ELECTROMAGNETIC LOADING OF DEWAR SHELLS

AS A RESULT OF TIME VARYING MAGNETIC FIELDS

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## Abstract

This paper analyses the currents induced in dewar shells due to time varying coil currents. The analyses are restricted to the central sections of the shells where end effects can be neglected. Solenoids and dipole coils are considered.

While a solenoid coil will result in loading either radially inward or radially outward (a loading similar to internal or external uniform pressure), it is concluded for dipole coils that, although the absolute values of surface loading are not generally large, the fact that they are non-uniformly distributed around the periphery may result in excessive deflections.

## I. Introduction

Rapid charge or discharge of superconducting or cryogenically cooled magnets causes induced currents in concentric dewar shells which can lead to electromagnetic loading of structural significance. The problem is particularly important in cases where the dewar or one of its components (e.g. radiation shield) is fabricated from a material with a relatively high electrical conductivity (typically copper or aluminum).

This paper contains an analysis of the induced currents and the resulting magnetic fields which result due to time varying coil currents. The magnetic fields are then used to calculate the resulting loads and deflections on dewar shells.

Although superconducting coils exist in many shapes, the two most common geometries are the solenoid and the dipole. The analyses presented will be restricted to two simplified cases -- the long thin solenoid and the long thin dipole with a sinusoidal winding distribution. Both of these cases have relatively simple closed form expressions for the magnetic field.

End effects are neglected and therefore the results are valid only for those cases where the shell diameter is smaller than the coil length.

The thin winding approximation is not very restrictive since it means only that an effective radius for the windings should be used.

The analyses are for the loads on cylindrical shells. The loads on end caps are beyond the scope of this paper.

## II. General Behavior of Shell Currents

It is shown in Appendices A and B that the central magnetic field generated by the currents induced in a shell by a time varying current in a coil are of the following form (all equations are in MKS units.):

For a shell outside the coil: ($r_s > r_o$)

$$\frac{dB_s}{d\,t} + \frac{2\rho}{\mu_o w_s r_s} B_s = -\left(\frac{r_o}{r_s}\right)^2 \frac{dB_o}{d\,t} \qquad (1)$$

For a shell inside the coil bore:

$$\frac{dB_s}{d\,t} + \frac{2\rho}{\mu_o w_s r_s} B_s = -\frac{dB_o}{d\,t} \qquad (2)$$

The equations are the same whether we are dealing with a solenoid or a dipole. (The equations for quadrupoles and higher order coils would be slightly different.)

When dealing with a solenoid, $B_o$ is the field in the bore generated by the coil currents alone and since it varies directly with the coil current, it is the independent variable. The field $B_s$ is the magnetic field produced in the bore of the shell by the shell currents alone. Due to the assumption of long length, the currents in the solenoid or in the shell produce a magnetic field which is zero outside the solenoid or shell and is uniform and parallel to the axis inside the bore of the solenoid or shell respectively.

A dipole with sinusoidally distributed windings produces a uniform magnetic field perpendicular to the axis of the dipole bore within the bore, and outside the bore, the magnetic field decays with distance in a typical dipole configuration.[1]

For a dipole coil, $B_o$ is the value of the uniform field generated inside the bore.

Currents induced in a co-axial shell by a dipole coil winding will also have a dipole current distribution. This means that the shell currents will produce a uniform field $B_s$ inside the bore of the shell, and external to the shell the field will decay with distance in a typical dipole manner.

The other quantities in Equations 1 and 2 are:

$w_s$ - thickness of the shell
$r_s$ - radius of the shell
$r_o$ - radius (effective) of the coil winding
$\rho$ - resistivity of the shell material

Equations 1 and 2 show that without any change in coil magnetic field, the shell magnetic field (or shell current) dies out with a time constant

$$\tau_s = \frac{\mu_o w_s r_s}{2\rho} \tag{3}$$

(The expression for the time constant is the same for the solenoid as well as for a dipole.)

Figure 1 shows a plot of the time constant for shells made of copper, aluminum and stainless steel operating at room temperature, liquid nitrogen temperature and at liquid helium temperature.

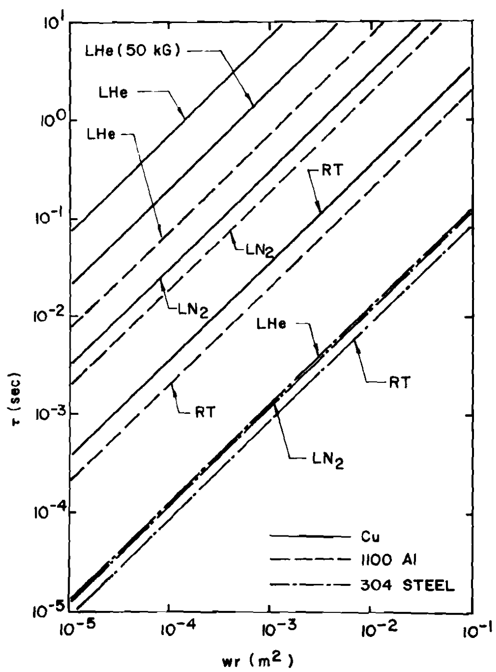


Fig. 1. Decay time constant for currents in a cylindrical shell of radius r and thickness w. This plot is valid for shell currents produced by either a long thin solenoid or a long thin dipole with sinusoidally distributed windings.

A typical large dewar might have a diameter of 1 m and a wall thickness of 0.25 cm, or a product of radius and thickness $w_s r_s = 2.5 \times 10^{-3}$ m. If the shell is made of stainless steel, it would have a time constant of a few milliseconds (at all temperatures). If the shell were made of aluminum, it would have a time constant of a few tens of milliseconds at room temperature and about 1 second at liquid helium temperature due to the decrease in resistivity with temperature. If the shell were made of copper, its time constant would be slightly larger than for aluminum, up to a few seconds if it were operated at liquid helium.

