PERTURBATION METHOD FOR $\cos\theta$ MAGNET WITH NONLINEAR IRON SHIELD[*]

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Abstract

Nonlinearities of the iron in high-field magnets are usually treated by the aid of mesh-type computer programs. In this paper an alternate approach using a quasi-analytical method is suggested. Since the field-current relationship of a $\cos\theta$ magnet deviates even at 40 kG only by a few percent from a linear law, it seems indicated to treat the nonlinearities as a perturbation of the infinite-permeability case. The method is, however, limited to a geometry with rotational symmetric iron shield. The application of this method to an actual magnet is in progress, but a definite statement as to its accuracy is not yet possible.

I. Introduction

The conventional and highly successful methods for solving two-dimensional nonlinear magnetostatic problems rely on some form of mesh-iteration procedure.[1-3] Analytical methods, on the contrary, have in the past received very little attention.[4,5] Since, however, the field-current relationship of a $\cos\theta$ magnet deviates even at 40 kG only by a few percent from a linear law and since the induced higher harmonics are also very small, it is tempting to treat the nonlinearities as a perturbation of the infinite permeability case.[6] The application of a perturbation method to $\cos\theta$ magnets with nonlinear iron shield is the subject of this paper.

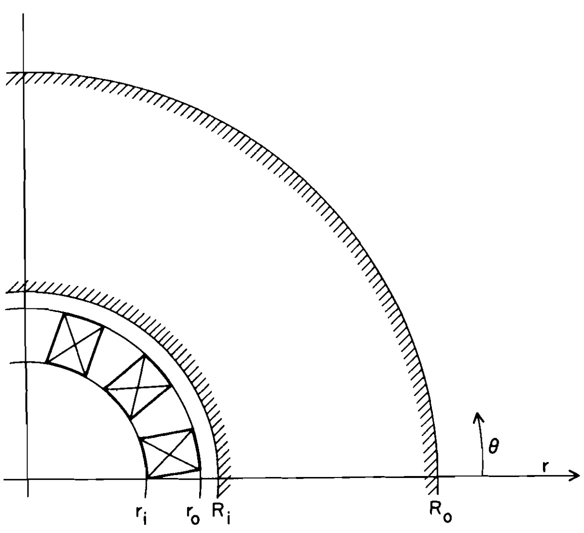

Fig. 1. Geometry of $\cos\theta$ magnet.

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The relative simplicity of the perturbation method suggested stems to a large extent from the simple geometry of a rotational symmetric iron shield (Fig. 1). The extension to magnets of different types, e.g. AGS magnet, picture-frame magnet, etc., is not straightforward. In this respect, the present method is inferior to mesh-type programs, which in principle can treat any geometry.

We limit our considerations to two-dimensional magnetic fields, $\vec{B} = \vec{u}_r B_r + \vec{u}_\theta B_\theta$. As usual, we search for a solution for the vector potential $\vec{A} = \vec{u}_z A$ rather than the field directly. The field is then obtained from

$$\vec{B} = \text{curl } \vec{A} \tag{1}$$

which reads in circular-cylinder coordinates

$$B_r = \frac{1}{r}\frac{\partial A}{\partial \theta} \tag{2a}$$

$$B_\theta = -\frac{\partial A}{\partial r} \quad . \tag{2b}$$

The vector potential A is the solution of the differential equation (DE) (natural units are used throughout),[7]

$$\text{div}(\gamma \ \vec{\text{grad}} \ A) = s^{ex} \tag{3}$$

with s^{ex} the conduction current density and γ the inverse of the permeability. γ is assumed to be a function of the absolute value of the magnetic field, $\gamma = \gamma(B)$, but otherwise isotropic. By using the vector identity

$$\text{div}(s\vec{v}) = s \ \text{div} \ \vec{v} + \vec{v} \cdot \vec{\text{grad}} \ s$$

with s and $\vec{v}$ arbitrary scalar and vector fields, one can transform the DE (3) into ($\gamma = 1$ at the coil location)

$$\text{div}(\vec{\text{grad}} \ A) = \Delta A = -s^{ex} - \gamma^{-1} \ \vec{\text{grad}} \ \gamma \cdot \vec{\text{grad}} \ A \quad . \tag{4}$$

The nonlinear DE (4) can be linearized by evaluating γ for a trial function which is not too different from the correct solution. A first approximation is obtained by taking the infinite permeability case as trial function. A better approximation is, in principle, obtainable by an iteration procedure.

