RESONANT INSTABILITY OF NONLINEAR OSCILLATIONS REPRESENTED BY FINITE DIFFERENCE EQUATIONS

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Abstract

The solution of nonlinear finite difference equations is found. The nonlinear frequency shift and the resonant instability produced by nonlinearity are calculated. The developed theory is applied to the calculations of the phase motion in microtron.

Transversal and longitudinal motion of particles in accelerators usually can be described by finite difference equations with constant or slowly changing coefficients. This method is very convenient if coefficients of the initial differential equations, representing the particle motion, are constant on every separate part of the trajectory. If the period of oscillations is much greater than the guide system period, the finite difference equations can be replaced by differential equations with constant or slowly changing coefficients. In this case weak constant nonlinearity of equations produces the frequency shift but no damping or antidamping of oscillations is caused. The typical example is the small phase oscillations in ring accelerators.

If the period of oscillations is of the same order as the guide system period it is impossible to replace finite difference equations by differential equations, and in this case the weak constant nonlinearity of equations can produce instability. This paper contains all necessary formulae describing the influence of nonlinearity and instability development for the case of the system of two first order equations with quadratic and cubical nonlinearity.

We are examining the system of equations

$$\begin{aligned} x_{n+1} &= \alpha_{11} x_n + \alpha_{12} y_n + \sum \beta_{jk} x_n^j y_n^k , \\ y_{n+1} &= \alpha_{21} x_n + \alpha_{22} y_n + \sum \gamma_{jk} x_n^j y_n^k , \\ & j,k = 0,1,2,\dots \quad j+k \geq 2 \end{aligned} \tag{1}$$

with constant coefficients satisfying the correlations

$$\begin{aligned} \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1 , \\ -2 < \alpha_{11} + \alpha_{22} < 2 . \end{aligned} \tag{2}$$

Let us put into equations the complex variable

$$\begin{aligned} W_n = K(x_n + \varkappa y_n) = \\ = U_n + i V_n = R_n e^{i\theta_n} . \end{aligned} \tag{3}$$

Here K is an arbitrary complex number, the value $\varkappa$ depends on the coefficients $\alpha_{11}, \alpha_{21}, \alpha_{22}$ in such a way that the system (1) in the variables W_n and $\overline{W}_n$ (we mark by line the complex conjugate values) has such a form

$$W_{n+1} = \lambda W_n + \sum_{\ell,m} F_{\ell m} W_n^{\ell-m} \cdot \overline{W}_n^{\,m} , \tag{4}$$

$$\ell = 2,3,\dots \quad m = 0,1,2,\dots \ell$$

where

$$\lambda = \frac{\alpha_{11}+\alpha_{22}}{2} + i\sqrt{1 - \frac{(\alpha_{11}+\alpha_{22})^2}{4}} = e^{i\nu} \tag{5}$$

– the eigenvalue of system (1), the coefficients $F_{\ell m}$ depend on the values of the coefficients β_{jk} and γ_{jk} .

The recurrent formula can be reduced to the equation

$$W_n = \lambda^n W_0 + \lambda^{n-1} \cdot \sum_k \lambda^{-k} \sum_{\ell,m} F_{\ell m} W_k^{\ell-m} \cdot \overline{W}_k^{\,m} . \tag{6}$$

As the nonlinear terms are small we have the solution in the form

$$W_n = W_n^{(1)} + W_n^{(2)} + W_n^{(3)} + \dots , \tag{7}$$

where $W_n^{(1)} = \lambda^n W_0$, the value $W_n^{(2)}$ is proportional to $|W_0|^2$ and so on. We can see that

every term of the formula (7) is the sum of geometric progressions and the denominator of every progression equals λ^q, where q - is an integer.
If

$$\nu q = 2\pi K \quad (K = 0, \pm 1, \pm 2, \ldots) \qquad (8)$$

the corresponding progression produces a resonant term, the module of the term is proportional to the number n. If $K = q = 0$ (for $W_n^{(3)}$) we have the resonant term