The behavior of a shell depends to a large extent on whether it is subjected to transients which are faster or slower than its time constant. If the transient is faster than the shell time constant, the shell currents will be large and may be comparable to the currents flowing in the main coil. From a structural point of view, this means that the loads on the shell will be of the same order of magnitude as the loads on the main coil.

If, on the other hand, the transients are slower than the shell time constants, the induced shell currents will be greatly reduced and the loads on the shell will be lower.

In designing a dewar, the estimates for maximum electromagnetic loading should be made for the most rapid transient.

## III. Loading Due To Long Solenoids

It is shown in Appendix A that the radial pressure (positive when outward radially) on a shell due to currents induced by a long solenoid is:

For an outer shell: ($r_s > r_o$)

$$p_n = \frac{B_s^2}{2\mu_o} \tag{4}$$

For an inner shell: ($r_o > r_s$)

$$p_n = \frac{B_s B_o}{\mu_o} + \frac{B_s^2}{2\mu_o} = \frac{B_s}{\mu_o}\left[B_o + \frac{B_s}{2}\right] \tag{5}$$

It is interesting to note that an outer shell ($r_s > r_o$) can be loaded only radially outward (whether the coil is being charged or discharged), due to the fact that the radial pressure is proportional to the square of the magnetic field produced by the shell currents.

For an inner shell during charging, the magnetic field due to the shell currents opposes the increasing central field so that $p_n$ will in general be negative which means that it is loaded radially inward. During discharge, or during a quench, the shell currents flow in such a direction as to make $B_s$ in the same direction as $B_o$, so that the radial pressure is positive and the shell is loaded radially outward.

One of the cases that reoccurs many times is that for a quenching solenoid in which the shell time constants are short compared with the quench time.

For this case we have:

$$B_s = \begin{cases} \left(\dfrac{r_o}{r_s}\right)^2 \dfrac{\tau_s}{\tau_q} B_o & r_s > r_o \\ \dfrac{\tau_s}{\tau_q} B_o & r_o > r_s \end{cases} \tag{6}$$

where it has been assumed that the quench of the coil can be approximated by a linearly decreasing coil magnetic field such that:

$$\frac{dB_o}{d\ t} = - \frac{B_o}{\tau_q} \tag{7}$$

where $\tau_q$ is the quench time.

Substitution of these values for $B_s$ into Equations 4 and 5 result in:

For an outer shell: $(r_s > r_o,\ \tau_q > \tau_s)$

$$p_n = \left(\frac{r_o}{r_s}\right)^4 \left(\frac{\tau_s}{\tau_q}\right)^2 \frac{B_o^2}{2\mu_o} \tag{8}$$

For an inner shell: $(r_o > r_s,\ \tau_q > \tau_s)$

$$p_n = \frac{\tau_s}{\tau_q} \frac{B_o^2}{\mu_o} \tag{9}$$

This loading is shown schematically in Figure 2.

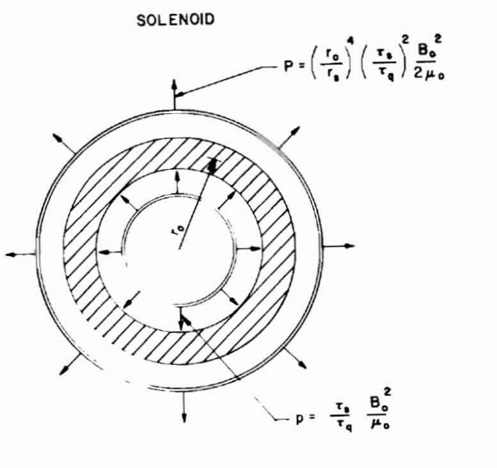


Fig. 2: Loading during the quench of a long solenoid.

It is important to note that the loading on the outer shell decreases rapidly as the ratio of coil radius to shell radius decreases. Also the loading on the outer shell is proportional to the second power of the ratio $\tau_s/\tau_q$ (which is less than unity) while that on the inner shell is proportional to the first power of this ratio.

If a general conclusion can be reached from these equations, it is that the inner shells of a dewar are likely to have higher loadings on them during a quench than outer shells made of the same materials. Further, the loading during a quench of a long solenoid on all shells is radially outward, so that while the loading may affect the design stress, the loading is not in suoh a direction as to buckle the shells.

## IV. Loading Due To Dipole Coils

Even before analyzing the loading of dewar shells due to time varying fields caused by a dipole coil, it is reasonable to expect that these loads will not be uniform radial loads (either inward or outward). Cylindrical shells are generally excellent structural elements when loaded radially outward by a uniform load. If they are subjected to loads radially inward, then buckling must be taken into account.

On the other hand, if a thin cylindrical shell is subjected to a load which varies in magnitude at different locations azimuthally, bending is introduced in the shell, and it is a much weaker structural element. We would therefore expect that non-uniform loading such as might be generated by a dipole coil winding would be considerably more severe than the uniform loading generated by a solenoid.

Appendix B derives the radial and shear or tangential loading for the case of a shell placed inside or outside the coil. The loading is:

For a shell outside $(r_s > r_o)$:

Radial:

$$p_n = - \frac{B_o B_s}{\mu_o} [1 + \cos 2\theta] \tag{10}$$

Tangential:

$$p_s = - \frac{B_s}{\mu_o} \left[ B_o \left(\frac{r_o}{r_s}\right)^2 + B_s \right] \sin 2\theta \tag{11}$$

For a shell inside $(r_s < r_o)$:

Radial:

$$p_n = \frac{B_o B_s}{\mu_o} [1 + \cos 2\theta] \tag{12}$$

Tangential:

$$p_s = -\frac{B_s}{\mu_o}[B_s + B_o] \sin 2\theta \qquad (13)$$

where $B_o$ and $B_s$ are the magnetic fields due to the coil and shell currents respectively. These magnetic fields are perpendicular to the axis of the dipole and point (when positive) along the $\theta = 90^o$ direction (see Figure 3). The radial loading has a constant component (which may buckle the shell if directed radially inward) as well as a component which varies as cos $2\theta$. For any given situation, the radial component on an outside shell is opposite in sign to the radial component on an inside shell. The tangential or shear loading is in the same direction in both inner and outer shells and varies as sin $2\theta$.