II. Air-Core Magnet

In this section a method for the analysis of the magnetic field due to an extraneous current distribution is elaborated, which matches the geometry of air-core $\cos\theta$ magnets (Fig. 1). The current density $s^{ex} = s^{ex}(r,\theta)$ is limited to the region $r_i < r < r_o$ and exhibits in the absence of fabrication errors the symmetry properties

$$s^{ex}(r,\theta) = s^{ex}(r,-\theta)$$

and $$s\ (r,\theta) = -s^{ex}(r,\theta+\pi) \quad .$$

Considerable simplifications are achieved by representing the current density through the Fourier series which has the general form

$$s^{ex} = \Sigma_o\, s_n^{ex}(r)\cos n\theta \quad \text{for } r_i < r < r_o \quad (5)$$

with n restricted to odd integers. (We will use the convention that Σ_o expresses a sum over odd n and Σ_e a sum over even n.) For certain geometries, the Fourier analysis of the current distribution can be done analytically, for others numerically only, but this question is irrelevant in the context of this paper and the knowledge of the s_n^{ex} is assumed.

The vector potential satisfies the DE (4) which in the absence of the iron shield is simply

$$\Delta A^{ex} = - s^{ex} \quad . \quad (6)$$

In view of the Fourier representation of the forcing term it is advantageous to make the ansatz for the vector potential

$$A^{ex} = \Sigma_o\, A_n^{ex}(r)\cos n\theta \ . \quad (7)$$

The components $A_n^{ex}(r)$ must then each satisfy the ordinary DE ($r_i < r < r_o$)

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dA_n^{ex}}{dr}\right) - \frac{n^2}{r^2}A_n^{ex} = - s_n^{ex}(r) \quad (8)$$

together with the boundary conditions that A_n^{ex} and $dA_n^{ex}/dr = A_n'^{ex}$ are continuous at r_i and r_o. A particular solution of (8) for the region $r_i < r < r_o$ is obtained in integral form[8]

$$A_n^{ex} = x\int_{r_i}^{r} s_n^{ex}\, y\, w^{-1}\, dr + y\int_{r}^{r_o} s_n^{ex}\, x\, w^{-1}\, dr \quad (9a)$$

$$A'^{ex} = x'\int_{r_i}^{r} s_n^{ex}\, y\, w^{-1}\, dr + y'\int_{r}^{r_o} s_n^{ex}\, x\, w^{-1}\, dr \quad (9b)$$

in which x and y are linearly independent solutions of the homogeneous equation and $w = xy' - yx'$ is the Wronskian determinant. Putting

$$x = (r_o/r)^n \quad ; \quad y = (r/r_i)^n$$

$$x' = - n(r_o/r)^{n+1}/r_o \ ; \ y' = n(r/r_i)^{n-1}/r_i$$

$$w = 2n(r_o/r_i)^n/r$$

leads to

$$A^{ex} = \left(\frac{r_o}{r}\right)^n \sigma_n^{ex}(r) + \left(\frac{r}{r_i}\right)^n \sigma^{+ex}(r) \quad (10a)$$

$$A'^{ex} = -\frac{n}{r_o}\left(\frac{r_o}{r}\right)^{n+1}\sigma_n^{ex}(r) + \frac{n}{r_i}\left(\frac{r}{r_i}\right)^{n-1}\sigma^{+ex}(r) \quad (10b)$$

with

$$\sigma_n^{ex} = \frac{1}{2n\, r_o^n}\int_{r_i}^{r} s_n^{ex}\, r^{n+1}\, dr$$

$$\sigma_n^{+ex} = \frac{r_i^n}{2n}\int_{r}^{r_o} s_n^{ex}\, r^{-(n-1)}\, dr$$

For use later on we define

$$S_n^{ex} = \sigma_n^{ex}(r_o)$$

and $S_n^{+ex} = \sigma_n^{+ex}(r_i)$.

Obviously, $\sigma_n^{ex}(r_i) = \sigma_n^{+ex}(r_o) = 0$.