$$\Phi_1 = n\lambda^n W_0^2 \overline{W}_0 \cdot \Big[\frac{\bar{\lambda} F_{21} F_{20} (2-\bar{\lambda}) - |F_{21}|^2}{1-\lambda} - \frac{2|F_{22}|^2}{1-\lambda^3} + \bar{\lambda} F_{31}\Big], \qquad (9)$$

produced by quadratic and cubical nonlinearity and existing at an arbitrary value ν. If the value ν is small there are no other resonant members, but when $\nu = \frac{\pi}{2}$, the term $W_n^{(3)}$ has an additional resonant member (if $K = -1, q = -4$)

$$\Phi_2 = n i^n \cdot \overline{W}_0^3 \cdot \Big[\frac{1+i}{2} F_{21} F_{22} - (1-i)\overline{F}_{20} F_{22} - i F_{33}\Big], \qquad (10)$$

produced by quadratic and cubical nonlinearity. When $\nu = \frac{2\pi}{3}$, the resonant member, produced by quadratic nonlinearity, occurs in the quadratic term $W_n^{(2)}$ and it equals

$$\Phi_3 = n F_{22} \cdot \lambda^{n-1} \cdot \overline{W}_0^2, \qquad (11)$$

while the cubical nonlinearity does not produce a resonant member. If we examine only quadratic and cubical approximations there are no particular cases over $\nu = \frac{\pi}{2}$ and $\nu = \frac{2\pi}{3}$.

Let us examine the properties of the solutions obtained. We shall admit that the functional determinant of the system (1) equals unity when nonlinear terms are taken into account. It is possible to prove that the expression in formula (9) in square brackets is the imaginary value. Therefore the value Φ_1 is a usual frequency shift and it does not cause change of amplitude. As for the terms Φ_2 and Φ_3, the case is somewhat different. Let us admit that there is the quadratic nonlinearity only and $\nu = \frac{\pi}{2} + \delta$, where $|\delta| \ll 1$. Then we can always admit (by choosing the value K) that $F_{20} = F_{22} = -\frac{1}{4}$, $F_{21} = -\frac{1}{2}$ and the solution follows from the formulae (9) and (10)

$$(V_n^2 - 2\delta)\cdot(V_n^2 - 2\delta) \equiv const. \qquad (12)$$

We can see from this formula that oscillations are unstable if $\delta \geqslant 0$ and the initial amplitude is greater than the threshold value $\sqrt{2\delta}$.
If $\nu = \frac{2\pi}{3} + \delta$, we can admit (by the same method) that $F_{22} = \frac{3}{2}\lambda$. The solution has a form

$$R_n^2 \cdot (\delta - R_n \sin 3\Theta_n) \equiv const. \qquad (13)$$

In this case the threshold amplitude is about $\frac{1}{3}|\delta|$ and instability takes place both when $\delta \geqslant 0$ and when $\delta \leqslant 0$. The dotted lines are plotted in Fig.1 and 2 in accordance with formulae (12) and (13), (when $\delta = 0$) and the continuous lines correspond to calculations of initial system (1) by the electronic computer. The agreement of results is good enough. If $\nu = \frac{\pi}{2}$ (fig.1), the high order terms, neglected in our theory, limit the amplitude increase following from formula (12) and the intense pulsation takes place.

We applied the developed theory to investigate phase motion in the microtron and found out that the instability described above develops speedily, during 10÷20 particle revolutions. This conclusion was affirmed by direct numerical calculations and our experiments[1]. Fig.3 illustrates the resonant instability (numerical calculations) when equilibrium phase corresponds to $\nu = \frac{2\pi}{3}$. The instability leads to large particle losses, occurring only due to the nonlinear dependence of the accelerating field on time (the sinusoidal law of the field change), when there are no perturbation factors and the accelerated current can be negligible. These losses cause minima on the accelerated current dependence on the equilibrium phase value. As a result the instability of the beam-resonator system occurs[1]. The consequence is the decrease (by factor 2) of the stable equilibrium phase region which is of importance for the work of both usual microtrons and racetrack microtrons on hundreds MeV energy. We must note that the resonant instability of phase oscillations in the microtron was independently discovered in the work[2] by numerical calculations.