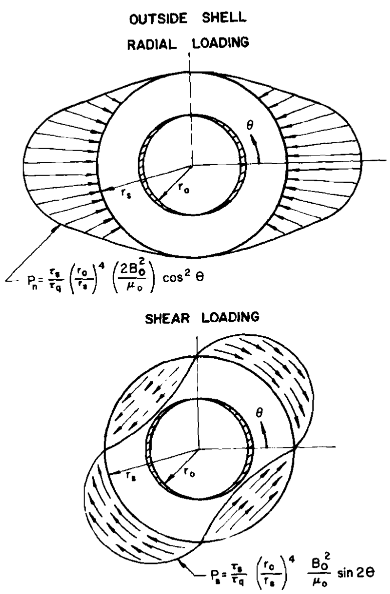


Fig. 3: Loading on an outside shell due to quench of a dipole coil.

The equations for the loading during a quench can be obtained in a manner similar to that used for the solenoid - namely, assuming a linear rate of decay of $B_o$ in a quench time $\tau_q$. Also for most practical cases $\tau_s < \tau_q$ so that a simple expression can be obtained for $B_s$ from Equations 1 and 2:

Outer Shell: $(r_s > r_o, \tau_s < \tau_q)$

$$p_n = -\frac{B_o^2}{\mu_o}\left(\frac{\tau_s}{\tau_q}\right)\left(\frac{r_o}{r_s}\right)^4 [\,1 + \cos 2\theta\,] \qquad (14)$$

$$p_s = -\frac{B_o^2}{\mu_o}\left(\frac{\tau_s}{\tau_q}\right)\left(\frac{r_o}{r_s}\right)^4 \sin 2\theta \qquad (15)$$

Inner Shell: $(r_s < r_o, \tau_s < \tau_q)$

$$p_n = \frac{B_o^2}{\mu_o}\left(\frac{\tau_s}{\tau_q}\right)(1 + \cos 2\theta) \qquad (16)$$

$$p_s = -\frac{B_o^2}{\mu_o}\left(\frac{\tau_s}{\tau_q}\right)\sin 2\theta \qquad (17)$$

The above loadings are shown in Figures 3 and 4 for an outer and inner shell respectively.

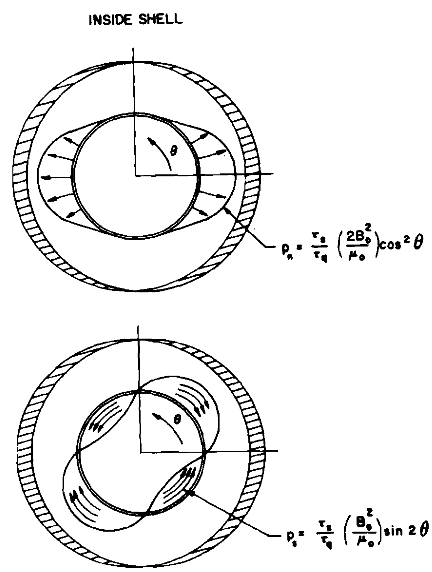


Fig. 4: Loading on an inner shell due to quench of a dipole coil.

The expected deflected shapes are shown in Figure 5. The shells deflect into an egg shape during the quench. The outer shell elongates in a direction parallel with the magnetic field while the inner shell elongates in a direction perpendicular to the main magnetic field.

TABLE I

| Material | I Steel | II Al | III Al | IV Steel | V Al |
|---|---|---|---|---|---|
| | 24 in Diameter Dewar | | | | 77 °K Radiation Shield Inside Bore |
| $r_s$ (in) | 12 | 12.75 | 14 | 15 | 3.875 |
| w (in) | 0.12 | 0.188 | 0.188 | 0.165 | 0.062 |
| $r_o$ (in) | 5.35 | 5.35 | 5.35 | 5.35 | 5.35 |
| E ($10^6$ psi) | 30 | 10.4 | 10.4 | 30 | 10.4 |
| Temp. (°K) | 4.2 | 77 | 77 | 300 | 77 |
| $\rho$ ($\mu\Omega$-m) (2) | 0.496 | 0.0166 | 0.0166 | 0.704 | 0.0166 |

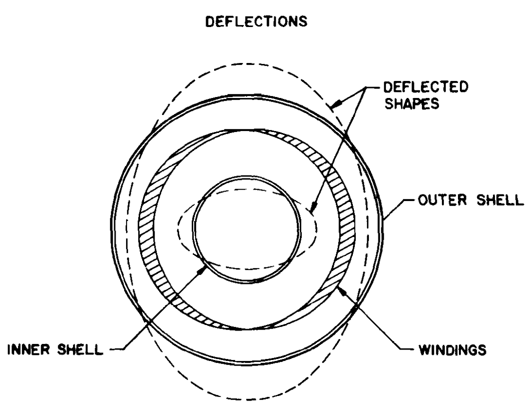


Fig. 5: Deflected shapes due to quenching of a dipole coil

Again, it is evident that the outer shells will in general be less critical due to the ratio $(r_o/r_s)^4$ which appears only in the equations for the outside shells. During the quench, the outside shell is loaded radially inward which could result in buckling. On the other hand, a very rapid charge could buckle the inner shell.

## V. Specific Results

The results of the previous analysis were applied to the calculation of the applied loads, and the resultant deflections and stresses for a particular case of a dipole coil quenching inside a bucket type test dewar.

A schematic diagram of the test dewar is shown in Figure 6 which also shows the four shells of which the dewar is made. The inner and outer shells are made of stainless steel and the liquid nitrogen shield is made up of two aluminum shells. A summary of the shell parameters is shown in Table I.

