COHERENT RADIAL INSTABILITY DUE TO HORIZONTAL RESISTIVE WALLS. EVALUATION FOR SATURNE.

by R. A. Beck, G. Gendreau, J.-L. Laclare, G. Leleux,
Centre d'Etudes Nucléaires de Saclay, France.

Abstract.

This paper gives the general expression for U and V defined by LNS[1] for coherent radial instabilities in a uniform rectangular pipe with horizontal resistive walls. The stability threshold is evaluated for Saturne ($N_{th} \sim 5\ 10^{14}$ p).

1 - Introduction.

Coherent radial instabilities have been observed within the flat vacuum chambers of several synchrotrons. A tentative explanation for these phenomena consists in taking into account the motion of induced charges in horizontal resistive walls, i. e., calculating the electromagnetic fields acting on the particles so as to enhance their radial betatron amplitude. This mechanism was extensively used by LNS[1] for the case of coherent vertical instabilities.

The present analysis is an extension of their paper leading to expressions for the characteristic quantities U and V connected with the stability threshold and the e-folding time. Using the LNS criterion $\Delta S \gtrsim U$, this threshold for Saturne, is $\sim$300 times larger than for vertical instabilities and far above the present limitation on the machine.

2 - Electromagnetic fields.

The unperturbed beam, with dimensions (Δ, τ), is supposed to be centred in a rectangular vacuum chamber (fig. 1). The unperturbed charge density is uniform along the x and y directions. The small radial perturbation of the closed orbit has the form $\exp[i\,(ky - \omega t)]$. This gives rise to charge and current sources :

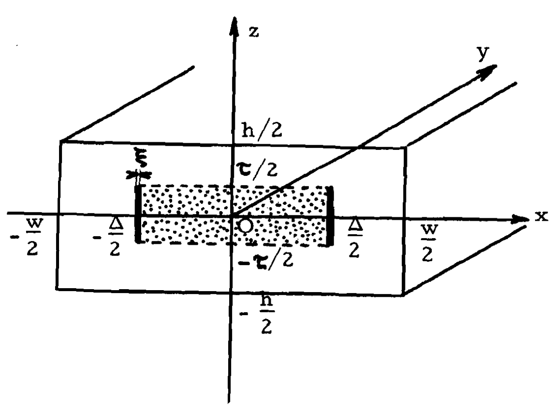

Fig 1

$$\rho_1 = \frac{\Lambda}{\Delta}\,\xi\;G(z)\left[\delta(x - \frac{\Delta}{2}) - \delta(x + \frac{\Delta}{2})\right] e^{i(ky-\omega t)}$$

$$j_{x1} = i(kv - \omega)\,\frac{\Lambda\,\xi}{\Delta}\;\varepsilon(x)\,G(z)\,e^{i(ky-\omega t)} \qquad (1)$$

$$j_{y1} = \rho_1 v$$

where :

Λ is the charge per unit length of the beam,
ξ is a constant ($\xi \ll \Delta$),
G(z) is the particle density distribution normalized so that $\int_{-h/2}^{+h/2} G(z)dz = 1$,

$$\varepsilon(x) = \begin{cases} 1 \text{ for } |x| < \frac{\Delta}{2} \\ 0 \text{ elsewhere.} \end{cases}$$

The resulting electromagnetic fields will be calculated[2] by superposition of :

- the fields $(\vec{E}_1, \vec{B}_1)$ generated by ρ_1 and $\vec{j}_1$ within perfectly conducting walls ($\sigma \to \infty$),
- the fields $(\vec{E}_2, \vec{B}_2)$ having no charge and current sources, within perfectly conducting walls, except for the wall at $z = -h/2$,
- the fields $(\vec{E}_3, \vec{B}_3)$ having no charge and current sources, within perfectly conducting walls, except for the wall at $z = +h/2$.