Taking into account the boundary conditions one obtains after simple manipulations

in the region $r < r_i$

$$A_n^{ex} = \left(\frac{r}{r_i}\right)^n S_n^{+ex} \quad (10c)$$

and in the region $r > r_o$

$$A_n^{ex} = \left(\frac{r_o}{r}\right)^n S_n^{ex} \quad . \quad (10d)$$

For the subsequent development only the coefficients S_n^{ex} are required, and it is immaterial how they were obtained. To simplify the expressions only "good" dipole magnets are considered in the sequel, that is, only S_1^{ex} will be retained.

As illustration we give here the results for a perfect dipole distribution, $s^{ex} = \hat{s}_1 \cos\theta$:

$$S_1^{ex} = \frac{\hat{s}_1}{6}\,\frac{r_o^3 - r_i^3}{r_o}$$

$$S_1^{+ex} = \frac{\hat{s}_1}{2}\, r_i(r_o - r_i) \quad .$$

One finds, indeed, that for the given coil geometry S_1^{ex} may be used as the single independent variable.

III. Presence of Infinite Permeability Iron Shield

In the presence of a constant permeability iron shield the solution for the vector potential is usually found from DE (6) together with the boundary condition that A and $\gamma dA/dr$ are continuous at the iron-air interfaces. An alternate approach, more suited to the subsequent development, consists in introducing induced magnetization currents at the interface. The vector potential is now determined by the DE

$$\Delta A = - s^{ex} - g^{Ri}\,\delta(R_i) - g^{Ro}\,\delta(R_o) \ , \quad (11)$$

where $\delta(R)$ is the delta function with the property

$$\int f(r)\,\delta(R) = f(R) \quad .$$

Consistency requires that the line currents g satisfy the boundary conditions

$$g^{Ri} = \left[(\gamma - 1)\ \frac{\partial A}{\partial r}\right]_{R_i + \epsilon} \tag{12a}$$

$$g^{Ro} = \left[(1 - \gamma)\ \frac{\partial A}{\partial r}\right]_{R_o - \epsilon} \tag{12b}$$

In the case of infinite permeability $\gamma = 0$ and (12) reduces to

$$g^{Ri} = - \left[\frac{\partial A}{\partial r}\right]_{R_i + \epsilon} \tag{12c}$$

and
$$g^{Ro} = \left[\frac{\partial A}{\partial r}\right]_{R_o - \epsilon} \tag{12d}$$

Separating the effects due to the extraneous currents and the induced magnetization currents one can write $A = A^{ex} + A^{Fe}$. Subtracting DE (6) from (11) leads to the DE for $A^{Fe} = \Sigma_O\ A_n^{Fe}(r)\ \cos n\theta$

$$\Delta A^{Fe} = - g^{Ri}\ \delta(R_i) - g^{Ro}\ \delta(R_o)\ . \tag{13}$$

One finds the solution in analogy to (9):

— for the region $R_i < r < R_o$

$$A_n^{Fe} = \frac{1}{2n}\left\{\left(\frac{R_i}{r}\right)^n g_n^{Ri}\ R_i + \left(\frac{r}{R_o}\right)^n g_n^{Ro}\ R_o\right\} \tag{14a}$$

— for the region $r < R_i$

$$A_n^{Fe} = \frac{1}{2n}\left\{\left(\frac{r}{R_i}\right)^n g_n^{Ri}\ R_i + \left(\frac{r}{R_o}\right)^n g_n^{Ro}\ R_o\right\} \tag{14b}$$

— and for the region $r > R_o$

$$A_n^{Fe} = \frac{1}{2n}\left\{\left(\frac{R_i}{r}\right)^n g_n^{Ri}\ R_i + \left(\frac{R_o}{r}\right)^n g_n^{Ro}\ R_o\right\}\ . \tag{14c}$$