The shells I - IV are the shells for the 24 inch diameter dewar. Shell number V is a fictitious shell of a size typical for a radiation shield in a warm hole dewar. Since the preliminary conclusions were that the inner shells were more critical, the calculations for shell V were carried out to determine whether a potential problem might exist.

A large dipole magnet was to be tested in the bucket dewar and the computations were carried out to determine the suitability of this particular dewar for the test.

Quench data for the dipole and for a model coil of the same construction are given in Figure 7, which shows the coil quench time as a function of the coil current.[3]

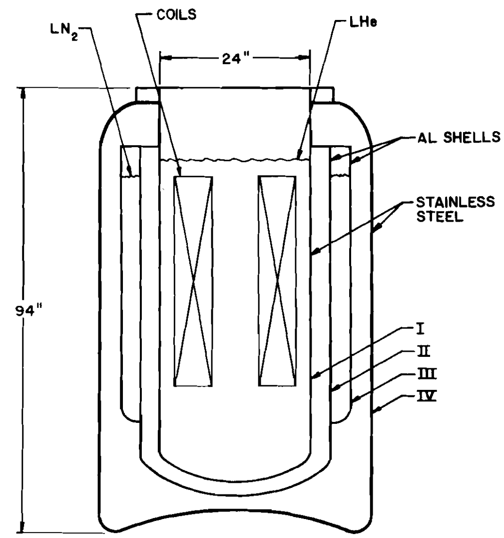


Fig. 6: Schematic diagram of bucket type test dewar.

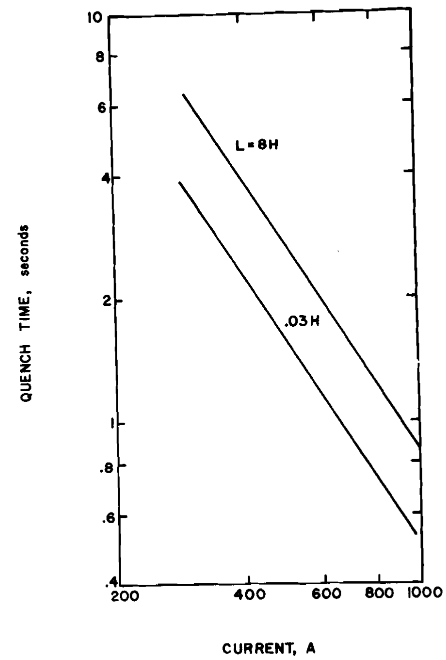


Fig. 7: Quench time as a function of coil current for the dipole magnet (8 H) as well as for a model solenoid of the same construction (.03 H).

The quench times shown were obtained from experimental data for the small coil and extrapolation of experimental data for the large coil. Although the actual quench is not a linearly decreasing magnetic field with time, this assumption does not lead to a large error. The data in Figure 7 are for the maximum rate of decrease of magnetic field divided by its initial value; as such, the data represent the quantity $\tau_q$ used in the analysis.

Using the quantities in Table I and Figure 7 as inputs, the loading amplitudes, deflections and stresses were computed for shells I-IV of the dewar and for shell V which would be typical of a warm bore construction.

The calculations were performed using formulas for loading, deflection and stresses derived in Appendix C, and are summarized in Figures 8, 9, and 10.

The basic loading of a shell is of the form:

$$p_n = \pm \sigma_o \cos^2\theta = \pm \frac{\sigma_o}{2} (1 + \cos 2\theta) \quad (18)$$

$$p_s = \pm \sigma_o \sin\theta \cos\theta = \pm \frac{\sigma_o}{2} \sin 2\theta \quad (19)$$

where $\sigma_o$ is related to the other quantities through Equations 10 to 13.

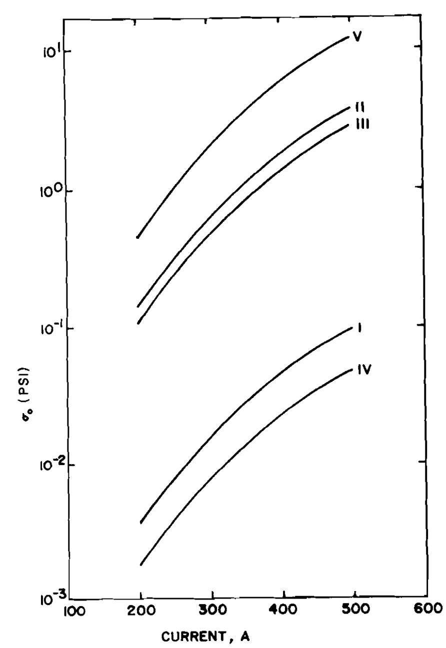


Fig. 8: Loading parameter $\sigma_o$ as a function of initial coil current.

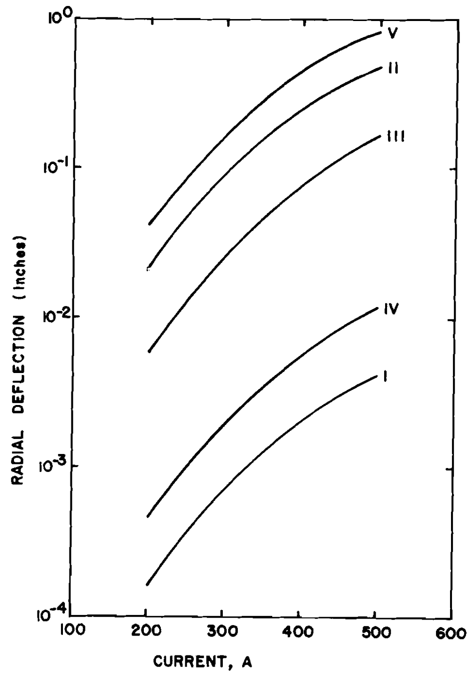


Fig. 9: Maximum radial deflection resulting in the dewar shells as a function of initial coil current.

