A NEW MATHEMATICAL TECHNIQUE TO COMPUTE THE MAGNETIC POTENTIAL PRODUCED BY PERMANENT MAGNETS IN PLANE OR CYLINDRICAL SYMMETRY

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Abstract

A new method is outlined to compute the magnetic potential produced by permanent magnets in plane and cylindrical symmetry. The space is divided by an orthogonal network of i lines and j columns. The potential in the vertices of each columns is considered as a vector of i components. The relations between these vectors are established. Using these relations it is possible to get equations to calculate the potentials in each point of the network in function of the potentials in the points of the contour of the space considered. To get the field in the permanent magnet space it is necessary to use an iterative method: this new iterative method is presented.

I. Introduction

To calculate the forces between a permanent magnet and an iron yoke in a cylindrical geometry, we needed to compute the magnetic field produced by the permanent magnet in the gap between it and the iron yoke. In Fig. 1 one sees the

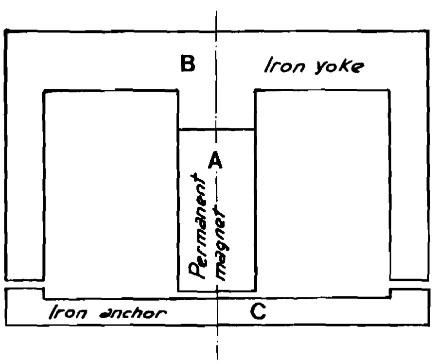

FIG. 1

geometry of our problem: A is the permanent magnet, B is an iron yoke of infinite magnetic permeability ($\mu_r = \infty$), C is the iron yoke on which is exerted the magnetic attraction to be calculated.

To resolve the problem, we divide the space D (see Fig. 3) as usual by a network of N rows and n+k columns (see Fig. 2). We assume the horizontal and vertical steps are equal; we do so only to explain more easily the procedure, but it is neither an essential assumption nor a more

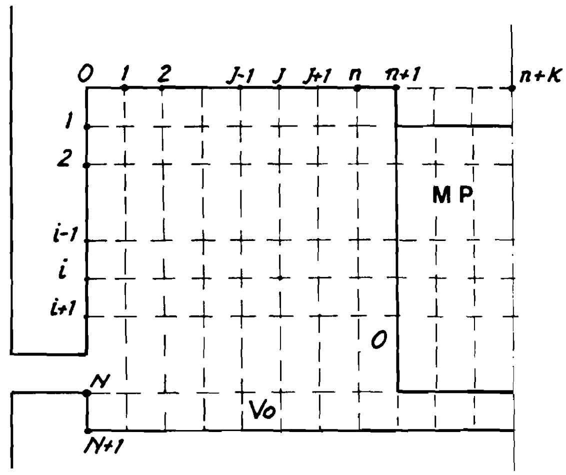

FIG. 2

convenient method to obtain more exact numerical calculations. We make two other assumptions:

a) The magnetic field is constant along the line a (Fig. 3);
b) The area D is rectangular.

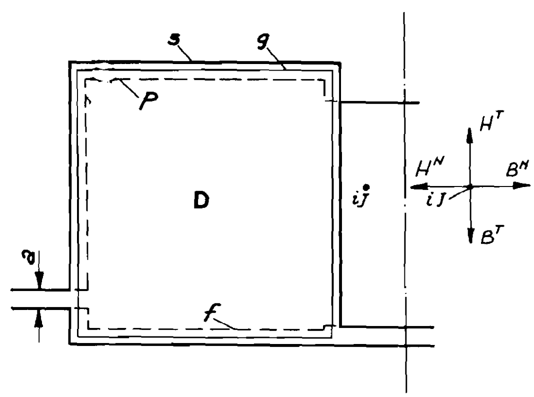

FIG. 3

Both of these statements simplify the program for the computer. However the mathematical statements outlined below would still be valid even if these assumptions were not made and the computer could still be used to do the calculations.