$(\vec{E}_1, \vec{B}_1)$ are solutions of Maxwell's equations :

$$\left\{\begin{array}{l} \mathrm{div}\,\vec{E}_1 = \dfrac{\rho_1}{\varepsilon_0} \\ \overrightarrow{\mathrm{rot}}\,\vec{E}_1 = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{E}_1 = -\overrightarrow{\mathrm{grad}}\,\phi_1 - \dfrac{\partial \vec{A}_1}{\partial t} \end{array}\right. \qquad \left\{\begin{array}{l} \mathrm{div}\,\vec{B}_1 = 0 \\ \overrightarrow{\mathrm{rot}}\,\vec{B}_1 = \mu_0 \vec{j}_1 + \dfrac{1}{c^2}\dfrac{\partial \vec{E}_1}{\partial t} \\ \vec{B}_1 = \overrightarrow{\mathrm{rot}}\,\vec{A}_1 \end{array}\right. \qquad (2)$$

$$\vec{E}_1 = -\frac{16\,\Lambda\,\xi\; e^{i(ky-\omega t)}}{\varepsilon_0\, w\, h\, \Delta}\sum_{p,q}\frac{g_c \sin\lambda\frac{\Delta}{2}}{\lambda^2+\eta^2-\frac{\omega^2}{c^2}}\left[\!\!\left[\left\{\lambda\left[1+\frac{\omega}{c^2}\,\frac{kv-\omega}{\lambda^2}\right]\cos\lambda x\,.\,\vec{x}\, + \right.\right.\right.$$

$$\left.\left.\left. + \, ik\left(1-\frac{\beta\omega}{kc}\right)\sin\lambda x\,.\,\vec{y}\right\}\cos\eta z - \eta\sin\lambda x\,.\,\sin\eta z\;.\;\vec{z}\,\right]\!\!\right]$$

$$\vec{B}_1 = +\frac{16\,\Lambda\,\xi\; e^{i(ky-\omega t)}}{\varepsilon_0\, w\, h\, \Delta\, c^2}\sum_{p,q}\frac{g_c \sin\lambda\frac{\Delta}{2}}{\lambda^2+\eta^2-\frac{\omega^2}{c^2}}\left\{v\eta\sin\lambda x\,.\,\sin\eta z\;.\;\vec{x}\right.$$

$$\left. -\frac{i\eta}{\lambda}(kv-\omega)\cos\lambda x\,.\,\sin\eta z\;.\;\vec{y} + \left[\frac{k}{\lambda}\,(kv-\omega)+v\,\lambda\right]\cos\lambda x\,.\,\cos\eta z\;.\;\vec{z}\right\} \qquad (3)$$

$$\text{with : }\left\{\begin{array}{l}\lambda = \dfrac{2\pi p}{w}\\[2mm] \eta = (2q+1)\,\dfrac{\pi}{h}\end{array}\right. \qquad \text{and} \qquad g_c = \int_0^{h/2} G(z)\cos\eta z\,dz \qquad (4)$$

while ($\vec{E}_{2,3}$, $\vec{B}_{2,3}$) depend only on $\vec{A}_{2,3}$ (i. e. $\Phi_{2,3} = 0$). Using the boundary conditions :

$$\left\{\begin{array}{l}E_{2,3\,x} = \pm\,(1-i)\,\mathcal{R}\,B_{1\,y}\\[1mm] E_{2,3\,y} = \pm\,(1-i)\,\mathcal{R}\,B_{1\,x}\end{array}\right. \qquad \mathcal{R} = \sqrt{\frac{\omega}{2\mu_0\sigma}} \qquad (5)$$

which are valid on the walls $z = \pm\, h/2$, one has :

$$\vec{E}_2+\vec{E}_3 = \frac{16\,\Lambda\,\xi\; e^{i(ky-\omega t)}}{\varepsilon_0\, w\, h\, \Delta\, c^2}\sum_{p,\,q}\frac{g_c \sin\lambda\frac{\Delta}{2}}{\lambda^2+\eta^2-\frac{\omega^2}{c^2}}\;\frac{\sin\eta\frac{h}{2}}{\cosh K\frac{h}{2}}\;Z_s\,\omega\left[\right.$$

$$\left.\left(-\frac{kv-\omega}{\lambda}\cos\lambda x\,.\,\vec{x} + i\,v\sin\lambda x\,.\,\vec{y}\right)\cosh Kz + \frac{\omega}{K}\sin\lambda x\,.\,\sinh Kz\,.\,\vec{z}\,\right]$$