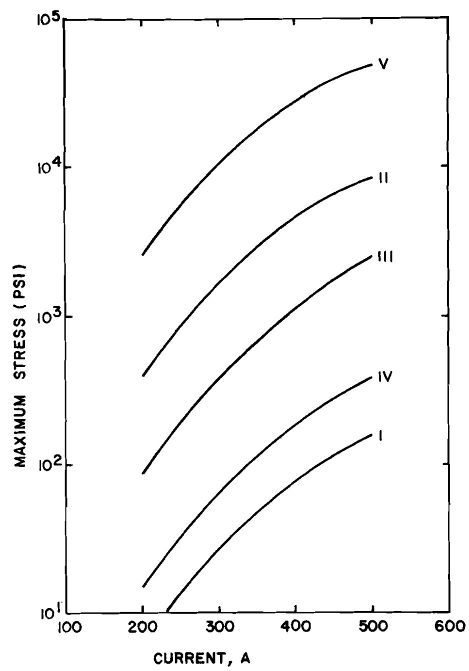


Fig. 10: Maximum stress in the shells during the quench as a function of initial coil current.

Figure 8 shows the absolute value of the quantity $\sigma_0$ in psi for various values of initial coil current prior to the quench. As expected, the stainless steel shells show relatively light loading, while the aluminum shells are more heavily loaded. Also, shell V, the aluminum inner shell at liquid nitrogen temperature, shows the highest loading of all. Near the 500A level in the coil, the aluminum shells II, III, and V are loaded with a $\sigma_0$ of a few psi (approximately 10 psi for shell V). The absolute magnitude of this load is small when compared with values of 60 psi to 100 psi internal pressure commonly used to design dewar vessels of this type. Nevertheless, the fact that the load varies azimuthally is an important difference from the uniform loading resulting from gas pressure.

Figure 9 shows the maximum radial deflection (see Figure 5 for location of maximum) as a function of the coil current. At the 500A level radial deflections of a fraction of an inch would result. These deflections are marginal, since radial clearances are generally of this order of magnitude. For the inner shell V, the predicted radial deflection would result in bumping against adjacent shells creating a temporary thermal short. Provided the design of the dewar is such that that bumping causes no permanent deformation locally, no serious problems exist with allowing the shells to bump.

Figure 10 shows the maximum stresses in the respective shells as a function of initial coil current - assuming elastic deformation of the shells. The stainless steel shells see very low stresses. The aluminum outer shells II and III see several thousand psi stress - very tolerable levels. The inner shell V, however, is exposed to several tens of thousands of psi stress level - and under normal circumstances would be overstressed. However, as has been mentioned above, the inner shell could not deflect in an unrestrained fashion without bottoming out on adjacent shells. This bottoming out would limit the deflection and reduce the stress level -it is important, however, to allow for this effect, since it would only take a slight permanent deformation to result in a permanent thermal short to the helium vessel which would render the dewar useless.

## VI. Conclusions

The main conclusions to be drawn from the results of this paper are that, although the magnitude of the electromagnetic loads in the quench of a dipole magnet are lower than the magnitude of the uniform gas pressure load for which most dewars are designed, the fact that this load is not uniform around the perimeter of the dewar shell can result in excessive deflections and stresses in low temperature copper or aluminum shells. Generally, the most critical shell is one within the coil bore.

Since the absolute magnitude of the load is small it is primarily the fact that a thin shell has little bending stiffness that may cause problems. Use of corrugations or other means of stiffening the shell can be used to limit the stresses and deflections. Unfortunately, this solution requires more room for the stiffening members. While space may be available in the region of the outer shells, it is generally at a premium for magnets with a warm bore dewar design which require inner shells.

## Appendix A - Long Solenoids

The behavior of long solenoids can be derived relatively simply, since both the solenoid and the shell produce magnetic field only inside their respective bores - they generate no field outside. In this appendix, we shall derive the magnetic fields and loading for conducting shells coaxial with a long solenoid which are larger in diameter (outside shell) and smaller in diameter (inside shell) than the solenoid.

### Outside Shell:

Consider a long thin solenoid of radius $r_o$ placed inside a thin conducting shell of radius $r_s > r_o$. Since the magnetic field due to either solenoid or shell currents is zero outside of its bore and is uniform inside of its bore respectively, the magnetic field in the region $r > r_o$ will be that due to the shell currents alone and that in the bore of the solenoid $r < r_o$ will be due to both the shell currents and solenoid currents.

If we denote by $B_o$ the field in the bore of the coil due to the coil current with no current in the shell and by $B_s$ the field generated in the bore of the shell due to its currents, the total flux linked by the shell is:

$$\phi = \pi( r_o^2 B_o + r_s^2 B_s ) \qquad \text{(A-1)}$$

Since the integral of the electric field around the circumference is equal to minus the rate of change of flux:

$$\int E \cdot d\ell = 2\pi r_s \rho j_s = - \frac{d\phi}{dt} \qquad \text{(A-2)}$$

where $\rho$ is the resistivity of the shell material and $j_s$ is the shell current density (assumed to be uniform).

The shell current density can be related to the magnetic field produced in its bore by:

$$B_s = \mu_o j_s w_s \qquad \text{(A-3)}$$

Making use of equations (A-1), (A-2) and (A-3), we can arrive at the following equation for $B_s$:

$$\frac{dB_s}{d\,t} + \frac{2\rho}{\mu_o r_s w_s} B_s = -\left(\frac{r_o}{r_s}\right)^2 \frac{dB_o}{d\,t} \qquad \text{(A-4)}$$

### Inner Shell:

The only difference in the derivation for an inner shell from that presented above is in the expression for the flux in the bore of the shell:

$$\phi = \pi r_s^2 (B_s + B_o) \qquad \text{(A-5)}$$

Making use of Equations (A-5) with (A-2) and (A-3) results in:

$$\frac{dB_s}{d\,t} + \frac{2\rho}{\mu_o r_s w_s} B_s = - \frac{dB_o}{d\,t} \qquad \text{(A-6)}$$

The only difference between Equations (A-4) and (A-6) is the factor $(r_o/r_s)^2$ by which the coil field is multiplied for the case of an outer shell ($r_o < r_s$).