Proceeding are performed for two geometries:

a) Plane with a symmetry axis ;
b) Cylindrical geometry.

The problem is resolved in the following steps :

1) One can find relations between magnetic potentials on the contour g (see Fig. 3) (potential along line p is zero, along line f is V_o = constant) and the magnetic fields on the contours not by iterative methods but by direct relations.
2) Through the magnetic properties of the permanent magnet, relations are found between the potential and the fields on the surface of the permanent magnet (inside the magnet). By assuming a linear relationship between magnetic field and the induction, that is

$$B = \mu_o \mu_r H + B_r . \qquad (1)$$

3) Using equation (1) and assuming an initial set of angles of the field in the different points of the network in the permanent magnet, the equations (written at finite differences)

$$\text{rot } H = 0 \qquad (2)$$

$$\text{div } B = 0 \qquad (3)$$

make possible to get the value of H (H^N, H^T) and B (B^N, B^T) (see Fig. 3) in the points of the second column inside the permanent magnet. This method can be repeated until the column n+k which is the axis of symmetry. Imposing the condition $B^N=0$, one obtain the values of $|V|_{n+1}$ that is the potential on the surface of the permanent magnet (column n+1). Thus one can find all the values of B^N, B^T, H^T, H^N at the different points of the permanent magnet: whith these values one calculates a new set of $\delta_{ij} = \text{arctg } B^N/B^T$ and one repeats the procedure. One steps the iterative procedure when the value of B^N/B^T is close enough to H^N/H^T.

II. Calculation of the relations between magnetic field and magnetic potential on the contour g in the plane geometry with an axis of symmetry

We define a vector $|V|_j$ as a vector of N components whose elements are the magnetic potentials at the points of the j^{th} column intersecting the different rows (see Fig. 2). We may easily establish relations between these vectors equivalent to the Maxwell equation for magnetic potential

$$\bar{\nabla}^2 V = 0 . \qquad (4)$$

Relation (4) may be written to finite differences

$$V_{ij} = \frac{1}{4}(V_{i-1,j} + V_{i+1,j} + V_{i,j+1} + V_{i,j-1}) \qquad (5)$$

and one gets the equation

$$M|V|_j = \frac{1}{4}|V|_{j-1} + \frac{1}{4}|V|_{j+1} + |V_o|_j \qquad (6)$$

where M is a matrix represented in Fig. 4 and $|V_o|_j$ is represented in Fig. 5.

$$M = \begin{vmatrix} 1 & -\frac{1}{4} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \\ -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \\ 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} \\ & & & & \cdot & \cdot & \cdot & 0 & 0 & -\frac{1}{4} & 1 \end{vmatrix}$$

FIG. 4

$$V_{oj} = V_o \begin{vmatrix} 0 \\ 0 \\ \\ \\ 0 \\ 1 \end{vmatrix}$$

V_{oj} = N COMPONENTS VECTOR

FIG. 5

Writing the eq. (6) for j=1, . . . , n one obtains a system of n linear equations in which we have n unknown vectors. Thus it is possible to get a direct symbolic relation between the magnetic potential $|V|_j$ (j^{th} column) and the potential on the contour. The system is shown in Fig. 6.

$$|V|_1 - \frac{M^{-1}}{4}|V|_2 = \frac{M^{-1}}{4}|V|_0 + \frac{M^{-1}}{4}|V_0|_1$$

$$-\frac{M^{-1}}{4}|V|_1 + |V|_2 - \frac{M^{-1}}{4}|V|_3 = \frac{M^{-1}}{4}|V_0|_2$$

$$-\frac{M^{-1}}{4}|V|_2 + |V|_3 - \frac{M^{-1}}{4}|V|_4 = \frac{M^{-1}}{4}|V_0|_3$$

- -

$$-\frac{M^{-1}}{4}|V|_{n-1} + |V|_n - \frac{M^{-1}}{4}|V|_{n+1} + \frac{M^{-1}}{4}|V_0|_{n+1}$$