The magnitude of the induced magnetization currents is given by the boundary conditions (12), from which follows the coupled set of linear equations in the g_n^{Ri} and g_n^{Ro}

$$g_n^{Ri} = \frac{1}{2}\ g_n^{Ri} - \frac{1}{2}\left(\frac{R_i}{R_o}\right)^{n-1} g_n^{Ro} + \frac{n}{r_o}\left(\frac{r_o}{R_i}\right)^{n+1} S_n^{ex} \tag{15a}$$

$$g_n^{Ro} = -\frac{1}{2}\left(\frac{R_i}{R_o}\right)^{n+1} g_n^{Ri} + \frac{1}{2}\ g_n^{Ro} - \frac{n}{r_o}\left(\frac{r_o}{R_o}\right)^{n+1} S_n^{ex}\ . \tag{15b}$$

Separation of the coefficients is possible, leading to

$$g_n^{Ri} = \frac{2n}{R_i}\left(\frac{r_o}{R_i}\right)^n S_n^{ex}\ \frac{1 + (R_i/R_o)^{2n}}{1 - (R_i/R_o)^{2n}} \tag{16a}$$

$$g_n^{Ro} = -\frac{2n}{R_o}\left(\frac{r_o}{R_o}\right)^n S_n^{ex}\ \frac{2}{1 - (R_i/R_o)^{2n}}\ . \tag{16b}$$

The presence of the iron shield increases the desired field inside the current coils. According to (14) the increase in the region $r < r_i$ is given by

$$A^{Fe} = \left(\frac{r}{R_i}\right)^n \left(\frac{r_o}{R_i}\right)^n S_n^{ex}\ . \tag{17}$$

It is worth noting that in agreement with intuition the outer radius does not appear in this equation. The magnetic flux density in the iron is, however, dependent on R_o.

The total field inside the current coil is given by ($r < r_i$)

$$A_n = \left(\frac{r}{r_i}\right)^n S_n^{+ex}\left[1 + \left(\frac{r_i}{R_i}\right)^n \left(\frac{r_o}{R_i}\right)^n \frac{S_n^{ex}}{S_n^{+ex}}\right]\ . \tag{18}$$

The second term in the square bracket expresses the relative gain in magnetic field due to the presence of the infinite permeability iron shield. As illustrative example we consider again the perfect dipole, for which the relative gain is $(r_o^2 + r_i r_o + r_i^2)/3R_i^2$.

In the subsequent nonlinear analysis the field level will be characterized by S_1^{ex}, which is directly related to the dipole field inside the coil for infinite permeability iron shield

$$B_1^{\infty} = S_1^{ex}\ \frac{r_o}{R_i^2}\ \left(1 + \frac{3R_i^2}{r_o^2 + r_o r_i + r_i^2}\right)\ . \tag{19}$$

The expressions for the total field inside the infinite permeability iron shield will be used as reference in the nonlinear analysis. They can be represented by ($R_i < r < R_o$):

$$A_n^{\infty} = \frac{2S_n^{ex}}{[1 - (R_i/R_o)^{2n}]}\left\{\left(\frac{r_o}{r}\right)^n - \left(\frac{r}{R_o}\right)^n \left(\frac{r_o}{R_o}\right)^n\right\} \tag{20a}$$

$$A_n'^{\infty} = \frac{-\ 2n\ S_n^{ex}}{r_o[1 - (R_i/R_o)^{2n}]}\left\{\left(\frac{r_o}{r}\right)^{n+1} + \left(\frac{r}{R_o}\right)^{n-1}\left(\frac{r_o}{R_o}\right)^{n+1}\right\} \tag{20b}$$

It should be noted that $A^{\infty} = 0$ but $A'^{\infty} \neq 0$ for $r \rightarrow R_o$, which means that, indeed, the magnetic field is parallel to the surface and vanishes outside.

IV. Perturbation Due to Nonlinear Iron

The exact DE for the vector potential in the presence of a nonlinear iron shield follows from (4) and (12) as

$$\Delta A = - s^{ex} - \gamma^{-1}\ \vec{grad}\ \gamma \cdot \vec{grad}\ A - \left[(\gamma - 1)\ \frac{\partial A}{\partial r}\right]_{R_i + \epsilon} \delta(R_i) - \left[(1 - \gamma)\ \frac{\partial A}{\partial r}\right]_{R_o - \epsilon} \delta(R_o)\ . \tag{21}$$