### General Behavior

Examination of Equations (A-4) and (A-6) shows that if the coil field is held steady ($dB_o/dt = 0$) following a transient, then any existing magnetic fields due to shell currents will decay with a time constant:

$$\tau_s = \frac{\mu_o r_s w_s}{2\rho} \qquad \text{(A-7)}$$

It is evident from the equations that if a coil is charged or discharged in a time short compared with $\tau_s$ then the shell currents do not have time to decay and the shell magnetic fields are proportional to the change of the coil magnetic field. If on the other hand the shell time constant is short compared with the magnetic field change, then the shell magnetic field will be proportional to the rate of change of the coil magnetic field.

As is shown in the main body of this paper for quenches of superconducting coils, the shell time constants $\tau_s$ are generally much smaller than the coil quench times $\tau_q$ so that the induced currents will be proportional to the rate of change of the coil magnetic field:

$\tau_q > \tau_s$

$$B_s = \begin{cases} -B_o \dfrac{\tau_s}{\tau_q}\left(\dfrac{r_o}{r_s}\right)^2 & r_o < r_s \\ -B_o \dfrac{\tau_s}{\tau_q} & r_o > r_s \end{cases} \qquad \text{(A-8)}$$

Since the currents induced in any shells produce magnetic fields that are uniform inside the bore, the resultant loads on the shells are uniform and are directed either radially in or out. The radial loading on the shell is simply the magnetic pressure inside minus the magnetic pressure on the outside.

The loading on an outer shell is:

$$p = \frac{B_s^2}{2\mu_o} \qquad \text{(A-9)}$$

(For an outer shell, the magnetic field on the outside is zero.)

The loading on an inner shell is:

$$p = \frac{(B_s + B_o)^2}{2\mu_o} - \frac{B_o^2}{2\mu_o} = \frac{B_s B_o}{\mu_o} + \frac{B_s^2}{2\mu_o} \qquad \text{(A-10)}$$

## Appendix B - Long Dipoles

### (Thin cosθ Winding Distribution)

The magnetic field due to a single dipole of radius R with a current density that varies as cosθ is given in polar coordinates by:(1)

$r < R$

$$B_r = -B \sin\theta \qquad B_\theta = -B\cos\theta \qquad \text{(B-1)}$$

$r > R$

$$B_r = -B\left(\frac{R}{r}\right)^2 \sin\theta \qquad B_\theta = B\left(\frac{R}{r}\right)^2 \cos\theta \qquad \text{(B-2)}$$

Where B is the uniform field created inside the bore and is related to the maximum current density $j_o$ and thickness of the winding or shell w by:

$$B = \mu_o j_o w/2 \qquad \text{(B-3)}$$

If we have two dipoles, one at radius $r_1$ generating a magnetic field $B_1$ and another at radius $r_2 > r_1$ generating a magnetic field $B_2$, then the magnetic field from both dipoles is:

$r < r_1$

$$B_r = -(B_1 + B_2)\sin\theta \qquad B_\theta = -(B_1 + B_2)\cos\theta \qquad \text{(B-4)}$$

$r_1 < r < r_2$

$$B_r = -\left[B_1\left(\frac{r_1}{r}\right)^2 + B_2\right]\sin\theta \qquad B_\theta = \left[B_1\left(\frac{r_1}{r}\right)^2 - B_2\right]\cos\theta \qquad \text{(B-5)}$$

$r > r_2$

$$B_r = -\left[B_1\left(\frac{r_1}{r}\right)^2 + B_2\left(\frac{r_2}{r}\right)^2\right]\sin\theta \qquad B_\theta = \left[B_1\left(\frac{r_1}{r}\right)^2 + B_2\left(\frac{r_2}{r}\right)^2\right]\cos\theta \qquad \text{(B-6)}$$

We can arrive at expressions for the load per unit area in terms of the fields $B_1$ and $B_2$ generated by each dipole if we recall that the current density for each shell is related to the magnetic field as follows:

$$j_1 = \frac{2B_1}{\mu_o w_1}\cos\theta \qquad \text{(B-7)}$$

$$j_2 = \frac{2B_2}{\mu_o w_2}\cos\theta \qquad \text{(B-8)}$$

The radial loading per unit area on the outer shell is then:

$$p_2 = -j_2 w_2 B_\theta \Big|_{r=r_2} = -\left(\frac{2B_2}{\mu_o}\cos\theta\right)\left(B_1\left(\frac{r_1}{r_2}\right)^2\cos\theta\right) \qquad \text{(B-9)}$$

$$p_2 = -\frac{2B_2B_1}{\mu_o}\left(\frac{r_1}{r_2}\right)^2\cos^2\theta$$

The radial loading on the inner shell is:

$$p_1 = -j_1 w_1 B_\theta \Big|_{r=r_1} = -\left(\frac{2B_1}{\mu_o}\cos\theta\right)\left(-B_2\cos\theta\right) \qquad \text{(B-10)}$$

$$p_1 = \frac{2B_1B_2}{\mu_o}\cos^2\theta$$

In addition to the radial load, there is a load in the θ-direction (shear). This shear loading on the outer dipole is:

$$p_{s2} = j_2 w_2 B_r \Big|_{r=r_2} =$$

$$-\left(\frac{2B_2\cos\theta}{\mu_o}\right)\left[B_1\left(\frac{r_1}{r_2}\right)^2 + B_2\right]\sin\theta \qquad \text{(B-11)}$$

$$p_{s2} = -\left(\frac{2B_2B_1}{\mu_o}\left(\frac{r_1}{r_2}\right)^2 + \frac{2B_2^2}{\mu_o}\right)\cos\theta\,\sin\theta$$

The shear load per unit area on the inner dipole is:

$$p_{s1} = j_1 w_1 B_r \Big|_{r=r_1} =$$

$$-\left(\frac{2B_1}{\mu_o}\cos\theta\right)\left(B_1 + B_2\right)\sin\theta \qquad \text{(B-12)}$$

$$p_{s1} = -\left(\frac{2B_1^2}{\mu_o} + \frac{2B_1B_2}{\mu_o}\right)\cos\theta\,\sin\theta$$

Outer Shell:

For a shell at a radius greater than the winding, the subscript 2 in the above equations is replaced by s (shell) and subscript 1 is replaced by o (coil).