FIG. 6

The normal components of the field on the surface of the permanent magnet at different points of the n+1th column is a vector of N components proportional to

$$\left| H^N \right|_n = \left| V \right|_{n+1} - \left| V \right|_n . \qquad (7)$$

Calling U the unitary matrix that diagonalize M^{-1} we may write the equations that connect $\left| H^N \right|_n$ to the contour potential: the complicated relation is reported in Fig. 7, where the symbol λ represent the eigenvalues of the matrix $M^{-1}/4$. The

$$\left| G_p(n) \right| \left| U \right| \left[\left| H^N \right|_n \right] + \left| (\lambda_p - 1) g_{pp}(n-1) + \lambda_p g_{pp}(n-2) \right| \left| V \right|_{n+1} =$$

$$= \left| (-1)^{n+1} \lambda_p g_{pp}(0) + \text{------} (-1)^{24} \lambda_p g_{pp}(n-1) \right| \left| U \right| \left| V_0 \right|,$$

FIG. 7

eigenvalues and eigenvectors of the matrix M or M^{-1} are easily calculated by the relation

$$\lambda_k = 1 - \frac{1}{2} \frac{\cos k\pi}{N+1} \qquad (k=1, 2, \ldots, N) \qquad (8)$$

The eigenvalues λ_k^* of matrix M^{-1} are

$$\lambda_k^* = \frac{1}{\lambda_k} . \qquad (9)$$

In this expression $G_p(k)$ are diagonal matrices whose elements $g_{pp}(k)$ may be computed by the recurring formula

$$g_{pp}(k) = g_{pp}(k-1) - \lambda_p^2 g_{pp}(k-2) . \qquad (10)$$

We have now a relation that connects the field on the permanent magnet surface and the magnetic potential on the contour p (see Fig. 3).

III. Iterative proceeding to compute the field in permanent magnet

As was stated in the introduction, we need two relations by which we correlate the magnetic field to the induction in the permanent magnet. That is

$$H_{ij}^T = \frac{1}{\mu_o \mu_r} B_{ij}^T - \frac{B_r}{\mu_o \mu_r} \cos \delta_{ij} \qquad a)$$

$$H_{ij}^N = \frac{1}{\mu_o \mu_r} B_{ij}^N - \frac{B_r}{\mu_o \mu_r} \sin \delta_{ij} \qquad b) \qquad (11)$$

where:

- the indices i, j indicate respectively the row and the column, that is the coordinates of the points considered;
- δ_{ij} is the angle of the field with respect to the column;
- B_r is the residual field characteristic of the material;
- $\mu_o \mu_r$ is the magnetic permeability;
- N, T indicate the normal or tangential component of field or induction (see Fig. 3).

On the surface just inside the permanent magnet it is possible to express H^T and B^N as functions of potentials on the surface s. Using relations (11) we get H^N and B^T, that is $\left| V \right|_{n+2}$ and $\left| B^T \right|_{n+1}$ (x).

Using the Maxwell equation at finite differences one obtains (12) and (13)

$$\left| B^N \right|_j = \left| B^N \right|_{j-1} + A \left| B^T \right|_{j-1} \qquad (12)$$

$$\left| H^T \right|_j = -A \left| H^N \right|_j + \left| H^T \right|_{j-1} \qquad (13)$$

The symbols used in equations (12) and (13) are defined in Fig. 8; a is the length of the side of a

$$\left| B^N \right|_j = \begin{vmatrix} B_{1j}^N \\ B_{2j}^N \\ \cdot \\ \cdot \\ \cdot \\ B_{nj}^N \end{vmatrix}$$

A VECTOR WHOSE COMPONENTS ARE THE NORMAL COMPONENTS OF INDUCTION B AT COLUMN j^{th} IN THE INTERSECTION OF THE COLUMN WITH ROW

a)

$$A = \begin{vmatrix} 1 & -1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & -1 & \cdot & \cdot & \cdot \end{vmatrix}$$

b)

