BOUNDARIES OF STOCHASTICITY DOMAINS IN HAMILTONIAN SYSTEMS

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Abstract

Intersecting invariant curves of an area-preserving mapping are computed using an improved algorithm. The extent of the stochasticity domains is estimated from the positions of the homoclinic points.

1. Introduction

Stochastic instability is now believed to be the most dangerous instability of non-linear oscillations[1]. This instability appears to increase with the amount of non-linearity. Since in an accelerator (and storage ring) the amount of non-linearity increases with the self-fields, and the latter increase with beam intensity, it is probable that some of the present accelerators (and storage rings) operate in conditions where stochastic instability plays a role. In order to facilitate eventual diagnostics, this paper intends to relate the notion of stochastic instability to the more usual notions of non-linear oscillations.

Stochastic instability is based on two phenomena in (mathematical) dynamic systems: i) the existence of solutions, involving in the mean an increase of energy, of:

$$\ddot{\phi}(t) + \omega^2 \sin \phi(t) = \varepsilon\, f(t) = \varepsilon \sum_n a_n \cos n\, \Omega t \ , \quad (1)$$

where ω, Ω, a_n are real constants, $0 < \varepsilon << 1$, and Ω is much smaller than the resonance width $\Delta\omega_n$ of (1) when only one $a_n \neq 0$; and ii) the existence of instability rings in area-preserving mappings. Both phenomena are completely deterministic. The adjective "stochastic" is used in this context to describe the qualitative structure of the "trajectories of motion". This structure is so complicated that from a casual point of view it appears to be random. On closer examination, however, several regular features can be discerned.

2. Some Properties of Dynamic Systems

Consider a real-valued continuous area-preserving point mapping $\bar{y} = f(x,y)$, $\bar{x} = g(x,y)$, $f(0,0) = g(0,0) = 0$, which can also be written in a discrete form (as a recurrence):

$$y_{n+1} = f(x_n, y_n) \ , \quad x_{n+1} = g(x_n, y_n) \ ,$$
$$n = 0, \pm 1, \pm 2, \ldots \ . \quad (2)$$

The invariant points of (2), i.e. the roots of the algebraic equations

$$y_{n+k} - y_n = 0 \ , \quad x_{n+k} - x_n = 0 \ ,$$
$$k = 1, 2, \ldots \ . \quad (3)$$

are called cycles (periodic points) of order k. If k = 1 they are called fixed points. Following the terminology of Poincaré*), the cycles were classified by Lattès[2] into saddles, centres, etc. The recurrence (2) admits also continuous invariant curves, possessing a certain number of continuous derivatives, which are also called trajectories. Let the point $x_n = y_n = 0$ be a centre C_0. It was already known to Poincaré and Birkhoff that in general the phase portrait of (2) around C_0 is as shown in Fig. 1. C_0 is first surrounded by a family of ordinary (free of cycles) invariant curves (elliptic zone); then by a family of invariant curves forming one or more "island structures", such as the one bounded by Γ_1 and Γ_2 which pass through the saddles S_1, S_2, S_3; and finally by one or more "rings of instability", such as the one characterized by the intersecting invariant curves Γ_4, Γ_5 which pass through the saddles S_4, S_5, S_6. According to Birkhoff, a ring of instability is always bounded by two closed invariant curves such as Γ_3 and Γ_6. The intersections of invariant curves at points which are not cycles (of a finite order), were called by Poincaré "homoclinic" points. The inner invariant curves of an instability ring may be closed, if they are sufficiently near to the inner centres, such as C_4, C_5, C_6.

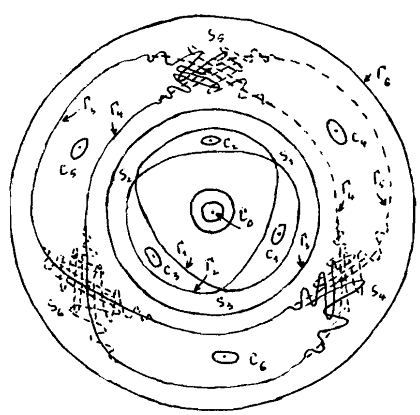

Fig. 1 Typical phase portrait of the recurrence (2) according to Birkhoff.

The relationship between an invariant curve and a "physical" trajectory is quite clear in the elliptic zone and in the zone of island structures, but it is not quite self-evident inside a ring of instability.

It was believed first on physical grounds, using a number-theoretical argument based on ratios of oscillation frequencies, that the instability rings were exceptional (negligibly few intersecting trajectories), i.e. that the homoclinic points were a rare mathematical curiosity. However, this line of argument turned out to be misleading, and it is now known that it is the absence of homoclinic points which is exceptional[3]), at least for motions having large amplitudes.

Birkhoff's (two-dimensional) instability rings cannot be the source of any "physical" instability, because these rings are mutually isolated and each is bounded by closed trajectories. It was shown by Arnold[4]) that higher-dimensional "rings" of instability may intersect, and that they contain trajectories which recede from C_0. The motion of particles on such wandering trajectories is now called Arnold diffusion. Other configurations of intersecting invariant curves appear also possible. Following present usage, regions in which such curves exist will be called stochastic. A sufficient criterion of stochasticity is thus the existence of homoclinic points.

