RESONANT COUPLING OF TRANSVERSAL OSCILLATIONS DUE TO POLARIZATION OF ELECTRON-ION RINGS

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Abstract

The resonant coupling between the oscillations of the odd moments of one beam and the even moments of another one has been considered which appear only due to accelerating field. The resonant condition is $nQ_e + (n \pm 1)Q_i = K$ where n and K are integer, Q_e and Q_i are betatron frequencies of electrons and ions. The analysis of the results obtained shows that the maximal polarization of the electron-ion rings is limited by these resonances.

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It has been noted in Reference (1) that, when the electron and ion rings are polarized, the resonant coupling between the oscillation of the odd moments of one beam and the even moments of another one becomes possible. The purpose of this paper is to investigate such resonances. The investigation is performed assuming that 1) stationary solution of the kinetic equation for polarized beams exists, 2). these stationary distribution functions coincide with those for nonpolarized beams.

The paper uses the same methods and the same notations as in [1].

1. Hydrodynamical Model

Let us consider the coupling resonance between the dipole oscillations of one beam and the quadrupole oscillations of another one. The coupling between these oscillations appears due to the nonlinearity of the electric field, i.e. in the presence of nonlinearity the shift of the centre of one beam changes the field gradient in the centre of another one.

On the other hand, the hydrodynamical model is not correct in the presence of nonlinearity because this model assumes the zero frequency shift. So the results using the hydrodynamical model are very approximate (except for the case when the resonance width exceeds the frequency spread).

In the presence of the accelerating field the linearized equations of the axial particle motion (in the noninertial system of rings' rest) are given by

$$\tilde{z}_e'' + \lambda^2 \tilde{z}_e = \frac{1}{m\gamma}\left(eE_{ac} - m\gamma a - eE_{iz} - e\frac{\partial E_{iz}}{\partial z}\tilde{z}_e\right), \tag{1}$$

$$\tilde{z}_i'' = \frac{1}{M}\left(-eE_{ac} - Ma + eE_{ez} + e\frac{\partial E_{ez}}{\partial z}\tilde{z}_i\right),$$

$$Z_e = Z_e - Z_e^0, \quad Z_i = Z_i - Z_i^0$$

where Z_e^0 and Z_i^0 are the coordinates of the equilibrium points of electrons and ions. E_{ac} is the strength of the accelerating field, $\frac{d}{dt} = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}$ (Ω is the frequency, θ is the azimuthal angle), E_{ez} and E_{iz} are the strengths of the electron and ion electric field, respectively, λ is the focusing frequency of the external electromagnetic field.

Supposing $Z_e = Z_i = 0$, the next equation for the electric fields in the equilibrium points E_{iz}^0 and E_{ez}^0 may be written from Eq.(1):

$$E_{iz}^0 = E_{ac}\frac{(M + m\gamma)N_i}{MN_i + m\gamma N_e} = E_{ez}^0\frac{N_i}{N_e} \tag{2}$$

where N_e and N_i are the numbers of electrons and ions divided on the length of the rings.

We suppose the electric field of each beam to be given by

$$E_z = (z - \bar{z})\left[1 - \frac{\alpha}{3}\frac{(z - \bar{z})^2}{g_z^2}\right]\frac{4Ne}{g_z(g_z + g_z)} \tag{3}$$

where $\overline{Z}$ is the coordinate of the centre of gravity, g_z and g_z are the effective beam dimensions on Z and z, respectively, α is the arbitrary coefficient of nonlinearity.