FIG. 8

square of network. Using equation (11) we again calculate H_j^N and B_j^T. All these quantities result functions of magnetic potential (not known) on per-

(x) - With this symbol we denote a vector at N_T component which elements are the values of B^T in the points i, n+1 with i from 1 to N.

manent magnet surface. Using the relation

$$\left| B^N \right|_{n+k} = 0$$

(we remember that column n+k is the axis of symmetry of our system) one calculates the value of potential and the numerical value of $|B^N|_j$ $|B^T|_j$ $|H^T|_j$ $|H^N|_j$ in the different columns. We may now compute the new set of δ_{ij} = $= \mathrm{arctg} B^N_{ij}/B^T_{ij}$. When the set of the new values does not change practically from one process to the next we stop the iterative procedure : we consider our problem resolved.

IV. Calculation of field produced by permanent magnet in case of cylindrical symmetry

To compute the field in the case of cylindrical symmetry the procedures are analogous to the plane case. In Figs. 9 and 10 we summarize how the relations already established for the plane symmetry are modified in cylindrical symmetry.

PLANE SYMMETRY

$$V_{ij} = \frac{V_{ij-1}}{4} + \frac{V_{ij+1}}{4} + \frac{V_{i+1j}}{4} + \frac{V_{i-1j}}{4}$$

$$M|V|_j = \frac{|V|_{j+1}}{4} + \frac{|V|_{j-1}}{4} + \frac{|V_0|_j}{4}$$

CYLINDRICAL SYMMETRY

$$V_{ij} = \frac{V_{ij+1}}{4}\left(1 - \frac{A}{2R_{ij}}\right) + \frac{V_{ij-1}}{4}\left(1 + \frac{A}{2R_{ij}}\right) + \frac{V_{i+1j}}{4} + \frac{V_{i-1j}}{4}$$

$$M|V|_j = \left(1 - \frac{A}{2R_j}\right)\frac{|V|_{j+1}}{4} + \left(1 + \frac{A}{2R_j}\right)\frac{|V|_{j-1}}{4} - \frac{|V_0|_j}{4}$$

(R_j = RADIUS OF THE COLUMN)

FIG. 9

Calculations result a little more complicated in cylindrical case, because of the lack of an easy recurring formula to calculate the polynomial terms $P_k(\lambda)$, differently from what happens to calculate $g_p(n)$ in the plane case.

V. Discussion of results

In the plane case, by means of a Fortran program in "double precision", we can resolve a network of nearly 700 points (24 x 28), with a precision of $1/10^3$. Really this method limits the number of columns (not of rows, that is practically illimited).

In fact we calculate the values of potential over the first column and then, using relations

RELATIONS BETWEN POTENTIALS AND GRADIENTS

A) PLANE CASE

$$G_p(N)U \begin{vmatrix} \mathrm{GRAD}\,V_1 \\ \mathrm{GRAD}\,V_1 \\ \mathrm{GRAD}\,V_N \end{vmatrix} + \left|(\lambda_p - 1)\,G_p(n-1) + \lambda_p^2 \left|G_p(n-2)\right|\right| \left|U\right| \begin{vmatrix} V_{1,n+1} \\ V_{2,n+1} \\ V_{N,n+1} \end{vmatrix} =$$

$$= V_0 \left|(-1)^{n+1} \lambda_p^n \, G_p(0) + \cdots + (-1)^{2n} \lambda_p \, G_p(n-1)\right| \left|U\right| \begin{vmatrix} 0 \\ 0 \\ \\ 1 \end{vmatrix} + (-1)^{n+1} \lambda_p V_0 G_p(0) \; U \begin{vmatrix} 0 \\ 0 \\ \\ 1 \end{vmatrix}$$

$$\left|G_p\right| = \mathrm{DET} \begin{vmatrix} 1-\lambda_p & 0 & 0 & \\ -\lambda_p & 1-\lambda_p & 0 & 0 \\ & & -\lambda_p & 1-\lambda_p \\ & & 0 & -\lambda_p \quad 1 \end{vmatrix}$$