The axial electric field at $r = r_s$ is obtained from Equation (B-5) or (B-6) as follows:

$$\frac{dE_z}{d\theta} = \begin{cases} -r\dfrac{dB_r}{dt}\Big|_{r=r_s} & \text{(B-13)} \\[2ex] r_s\left[\left(\dfrac{r_o}{r_s}\right)^2\dfrac{dB_o}{dt} + \dfrac{dB_s}{dt}\right]\sin\theta & \text{(B-14)} \end{cases}$$

where $B_o$ and $B_s$ represent the uniform magnetic fields generated in the bores of the coil and the shell respectively.

Integrating with respect to θ yields:

$$E_z = -r_s\left[\left(\frac{r_o}{r_s}\right)^2\frac{dB_o}{dt} + \frac{dB_s}{dt}\right]\cos\theta \qquad \text{(B-15)}$$

The constant of integration is zero since the average value of $E_z$ over the whole circumference must be zero.

By Ohm's law we can find the current flowing in the shell:

$$j_s = -\frac{r_s}{\rho}\left[\left(\frac{r_o}{r_s}\right)^2\frac{dB_o}{dt} + \frac{dB_s}{dt}\right]\cos\theta \qquad \text{(B-16)}$$

The maximum current density (θ = 0, π) is related to the uniform magnetic field produced in the bore of the shell by the shell currents by Equation (B-3).

$$B_s = -\mu_o\frac{w_s}{2}\frac{r_s}{\rho}\left[\left(\frac{r_o}{r_s}\right)^2\frac{dB_o}{dt} + \frac{dB_s}{dt}\right] \qquad \text{(B-17)}$$

which can be rearranged to give:

$$\frac{dB_s}{dt} + \frac{2\rho}{\mu_o w_s r_s}B_s = -\left(\frac{r_o}{r_s}\right)^2\frac{dB_o}{dt} \qquad \text{(B-18)}$$

As far as magnetic fields are concerned, the dipole has a behavior which is similar to that of the solenoid (see Equation (A-4).)

Inner Shell:

For a shell placed inside the bore of a dipole, the derivation of the equations follows a parallel development. The subscripts 2 and 1 in Equations (B-4) through (B-8) are replaced by $r_o$ and $r_s$ respectively.

The axial electric field at $r = r_s$ is obtained from (B-4) or (B-5) as follows:

$$\frac{dE_z}{d\theta} = -r\frac{dB_r}{dt}\Big|_{r=r_s} = r_s\left[\frac{dB_o}{dt} + \frac{dB_s}{dt}\right]\sin\theta \qquad \text{(B-19)}$$

Integrating and using Ohm's law as with the outer shell:

$$j_s = -\frac{r_s}{\rho}\left[\frac{dB_o}{dt} + \frac{dB_s}{dt}\right]\cos\theta \qquad \text{(B-20)}$$

The magnetic field due to the shell is:

$$B_s = -\frac{\mu_o w_s r_s}{2\rho}\left[\frac{dB_o}{dt} + \frac{dB_s}{dt}\right] \qquad \text{(B-21)}$$

and:

$$\frac{dB_s}{dt} + \frac{2\rho}{\mu_o w_s r_s}B_s = -\frac{dB_o}{dt} \qquad \text{(B-22)}$$

Appendix C - Elastic Deflections and Stresses

The loading on a shell induced by a dipole are of the form:

$$p_n = \sigma_n \cos^2 \theta = \sigma_n \left( \frac{1}{2} + \frac{\cos 2\theta}{2} \right) \qquad \text{(C-1)}$$

$$p_s = \sigma_s \cos \theta \sin \theta = \frac{\sigma s}{2} \sin 2\theta \qquad \text{(C-2)}$$

where the appropriate values for $\sigma_n$ and $\sigma_s$ (together with appropriate signs) can be obtained from the expressions given in Appendix B or the main body of this report. The convention used in Equation (C-1) is that $p_n$ is positive radially outward and $p_s$ positive in the positive $\theta$direction.

This appendix will derive expressions for the radial deflection and the resultant stresses using methods given by den Hartog.[4]

The general form of the radial deflection u is:

$$u = - u_o \cos 2\theta \qquad \text{(C-3)}$$

For compatibility the deflection v in the $\theta$ direction is obtained from:

$$dv = -u d\theta \qquad \text{(C-4)}$$

$$v = \frac{u_o}{2} \sin 2\theta \qquad \text{(C-5)}$$

We shall proceed to find the deflections by means of the principle of virtual work: We shall find the work performed by the loads if the shell were deflected an incremental amount from its equilibrium position. This work will then be equated to the change in elastic energy of the shell for the same incremental deflection.

The work done by the radial load $p_n$:

$$\delta W_n = \begin{cases} \displaystyle\int_o^{2\pi} \sigma_n \left[\frac{1}{2} + \frac{\cos 2\theta}{2}\right] \left[-\delta u_o \cos 2\theta\right] r_s d\theta \\ - \frac{\pi}{2} \sigma_n r_s \delta u_o \end{cases} \qquad \text{(C-6)}$$

The work done by the shear load $p_s$:

$$\delta W_s = \begin{cases} \displaystyle\int_o^{2\pi} \left[\frac{\sigma_s}{2} \sin 2\theta\right] \left[\frac{\delta u_o}{2} \sin 2\theta\right] r_s d\theta \\ \frac{\pi}{4} \sigma_s r_s \delta u_o \end{cases} \qquad \text{(C-7)}$$

In addition to the two above contributions to the work done, there is additional work performed by the constant component of the normal force due to the fact that the cross sectional area of the shell changes.