Before attempting the determination of homoclinic points, it is good to recall that a dynamic system can be described by differential equations, recurrences, functional equations, etc. Which formulation is chosen depends on circumstances. It is known, however, since Poincaré that differential equations of the form

$$\dot{y}(t) = f_1(x,y,t) \ , \quad \dot{x}(t) = g_1(x,y,t) \ , \qquad (4)$$

with a periodic dependence on $t^{*)}$, determine a unique recurrence of form (2). The inverse is known not to be true in general. The practical determination of f,g from f_1,g_1 is extremely laborious, except when f,g depend on t by means of Dirac impulses (a not very realistic case, but used usually in illustrative examples). From the equivalence of (2) and (4) the phase portrait of Fig. 1 becomes intuitively evident. In fact, if the solutions of (4) are unique this uniqueness (non-intersecting trajectories) takes place in the space (x,y,t). The projection of curves in (x,y,t) on the phase plane (x,y) can be only exceptionally non-intersecting.

3. Estimation of Stochasticity Boundaries

In order to test for the existence of homoclinic points in a part G of the (x,y) plane, it is first necessary to find some cycles of (2) in G, i.e. to find some roots of (3). This essential first step does not constitute a trivial problem[5]), as it is commonly believed. If among the cycles one has succeeded in finding one saddle the next step becomes possible: the determination of invariant curves passing through this saddle*), Suppose that $y = y(x)$ is the equation of a continuous invariant curve possessing $n \geq 1$ continuous derivatives. It is known[2]) that $y(x)$ is a solution of the non-linear functional equation

$$f[x,y(x)] = y[g\{x,y(x)\}] \ , \quad y(x_0) = y_0 \ , \qquad (5)$$

where (x_0,y_0) is a given ordinary point. If (x_0,y_0) is on a cycle, or some more complex singularity, then it is also necessary to prescribe at x_0 the values of some derivatives of $y(x)$. For a saddle it is sufficient to prescribe $\dot{y}(x_0)$. The derivative $\dot{y}(x)$ satisfies the recurrence

$$\left[\frac{\partial g(x_n,y_n)}{\partial x_n} + \frac{\partial g(x_n,y_n)}{\partial y_n}\dot{y}(x_n)\right]\dot{y}(x_{n+1}) = \qquad (6)$$
$$= \frac{\partial f(x_n,y_n)}{\partial x_n} + \frac{\partial f(x_n,y_n)}{\partial y_n}\dot{y}(x_n) \ ,$$

which reduces to the eigenslope equation at a fixed point.

Knowing the solution of (5) it is possible to decide whether one has an island structure, an instability ring, or something more complicated. The decision is often difficult because both the location of the saddles and the shape of the invariant curves are known only with a finite precision. The numerical computations were carried out for the special but qualitatively rather general recurrence

$$x_{n+1} = y_n + F(x_n) \ , \quad y_{n+1} = - x_n + F(x_{n+1}) \ , \ (7)$$

where $F(x) = \mu x + (1 - \mu)x^2$ and $F(x) = \mu x + (1 - \mu)x^3$, $0 \leq \mu < +1$. This special form was chosen because in the case of the quadratic $F(x)$ some cycles were already known[6]), and because it has a saddle S at $x_n = 1$, $y_n = 0$ not depending on μ, $-1 < \mu < 1$.

The qualitative phase portrait of (7) is given in Fig. 2. Stochasticity develops first near the invariant curves passing through the saddle S, and then spreads inwards as $(1 - \mu)$ increases, by developing on cycles of various orders $k > 1$. The influence domains of the various cycles overlap, and there appears to be no evidence of any bounding curves like Γ_3 and Γ_6 in Fig. 1. There appears to exist only a lower limit for the complete stochastic region for a given value of μ (shown by the curve Γ in Fig. 2) [7]). It could not be ascertained yet

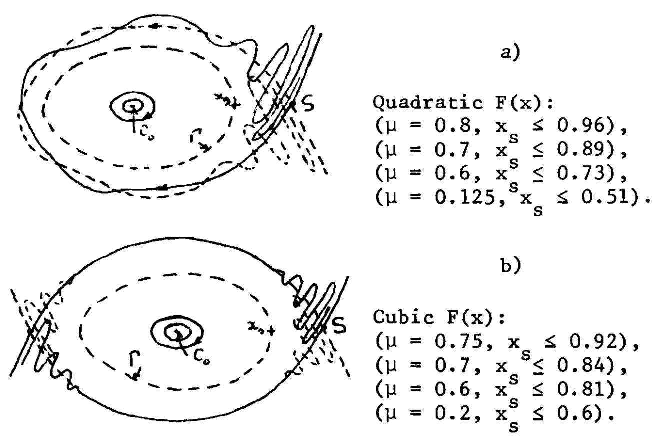

Fig. 2 Phase portrait of the recurrence (7).

whether Γ is defined by a unique closed invariant curve, or by the envelope of a family of intersecting invariant curves. The successive iterations x_n, y_n of (2), when started outside of Γ, eventually may become much larger than one (apparently $|x_n| \to \infty$). The region sufficiently outside of Γ appears to be the region of stochastic instability first noted in (1). This instability will be dangerous in accelerators (or storage rings), when the beam size becomes appreciably larger than the interior of Γ.

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References

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*) Subject to appropriate continuity conditions.