The inverse permeability is now a function of the absolute value of the local magnetic field B or, in view of (2), a function of the absolute value of the gradient of the vector potential, $|\vec{grad}\ A|$, since

$$B = \sqrt{\left(\frac{\partial A}{\partial r}\right)^2 + \left(\frac{1}{r}\ \frac{\partial A}{\partial \theta}\right)^2}\ .$$

It was pointed out in the introduction that the nonlinear solution differs not too much from the infinite permeability case. It is, therefore, natural to consider the finite permeability as well as the nonlinear case as a perturbation of the infinite permeability case,

$$A = A^{\infty} + \delta A$$

with A^{∞} in the iron region given by (20). In order to obtain the DE for the perturbation δA we subtract (11), that is

$$\Delta A^{\infty} = - s^{ex} - \left[- \frac{\partial A^{\infty}}{\partial r} \right]_{R_i + \epsilon} \delta(R_i) - \left[\frac{\partial A^{\infty}}{\partial r} \right]_{R_o - \epsilon} \delta(R_o)$$

from DE (21) and find the rigorous nonlinear DE

$$\Delta \delta A = - \gamma^{-1} \ \vec{grad} \ \gamma \cdot \vec{grad} \ A - \left[- \frac{\partial \delta A}{\partial r} + \gamma \frac{\partial A}{\partial r} \right]_{R_i + \epsilon} \delta(R_i) - \left[\frac{\partial \delta A}{\partial r} - \gamma \frac{\partial A}{\partial r} \right]_{R_o - \epsilon} \delta(R_o) \ . \quad (22)$$

It is now possible to linearize DE (22) by evaluating γ for a known approximate trial solution A^{tr}. It is tempting to start with $A^{tr} = A^{\infty}$, but if by some other way (mesh iteration) a better solution would be known, it also could be used. The approximate, but linear DE describing the nonlinear iron takes the form

$$\Delta \delta A + \gamma^{-1} \ \vec{grad} \ \gamma \cdot \vec{grad} \ \delta A = - \gamma^{-1} \ \vec{grad} \ \gamma \cdot \vec{grad} \ A^{\infty} - G^{Ri} \ \delta(R_i) - G^{Ro} \ \delta(R_o) \quad (23)$$

together with the boundary conditions

$$G^{Ri} = \left[(\gamma - 1) \frac{\partial \delta A}{\partial r} + \gamma \frac{\partial A^{\infty}}{\partial r} \right]_{R_i + \epsilon} \quad (24a)$$

$$G^{Ro} = - \left[(\gamma - 1) \frac{\partial \delta A}{\partial r} + \gamma \frac{\partial A^{\infty}}{\partial r} \right]_{R_o - \epsilon} \ . \quad (24b)$$

Making the usual ansatz $\delta A = \Sigma_o \ \delta A_n \cos n\theta$ results in a coupled set of linear DE in the δA_n. Proceeding towards this goal one has to obtain the Fourier representation of

$$\gamma(R_o) = \gamma_o^{Ro} + \Sigma_e \ \gamma_m^{Ro} \cos m\theta \quad (25a)$$

$$\gamma(R_i) = \gamma_o^{Ri} + \Sigma_e \ \gamma_m^{Ri} \cos m\theta \quad (25b)$$

and, correspondingly,

$$\gamma^{-1} \ \vec{grad} \ \gamma = \gamma^{-1} \frac{d\gamma}{dB} \ \vec{grad} \ B = \vec{u}_r \left\{ M_o^r(r) + \Sigma_e \ M_m^r(r) \cos m\theta \right\} + \vec{u}_\theta \left\{ \Sigma_e \ M_m^\theta(r) \sin m\theta \right\} \ . \quad (26)$$

The determination of the functions M_m^r, M_m^θ, and coefficients γ_m^{Ri}, γ_m^{Ro} represents a considerable fraction of the computational work.