Then we may obtain from (1)-(3) the following system of linearized equations

$$
\begin{aligned}
&\overline{\Delta z_e}'' + \Omega_e^2 \overline{\Delta z_e} = \Omega_1^2 \overline{\Delta z_i} + 1{,}5\,\Omega_1^2 \Delta g_{iz}\, d,\\
&\overline{\Delta z_i}'' + \Omega_i^2 \overline{\Delta z_i} = \Omega_i^2 \overline{\Delta z_e} - 1{,}5\,\Omega_i^2 \Delta g_{ez}\, d,\\
&\Delta g_{ez}'' + 4\Omega_e^2 \Delta g_{ez} = 1{,}5 \Delta g_{iz} + \\
&+ 2\alpha d \varkappa^2 \Omega_1^2 (\overline{\Delta z_e} - \overline{\Delta z_i}), \qquad (4)\\
&\Delta g_{iz}'' + 4\Omega_i^2 \Delta g_{iz} = 1{,}5 \Delta g_{ez} + 2\alpha d \varkappa^2 \Omega_i^2 (\overline{\Delta z_e} - \overline{\Delta z_i})
\end{aligned}
$$

where $\overline{\Delta z_i} = \overline{\tilde{z}_i} = \overline{z_i} - z_{io},\ \Delta z_e = z_e - z_e^o,$

$$\Delta g_{zi} = g_{zi} - g_{zi}^o,\ \Delta g_{ez} = g_{ez} - g_{ez}^o,$$

$$d = \frac{z_i^o - z_e^o}{g_z^o},\ \Omega_e^2 = \lambda^2 + \frac{e}{m\gamma}\frac{\partial E_{iz}}{\partial z}\Big|_{z=z_e^o} =$$

$$\lambda^2 + \Omega_1^2,\ \Omega_i^2 = -\frac{e}{M}\frac{\partial E_{ez}}{\partial z}\Big|_{z=z_i^o},\ \varkappa^2 = (1-\alpha d)^{-1}$$

Let us look for solutions of the system (4) in the form $a\, e^{i(K\theta - \omega t)}$ where a is a constant, K is an integer number. After substituting the exponent system (4) becomes that of linear equations with the constant coefficients. And we obtain the dispersion equation

$$
\begin{aligned}
&\{[Q_e^2 - (x-K)^2](Q_i^2 - x^2) - Q_1^2 Q_i^2\}\times\\
&\times\{[4Q_e^2 - (x-K)^2](4Q_i^2 - x^2) - \tfrac{9}{4} Q_1^2 Q_i^2\} =\\
&= -3\alpha^2 d^2 \varkappa^2 Q_1^2 Q_i^2 \{x^2[1{,}5 Q_1^2 + \qquad (5)\\
&+ 4Q_e^2 - (x-K)^2] - [\lambda_1^2 - (x-K)^2][5{,}5 Q_i^2 - x^2]\},
\end{aligned}
$$

where

$$Q_e = \Omega_e/\Omega,\ x = \omega/\Omega,\ Q_i = \Omega_i/\Omega,$$

$$\lambda_1 = \lambda/\Omega,\ Q_1^2 = Q_e^2 - \lambda_1^2.$$

Analysis of Eq.(5) shows that the "even-odd" resonances appear when one of the dipole self-frequencies of some beam is equal to that of the quadrupole self-frequencies of another beam. When the coupling between the dipole and quadrupole oscillations of different beams is negligible the resonant conditions of the "even-odd" resonance are given by

$$Q_e + 2Q_i = K, \quad Q_i + 2Q_e = K \qquad (6)$$

Near the "even-odd" resonance the approximate solution of Eq.(5) is

$$x_{1,2} = \frac{V_1 + V_2}{2} \pm \sqrt{\left(\frac{V_1 - V_2}{2}\right)^2 - \frac{3}{8}\alpha^2 d^2 \varkappa^2 \frac{Q_1^2 Q_i}{Q_e}} \qquad (7)$$

where if $Q_e + 2Q_i = K$ $V_1 = K - Q_e$, $V_2 = 2Q_i$

if $Q_i + 2Q_e = K$ $V_1 = K - 2Q_e$, $V_2 = Q_i$

In the centre of the resonance region the increment is

$$Im(x) = 0{,}56\,\alpha d \varkappa \sqrt{\frac{Q_1^2 Q_i}{Q_e}} \qquad (8)$$

2. Solution of Kinetic Equation

The "even-odd" resonance of arbitrary order may be investigated using the system of Vlasov's equations. The solution of this system was obtained under the following assumptions: 1) the beam is infinite in the direction normal to z-axis; 2) only the resonance harmonics of the Fourier expansion on the betatron oscillations phase are taken into account; 3) in the $\frac{\partial f_o}{\partial I} = \delta(I - I_o)/I_o$ where $f_o(I)$ is the stationary distribution function, I is the energy of the betatron oscillations. So we get

