q_n N_n \cos n (\varphi_n - \theta_n). \quad (15)$$

Thus each harmonic of the sheet coil only links flux of the corresponding multipole of the field.

The emf induced in the coil is proportional to $-\dot{\psi}$. If the field changes, Q_n and/or φ_n will change; if the coil is rotated θ_n will change and if the coil is translated the Q_n in (7) will change because the origin for the expansion (6) moves with the axis of the coil. In any case the time integral of the induced emf per unit length, e is a measure of the change in the total flux linkages per unit length of the coil, $\Delta\psi = \Sigma\Delta\psi_n$.

Thus $\int$ edt of an ideal harmonic coil an open circuit is a measure of the change in the flux linkages (15) during the time over which the integral is taken. There are basically two methods for making such a field measurement: to pulse the field while the coil is stationary[5] or to rotate the coil about its axis while the field is held constant[6].

If the measuring coil contains other harmonics in its turn function (11) it will measure that combination of harmonics of the field according to the sum of the changes in (15). In general, a coil will measure the combination of harmonics which a current through it would produce; this field can be calculated by the methods of the next section.

5. Sheet Coil Fields and Inductances

A sheet coil becomes a current sheet if a current, say I abamperes, flows through it. The complex potentials, $W_{in}(Z)$ and $W_{out}(Z)$, of the field produced in the regions inside and outside the cylinder are related at the cylinder where $Z = z = ae^{i\varphi}$, by the current sheet theorem[7].

$$W_{out}(z) - W_{in}(z) = 4\pi i NI + \text{const.} \quad (16)$$

where N is the cumulative turn function.

If we let $z_m = ae^{i\theta_m}$, then $e^{i(\varphi - \theta_m)} = \frac{z}{z_m}$

and, using the turn function (11) for N, we can write the Fourier series for NI in the form

$$NI = \frac{I}{2i} \sum N_m [(\frac{z}{z_m})^m - (\frac{z_m}{z})^m] \quad (17)$$

Since the potentials produced by the current sheet must be regular at Z = o and Z = ∞, we deduce from (16) and (17) that they are

$$W_{in}(Z) = - 2\pi I \sum N_m (\frac{Z}{z_m})^m \quad (18)$$

$$W_{out}(Z) = - 2\pi I \sum N_m (\frac{z_m}{Z})^m \quad (19)$$

The resulting fields are given by $H_Y + iH_X = \frac{dW}{dZ}$ in each region.

The potential W_{in} inside the cylinder is regular at Z = 0 and of the form (6) with $W_m = - 2\pi I N_m / z_m^m$. In (7) we identify $\varphi_m = \theta_m$ and find

$$Q_m = - 2\pi I N_m / a^m \quad (20)$$

The interior potentials A_{in} and Ω_{in} are then given by (9) and (10). The self-flux linked per unit current for the mth harmonic component is found from (15) to be, since $\varphi_m = \theta_m$,

$$L_m = \frac{\psi_m}{I} = 2\pi^2 m N_m^2 \quad (21)$$

This is the self-inductance per unit length of the mth coil component in nanohenries/cm (e.m. units). Since the cross flux linkages between different harmonic coil components are zero the total self inductance per unit length of the sheet coil (11) is

$$L = \sum L_m = 2\pi^2 \sum m N_m^2 \text{ nanohenries/cm} \quad (22)$$

Note that this L is a pure number as it should be and that it does not depend on the radius a of the coil cylinder. In this sense thick coils may be thought of as a superposition of thin sheet coils.

In the same way we can find the mutual inductance per unit length of coaxial cylindrical sheet coils of radii a > b:

$$L_{ab} = 2\pi^2 \sum n(\frac{b}{a})^n (N_a)_n (N_b)_n \cos [(\theta_a)_n - (\theta_b)_n] \quad (23)$$

in nanohenries per cm.

6. Superconducting Multipole Shield

Imagine an ideal mth harmonic superconducting cylindrical sheet coil enclosing an aperture with its circuit closed. Let the turn function be

$$N = N_m \sin n (\varphi - \theta_m) \quad (24)$$

If, starting from zero field, we impose any field (6) on the aperture from outside the cylinder an induced current will flow in the superconducting coil so as to prevent (by adding its own field) any change in the flux linkage from the initial zero value.

The flux linkage (15) with the nth component, $W_n Z^n$, of the imposed field is automatically zero for $n \neq m$. Therefore, all $n \neq m$ field components enter the aperture without hindrance.

The mth component of the imposed field (6), (7)

$$W_m Z^m = Q_m e^{-im\varphi_m} Z^m \quad (25)$$

can, however, be blocked out of the aperture completely by (24) if we set $\theta_m = \varphi_m$. The field of the induced current I has the potential (18)

$$- 2\pi I N_m (\frac{Z}{z_m})^m = - \frac{2\pi I N_m}{a^m} e^{-im\theta_m} Z^m$$

which cancels the externally imposed field (25) when the induced current is

$$I = \frac{a^m Q_m}{2\pi N_m} \quad (26)$$

In this way a carefully made superconducting harmonic coil (24) can be used as a self-powered shield to prevent the intrusion into the aperture of the like harmonic field from the outside.

It will be seen that this total 2m-pole shielding will take place whenever

$$\Delta_m = m (\varphi_m - \theta_m)$$

is an integral multiple of π and that the shield coil has no effect whenever Δ_m is an odd multiple of $\pi/2$.

The current (26) induced in the coil on closed circuit is a measure of the imposed field change just as the time integral of the induced emf is an open circuit.

7. Concluding Remarks

By introducing the concept of sheet coils as the passive equivalent of current sheets we have been able to show that harmonic coils can be used either to measure or to shield multipole components of a two dimensional field in an aperture.

While very good approximations to harmonic turn distributions can be made the three-dimensional problem of coils ends has not been discussed here. For a coil to be used on open circuit as a measuring coil the problem is usually solved by making the coil appreciable longer than the field to be measured.

On the other hand, a coil to be used as a simple shield against any multipole component should be the same length as the imposed field. If a superconducting shield coil extends beyond the imposed field region, so will the cancelling flux field.

References

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