Limiting the solution to "good" dipole magnets, i.e., $A^{\infty} = A_1^{\infty} \cos\theta$ simplifies Eq. (23) which is now replaced by the following set of coupled ordinary linear DE:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d \delta A_n}{dr} \right) - \frac{n^2}{r^2} \delta A_n + D_n = - s_n - G_n^{Ri} \ \delta(R_i) - G_n^{Ro} \ \delta(R_o) \quad (27)$$

with

$$s_1 = \left(M_o^r + \frac{1}{2} M_2^r \right) A_1^{\infty\prime} - \left(\frac{1}{2} M_2^\theta \right) A_1^{\infty}/r \quad (28a)$$

$$s_n = \frac{1}{2} \left(M_{n-1}^r + M_{n+1}^r \right) A_1^{\infty\prime} + \frac{1}{2} \left(M_{n-1}^\theta - M_{n+1}^\theta \right) A_1^{\infty}/r \quad (28b)$$

and

$$D_n = \frac{1}{2} \sum_i \sum_m \delta_{n(i+m)} \left(M_m^r \ \delta A_i^{\prime} + i \ M_m^\theta \ \delta A_i/r \right) + \frac{1}{2} \sum_i \sum_m \delta_{n|i-m|} \left(M_m^r \ \delta A_i^{\prime} - i \ M_m^\theta \ \delta A_i/r \right) \quad (29)$$

in which the Kronecker symbols have their usual meaning

$$\delta_{n(i+m)} = \begin{cases} 1 & n = i + m \\ 0 & n \neq i + m \end{cases}$$

and

$$\delta_{n|i-m|} = \begin{cases} 1 & n = |i - m| \\ 0 & n \neq |i - m| \end{cases} \ .$$

The boundary conditions (24) must be replaced by a set of coupled linear equations in the G_n^{Ri} and G_n^{Ro}

$$G_n^{Ri} = - \left[\delta A_n^{\prime} \right]_{R_i + \epsilon} + E_n^{Ri} + F_n^{Ri} \quad (30a)$$

$$G_n^{Ro} = + \left[\delta A_n^{\prime} \right]_{R_o - \epsilon} - E_n^{Ro} - F_n^{Ro} \quad (30b)$$

with

$$F_1^{Ri} = \left(\gamma_o^{Ri} + \frac{1}{2} \gamma_2^{Ri} \right) \left[A_1^{\infty\prime} \right]_{R_i + \epsilon} \quad (31a)$$

$$F_n^{Ri} = \frac{1}{2} \left(\gamma_{n-1}^{Ri} + \gamma_{n+1}^{Ri} \right) \left[A_1^{\infty\prime} \right]_{R_i + \epsilon} \quad (31b)$$

$$E_n^{Ri} = \frac{1}{2} \sum_i \sum_m \delta_{n(i+m)} \ \gamma_m^{Ri} \left[\delta A_i^{\prime} \right]_{R_i + \epsilon} + \frac{1}{2} \sum_i \sum_m \delta_{n|i-m|} \ \gamma_m^{Ri} \left[\delta A_i^{\prime} \right]_{R_i + \epsilon} \quad (32)$$

and corresponding expressions for F_n^{Ro} and E_n^{Ro}.

The exact general solution of (27) can be found by numerical methods only. A first approximation may be obtained by truncating the Fourier series for δA and retaining only the dipole term $\delta A = \delta A_1 \cos\theta$. After having obtained this solution, one extends the series to the sextupole term and finds the solution for δA_3 while keeping δA_1 unchanged. This procedure can be repeated until all harmonics of interest are determined. In this paper the solution will be limited to the dipole term, the details of which will be elaborated in the subsequent section.

V. The Dipole Approximation

In this section, a solution of DE (27) will be derived under the assumption that $\delta A \approx \delta A_1 \cos\theta$ represents an adequate description of the nonlinear effects. We now have the DE

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\delta A_1}{dr}\right) - \frac{\delta A_1}{r^2} + D_1(\delta A_1) = - s_1 - G_1^{Ri}\,\delta(R_i) - G_1^{Ro}\,\delta(R_o) \qquad (33)$$

in which s_1 is given by (28a) and

$$D_1(\delta A_1) = \left(M_o^r + \frac{1}{2} M_2^r\right) \delta A_1' - \frac{1}{2} M_2^\theta\, \delta A_1/r \quad . \qquad (34)$$