$$x^2 - (u_1 + u_2)x + u_1 u_2 + \varepsilon_{n,e} = 0, \qquad (9)$$

where $U_1 = K - \ell Q_e,\ U_2 = n Q_i,$

$$\varepsilon_{n,e} = Q_1^2 Q_i Q_e^{-1} \beta_{n,e}^2 d^2, \quad (10)$$

$$\beta_{n,e}^2 = \frac{(8\pi-3)^2}{n \ell d^2}\left[\int_{\chi_0}^{\pi} \sin n\varphi \sin \ell\chi \sin\chi \, d\chi\right] \times \left[\int_{\chi_0}^{\pi} \sin \ell\chi \sin n\varphi \sin\chi \, d\chi\right], \quad (11)$$

$$\cos\varphi = \cos\chi + d, \quad \cos\chi_0 = 1 - d.$$

The coefficients $\beta_{n,n-1}$ and $\beta_{n,n-3}$ numerically calculated. The calculation results $\beta_{n,n-1}$ are given in the Table I. The coefficients $\beta_{n,n-3}$ are not tabulated as they are by an order of magnitude less than $\beta_{n,n-1}$.

Table I

d \ n	2	3	4	5
0,05	0,23	0,24	0,24	0,24
0,1	0,23	0,23	0,23	0,24
0,2	0,22	0,22	0,20	0,18
0,3	0,21	0,20	0,16	0,13

6	7	8	9	10
0,24	0,23	0,23	0,23	0,22
0,22	0,21	0,20	0,19	0,17
0,16	0,13	0,11	0,10	0,10
0,10	0,10	0,04	0,04	0,01

If $n < d^{-1}$ the coefficients $\beta_{n,n-1}$ may be calculated by approximate formula

$$\beta_{n,n-1} = \frac{16\sqrt{n(n-1)}}{\pi^3 (2n-1)}. \quad (12)$$

If $n > d^{-1}$ the coefficients $\beta_{n,n-1}$ rapidly decrease as compared to Eq.(12).

From (9)-(12) we obtain that the resonances appear if $U_1 = U_2$, and, consequently, $\ell Q_e + n Q_i = K$. The expression for the increment of the dipole-quadrupole resonance in the centre of the resonance region is

$$\operatorname{Im}(x) = 0{,}26\, d \sqrt{\frac{Q_1^2 Q_i}{Q_e}} \quad (13)$$

Practically this formula coincides with (8), so α is equal to 0,5 for the chosen stationary distribution function.

3. Conclusion

The analysis of the "even-odd" resonances shows that their increments (if $n < d^{-1}$) are constant and the widths in the betatron frequency diamond decrease with n proportionaly to $[n^2+(n-1)^2]^{-\frac{1}{2}}$. The distance between the resonances decreases as n^{-2}, thus, for some values of n the "even-odd" resonances begin to "overlapp". This effect is very complicated but an essential decrease of stability in such systems should be expected.

The "even-odd" resonances are stabilized by the Landau damping. The stability limit decreases as n^{-1} and therefore the dipole-quadrupole resonance is the most dangerous. At reasonable values of parameters α and d ($\alpha \sim 1$ and $d \sim 0{,}3$) we find that $\operatorname{Im} x = 0{,}15\sqrt{Q_1^2 Q_i / Q_e}$ and the stabilizing frequency spread is large and can hardly be realized. Thus, the "even-odd" resonances are one of important factors limiting tolerable polarization and the maximal strength of the accelerating field in the electron-ion ring accelerator.

References

1. P.R.Zenkevich, D.G.Koshkarev, Preprint ITEP No 841 (1970).