B) CYLINDRICAL CASE

$$P(N)U \begin{vmatrix} \mathrm{GRAD}\,V_1 \\ \mathrm{GRAD}\,V_2 \\ \\ \mathrm{GRAD}\,V_N \end{vmatrix} - \left[\left(1 - \frac{a}{2R_n}\right) P(n-1) - P_p(n)\right] U \begin{vmatrix} V_{1,n+1} \\ V_{2,n+1} \\ \\ V_{N,n+1} \end{vmatrix} =$$

$$= V_0 \left[(-1)^{n+1} P_p(0) + P_p(\lambda) + \cdots + P_p(n+1)\right] U \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

$$P_p(H) = \begin{vmatrix} 1 & A_1^* & 0 & 0 & & \\ A_2 & 1 & A_2^* & 0 & & \\ . & . & . & . & 1 & A_{n-1}^* \\ . & . & . & . & A_n & 1 \end{vmatrix}$$

$$A_J = -\left(1 + \frac{a}{2R_J}\right)\lambda_p$$

$$A_J^* = -\left(1 - \frac{a}{2R_J}\right)\lambda_p$$

FIG. 10

reported in Fig. 6 it is possible to calculate, step by step, the value of potential over all the following columns.

In each operation we have an error that is nearly seven times larger than the error in each one of the points of the preceding column.

Over the N^{th} column we may have an error 7^{N-1} times larger than over the first column (if we consider statistics errors we have only $7^{(N-1)/2}$).

We can get better results (and a larger number of columns for the network), using two different methods :

1) We calculate potentials from both the sides of network, until covering central columns. The

error becomes $\propto 7^{N/2}$.

2) Using a different fundamental equation. Instead of expressing each value of potential as function of potential of four surrounding points, i. e.

$$V_{i,j} = \frac{1}{4}\left[V_{i+1,j} + V_{i-1,j} + V_{i,j-1} + V_{i,j+1}\right], \qquad (14)$$

we may express the potential $V_{i,j}$ as function of potentials of eight or more points in the neighbourhood of point $P_{i,j}$. Naturally we must take into account the different distances these points have from $P_{i,j}$.

In the cylindrical case coefficients (see Fig. 9) as $(1 - a/2r)$ and $(1 + a/2r)$, make worse precision and limit more the number of columns.

If we use a rectangular ($a \times b$ = dimension of a single rectangle) network we found the same a stronger limitation on the number of columns : this may be understood, because, now, coefficients of fundamental equation are modified, i. e. instead of equation (5) we have

$$V_{i,j} = (V_{i,j+1} + V_{i,j-1}) \frac{b^2}{2(a^2+b^2)} + (V_{i-1,j} + V_{i+1,j}) \frac{a^2}{2(a^2+b^2)} . \qquad (15)$$

Fortran programs are at concerned's disposal. In Fig. 11 is reported as example a plot of potential calculated.

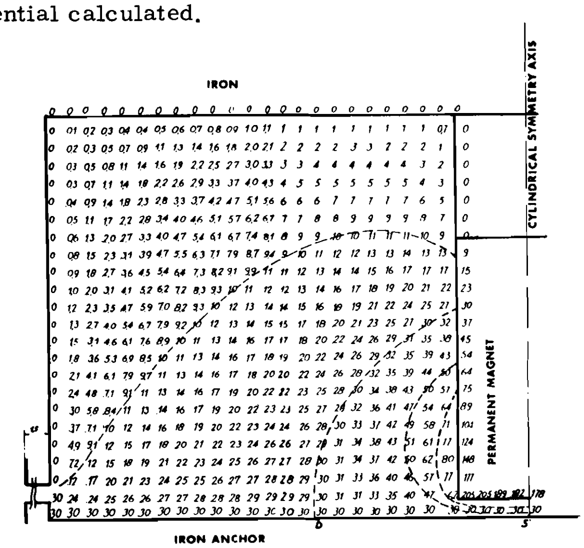

FIG. 11

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