The coefficients G_1^{Ri} and G_1^{Ro} follow from the boundary conditions

$$G_1^{Ri} = \Gamma_1^{Ri}\left[\delta A_1'\right]_{R_i+\epsilon} + F_1^{Ri} \qquad (35a)$$

$$G_1^{Ro} = -\Gamma_1^{Ro}\left[\delta A_1'\right]_{R_o-\epsilon} - F_1^{Ro} \qquad (35b)$$

with the F_1^{Ri} and F_1^{Ro} as previously defined by (31) and

$$\Gamma_1^R = \gamma_o^R + \frac{1}{2}\gamma_2^R - 1 \quad .$$

To obtain the general solution of DE (33) it is necessary to first solve the homogeneous part of (33), the solutions of which may be written as

$$x_1 = R_o/r + \xi_1 \;;\quad y_1 = r/R_i + \eta_1$$

$$x_1' = - R_o/r^2 + \xi_1' \;;\quad y_1' = 1/R_i + \eta_1'$$

$$w_1 = x_1 y_1' - y_1 x_1' \quad .$$

To satisfy boundary conditions one must impose

$$\xi_1(R_o) = \xi_1'(R_o) = 0 \qquad (36a)$$

and

$$\eta_1(R_i) = \eta_1'(R_i) = 0 \quad . \qquad (36b)$$

In general, ξ_1 and η_1 must be determined by numerical methods. One possible approach is to rearrange the homogeneous DE into the DE

$$\frac{1}{r}\frac{d}{dr}\left(r\,\xi_1'\right) - \xi_1/r^2 = - D_1(x_1) \qquad (37)$$

and a corresponding equation in η_1.

It is now possible to derive integral equations which are readily solved by point-by-point integration.[9] Again using (9) one finds

$$\xi_1 = \frac{1}{2} r^{-1} \int_{R_o}^{r} r^2\, D_1(x_1)\, dr - \frac{1}{2} r \int_{R_o}^{r} D_1(x_1)\, dr \qquad (38a)$$

$$\xi_1' = - \frac{1}{2} r^{-2} \int_{R_o}^{r} r^2\, D_1(x_1)\, dr - \frac{1}{2} \int_{R_o}^{r} D_1(x_1)\, dr \qquad (38b)$$

and a corresponding solution for η_1,

$$\eta_1 = \frac{1}{2} r^{-1} \int_{R_i}^{r} r^2\, D_1(y_1)\, dr - \frac{1}{2} r \int_{R_i}^{r} D_1(y_1)\, dr \qquad (38c)$$

$$\eta_1' = - \frac{1}{2} r^{-2} \int_{R_i}^{r} r^2\, D_1(y_1)\, dr - \frac{1}{2} \int_{R_i}^{r} D_1(y_1)\, dr \qquad (38d)$$

After having determined ξ_1 and η_1 as outlined one proceeds to write the solution of (33) in the form

$$\delta A_1 = x_1(r)\, G_1^{Ri}/w_1(R_i) + y_1(r)\, G_1^{Ro}/w_1(R_o) + x_1(r)\int_{R_i}^{r} s_1 y_1 w_1^{-1}\, dr + y_1(r)\int_{r}^{R_o} s_1 x_1 w_1^{-1}\, dr \qquad (39a)$$

$$\delta A_1' = x_1'(r)\, G_1^{Ri}/w_1(R_i) + y_1'(r)\, G_1^{Ro}/w_1(R_o) + x_1'(r)\int_{R_i}^{r} s_1 y_1 w_1^{-1}\, dr + y_1'(r)\int_{r}^{R_o} s_1 x_1 w_1^{-1}\, dr \qquad (39b)$$

The as yet unknown coefficients G_1^{Ri} and G_1^{Ro} are now determined by substituting (39) into (35) which leads to the coupled linear equations

$$G_1^{Ri} = \Gamma_1^{Ri}\left[X_1'^{Ri}\, G_1^{Ri} + Y_1'^{Ri}\, G_1^{Ro} + y_1'(R_i)\, S_1^{\dagger}\right] + F_1^{Ri} \qquad (40a)$$

$$G_1^{Ro} = - \Gamma_1^{Ro}\left[X_1'^{Ro}\, G_1^{Ri} + Y_1'^{Ro}\, G_1^{Ro} + x_1'(R_o)\, S_1\right] - F_1^{Ro} \qquad (40b)$$