The developed theory may be applied not only to the microtron, but to some other cases. It is worth to emphasize that in strong -focusing systems the frequency value $\nu = \frac{\pi}{2}$ corresponds to the centre of stable zone (in linear approximation) and this point is the best in many respects. However, the nonlinear instability takes place in this point. Using the present theory we can calculate the initial frequency drift sufficient to eliminate the instability.

We described above the final results of calculations. The whole work will be published in "The Journal of Experimental and Theoretical Physics" (v. 61, N 10).

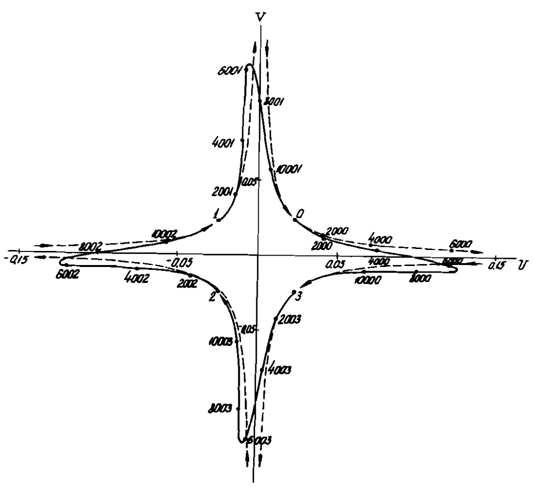

Fig.1 The pulsation of amplitude - $\nu = \frac{\pi}{2}$ $\alpha_{11} = \alpha_{12} = 1,\ \alpha_{21} = -2,\ \alpha_{22} = -1,\ K = 1$.

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<u>REFERENCE</u>

1. **V.N.Melekhin, L.B.Lugansky "Journal of Technical Physics", v.40,p.2465 (1970).**
2. **G.Bathow, E.Freytag "Diskontinuierliche Phasenschwingungen in Teilchenbeschleunigern", DESY 69/29.**

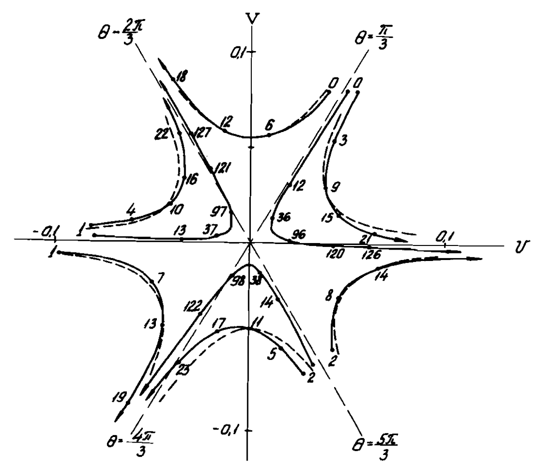

Fig.2 The instability of oscillations - $\nu = \frac{2\pi}{3}$, $\alpha_{11} = 0,\ \alpha_{21} = 1,\ \alpha_{12} = \alpha_{22} = -1,$ $F_{20} = F_{21} = 0,\ K = 1$.

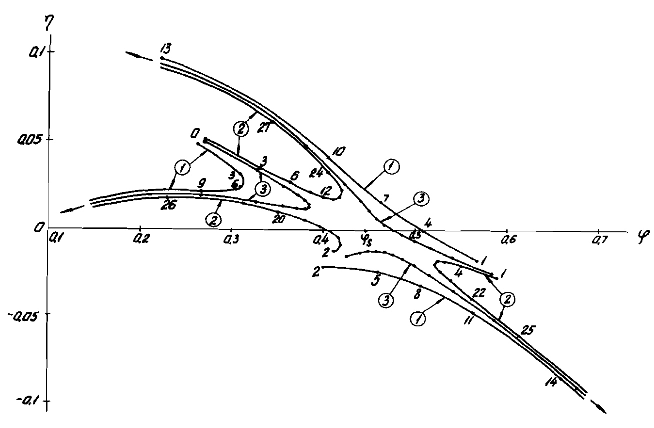

Fig.3 The instability of phase motion in a microtron-$\varphi_s = arctg\frac{3}{2\pi}$, ①,②,③-phase trajectories corresponding to different positions of initial point.