In order to arrive at a change in area second order effects need to be taken into account. Specifically, instead of using Equation (C-3) for the radial deflection, an additional term needs to be added. This additional term results from the requirement that the perimeter of the shell remain constant:

$$u_a = - u_o \cos 2\theta - \frac{u_o^2}{r_s} \qquad \text{(C-8)}$$

Using this equation for the deflection the change in area is then:

$$\Delta \text{ Area} = - \frac{3\pi u_o^2}{2} \qquad \text{(C-9)}$$

So that the work done by a constant loading is simply the loading times the above change in area. If we include an external pressure p in addition to the constant outward directed radial load $\sigma_n/2$, the work done is:

$$W_c = \frac{3}{2} \pi u_o^2 \left[ p - \frac{\sigma_n}{2}\right] \qquad \text{(C-10a)}$$

which on an incremental basis becomes:

$$\delta W_c = 3\pi u_o \delta u_o \left[ p - \frac{\sigma_n}{2}\right] \qquad \text{(C-10b)}$$

The total incremental work done by all components of the loading is then:

$$\delta W = \pi r_s \delta u_o \left[ - \frac{\delta_n}{2} + \frac{\sigma_s}{4} + 3 \frac{u_o}{r_s} \left(p - \frac{\sigma_n}{2}\right)\right] \qquad \text{(C-11)}$$

The total elastic energy for the assumed deflection is:[4]

$$U = \frac{9\pi}{2} \frac{EIu_o^2}{r_s^2} \tag{C-12}$$

where E is Young's modulus of elasticity and I is the moment of inertia of the cross section. The change in energy due to a change in deflection $\delta u_o$ is:

$$\delta U = \frac{9\pi EIu_o \delta u_o}{r_s^2} \tag{C-13}$$

For equilibrium the change in energy must equal the work done:

$$\delta W = \delta U \tag{C-14}$$

making use of Equations (C-11) and (C-13) the following expression results for the deflection after some simplification:

$$\frac{u_o}{r_s} = \frac{1}{3} \frac{\frac{\sigma_s}{4} - \frac{\sigma_n}{2}}{\frac{3EI}{r_s^3} - p + \frac{\sigma_n}{2}} \tag{C-15}$$

The above expression can be used to compute the deflections under any set of normal and shear loading conditions. However, for the case of a quenching dipole coil with a quench time $\tau_q$ longer than the shell time constant $\tau_s$:

For an inner shell: $(r_o > r_s, \tau_q > \tau_s)$

$$\sigma_n = \frac{2B_o^2}{\mu_o}\left(\frac{\tau_s}{\tau_q}\right) = \sigma_o \tag{C-16}$$

$$\sigma_s = -\sigma_o \tag{C-17}$$

using these values results in:

$$\frac{u_o}{r_s} = -\frac{\frac{\sigma_o r_s^3}{12EI}}{1 - \left(p - \frac{\sigma_o}{2}\right)\frac{r_s^3}{3EI}} \tag{C-18}$$

The moment of inertia for a plain shell of thickness w is:

$$I = \frac{1}{12} w^3 \tag{C-19}$$

The stress is found by first computing the bending moment from Equation (C-8) and the expression for the bending moment:

$$M = \frac{EI}{r_s^2}\left[u_a + \frac{d^2 u_a}{d\theta^2}\right] \tag{C-20}$$

$$M' = \frac{EI}{r_s}\left(\frac{u_o}{r_s}\right)\left(3 \cos 2\theta - \frac{u_o}{r_s}\right) \tag{C-21}$$

The stress becomes:

$$\sigma = \frac{Mc}{I} = E\left(\frac{c}{r_s}\right)\left(\frac{u_o}{r_s}\right)\left(3 \cos 2\theta - \frac{u_o}{r_s}\right) \tag{C-22}$$

where c is the distance from the neutral axis to the highest stressed fiber. For a simple shell c is equal to one half the thickness and the stress becomes

$$\sigma = \frac{3}{2}\left(\frac{w}{r_s}\right)\left(\frac{u_o}{r_s}\right) E\left[\cos 2\theta - \frac{1}{3}\frac{u_o}{r_s}\right] \tag{C-23}$$

where $u_o/r_s$ is obtained from (C-18) with the use of (C-19).

For an outer shell:

$$\sigma_n = -\frac{2B_o^2}{\mu_o}\left(\frac{r_o}{r_s}\right)^4 \frac{\tau_s}{\tau_q} = -\sigma_o \tag{C-24}$$

$$\sigma_s = -\sigma_o \tag{C-25}$$

Making these substitutions into (C-15) gives:

$$\frac{u_o}{r_s} = \frac{\frac{\sigma_o r_s^3}{36EI}}{1 - \left(p + \frac{\sigma_o}{2}\right)\frac{r_s^3}{3EI}} \tag{C-26}$$

The expressions for bending moment and stress are obtained from (C-21), (C-22), or (C-23).

A comparison of Equations (C-18) for the deflection of the inner shell with (C26) for the deflection of the outer shell reveals several interesting facts:

1. The deflections are of opposite sign ($\sigma_o$ is positive in all cases) so that the major axis of deformation will be different for an inner shell than for an outer shell.

2. The deflections for an inner shell are larger (for the same quench time, shell radius and moment of inertia).

3. For purposes of determining buckling loads for an inner shell, an effective pressure $\sigma_o/2$ can be subtracted from the external pressure p, while for an outer shell an effective pressure $\sigma_o/2$ must be added.

4. Due to the fact that $\sigma_o$ contains the term $(r_o/r_s)^4$ for outer shells and does not contain it for the inner shells means that outer shell loading decreases very rapidly as the shell radius increases.

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