with

$$S_1 = \int_{R_i}^{R_o} s_1 y_1 w_1^{-1}\, dr$$

$$S_1^{\dagger} = \int_{R_i}^{R_o} s_1 x_1 w_1^{-1}\, dr$$

and

$$X_1^R = x_1(R)/w_1(R_i)$$

$$X_1'^R = x_1'(R)/w_1(R_i)$$

$$Y_1^R = y_1(R)/w_1(R_o)$$

$$Y_1'^R = y_1'(R)/w_1(R_o) \quad .$$

By solving (40) one obtains for the coefficients

$$G_1^{Ri} = \frac{\left(1 + \Gamma_1^{Ro}\, Y_1'^{Ro}\right)\left\{\Gamma_1^{Ri}\, y_1'(R_i)\, S_1^{\dagger} + F_1^{Ri}\right\} - \Gamma_1^{Ri}\, Y_1'^{Ri}\left\{\Gamma_1^{Ro}\, x_1'(R_o)\, S_1 + F_1^{Ro}\right\}}{\left(1 - \Gamma_1^{Ri}\, X_1'^{Ri}\right)\left(1 + \Gamma_1^{Ro}\, Y_1'^{Ro}\right) + \Gamma_1^{Ri}\,\Gamma_1^{Ro}\, Y_1'^{Ri}\, X_1'^{Ro}} \tag{41a}$$

$$G_1^{Ro} = -\frac{\Gamma_1^{Ro}\, X_1'^{Ro}\left\{\Gamma_1^{Ri}\, y_1'(R_i)\, S_1^{\dagger} + F_1^{Ri}\right\} + \left(1 - \Gamma_1^{Ri}\, X_1'^{Ri}\right)\left\{\Gamma_1^{Ro}\, x_1'(R_o)\, S_1 + F_1^{Ro}\right\}}{\left(1 - \Gamma_1^{Ri}\, X_1'^{Ri}\right)\left(1 + \Gamma_1^{Ro}\, Y_1'^{Ro}\right) + \Gamma_1^{Ri}\,\Gamma_1^{Ro}\, Y_1'^{Ri}\, X_1'^{Ro}} \tag{41b}$$

Actually, the change of the field in the vicinity of the axis due to saturation represents a quantity of interest and is directly accessible to experimental verification. This change is given by $(r < R_i)$

$$\delta A_1 = \frac{r}{R_i}\left\{\frac{x_1(R_i)}{w_1(R_i)}\, G_1^{Ri} + \frac{1}{w_1(R_o)}\, G_1^{Ro} + S_1^{\dagger}\right\} . \tag{42a}$$

The leakage field outside the iron shield, which is also accessible to measurement, is given by $(r > R_o)$

$$\delta A_1 = \frac{R_o}{r}\left\{\frac{1}{w_1(R_i)}\, G_1^{Ri} + \frac{y_1(R_o)}{w_1(R_o)}\, G_1^{Ro} + S_1\right\} . \tag{42b}$$

The numerical evaluation of the dipole solution is in progress. The results obtained so far seem to indicate that the use of the infinite-permeability case as trial function does not yield results of sufficient accuracy, limiting considerably the applicability of this method. It is possible that a simple iteration procedure could overcome this limitation. However, a definite statement as to the accuracy of the perturbation method is not possible at the present time.

References

1. A.M. Winslow, J. Comput. Phys. 2, 149 (1967).
2. N.I. Doinikov and A.S. Simakov, Sov. Phys. — Tech. Phys. 14, 1097 (1970).
3. G. Parzen and K. Jellett, Particle Accelerators 2, 169 (1971).
4. K. Halbach, Nucl. Instr. and Meth. 78, 185 (1970).
5. N.I. Doinikov, Sov. Phys. — Tech. Phys. 15, 2030 (1971).
6. H. Hahn, BNL Report CRISP 72-14 (1972).
7. L.D. Landau and E.M. Lifshitz, Electrodynamics of Continuous Media (Pergamon Press, 1960), p. 121.
8. R. Rothe, Höhere Mathematik (Teubner, Leipzig, 1953), Vol. 3, p. 194.
9. B. Carnahan, H.A. Luther, and J.O. Wilkes, Applied Numerical Methods (J. Wiley & Sons, 1969), p. 341.