$$\vec{B}_2+\vec{B}_3 = \frac{16\,\Lambda\,\xi\; e^{i(ky-\omega t)}}{\varepsilon_0\, w\, h\, \Delta\, c^2}\sum_{p,\,q}\frac{g_c \sin\lambda\frac{\Delta}{2}}{\lambda^2+\eta^2-\frac{\omega^2}{c^2}}\;\frac{\sin\eta\frac{h}{2}}{\cosh K\,\frac{h}{2}}\;Z_s\left[\!\!\left[\right.\right.$$

$$\left\{\left(\frac{k\omega}{K}-Kv\right)\sin\lambda x\,.\,\vec{x} + i\left[\frac{K(kv-\omega)}{\lambda}+\frac{\omega\lambda}{K}\right]\cos\lambda x\,.\,\vec{y}\right\}\sinh Kz\; +$$

$$\left.\left. + \,\lambda\left[v+\frac{k(kv-\omega)}{\lambda^2}\right]\cos\lambda x\,.\,\cosh Kz\,.\,\vec{z}\,\right]\!\!\right] \qquad (6)$$

$$\text{with } Z_s = (1+i)\,\frac{\mathcal{R}\eta}{\omega} \quad \text{and } K = \sqrt{\lambda^2+k^2-\frac{\omega^2}{c^2}} \qquad (7)$$

The averaged radial force per unit charge is then

$$\left\langle\frac{F_x}{e}\right\rangle = \int_{-h/2}^{+h/2}\sum_{i=1}^{3}\left(E_{ix}+vB_{iz}\right)G(z)\,dz$$

$$= \frac{16\,\Lambda\,\xi\; e^{i(ky-\omega t)}}{\varepsilon_0\, w\, h\, \Delta}\sum_{p,\,q}\frac{g_c\,\lambda\sin\lambda\Delta}{\lambda^2+\eta^2-\frac{\omega^2}{c^2}}\left\{\left[-\frac{1}{\gamma^2}+\frac{k^2}{\lambda^2}(\beta-\beta_w)^2\right]g_c\; +\right.$$

$$\left. + \, Z_s\left[\beta^2+\frac{k^2}{\lambda^2}(\beta-\beta_w)^2\right]g_{ch}\;\frac{\sin\eta\frac{h}{2}}{\cosh K\,\frac{h}{2}}\right\} \qquad (8)$$

$$\text{where : } \beta_w = \frac{\omega}{kc} \qquad \text{and } g_{ch} = \int_0^{h/2} G(z)\cosh Kz\,dz. \qquad (9)$$

3 - Coefficients of the dispersion relation.

The quantities U and V defined by LNS[1] in the dispersion relation

$$1 = \nu_x \omega_o \left[U + (1+i) V \right] \mathcal{J} \tag{10}$$

with :

$$\mathcal{J} = \int \frac{h'(a)\, f(W)\, a^2\, da\, dW}{(\omega - n\Omega)^2 - \nu_x^2 \Omega^2} \tag{11}$$

are :

$$U = \frac{16\, r_o\, c\, N}{\pi\, w\, h\, \Delta\, \nu_x \beta} \sum_{p,q} \frac{g_c^2\, \lambda \sin \lambda\Delta}{\lambda^2 + \eta^2 - \frac{\omega^2}{c^2}} \left[-\frac{1}{\gamma^2} + \frac{k^2}{\lambda^2} (\beta - \beta_w)^2 \right]$$

$$V = \frac{16\, r_o\, c\, N}{\pi\, w\, h\, \Delta\, \nu_x \beta} \frac{\mathcal{R}}{\omega} \sum_{p,q} \frac{g_c\, g_{ch}\, \lambda \eta\, \sin \lambda\Delta}{\lambda^2 + \eta^2 - \frac{\omega^2}{c^2}} \left[\beta^2 + \frac{k^2}{\lambda^2} (\beta - \beta_w)^2 \right] \frac{\sin \eta \frac{h}{2}}{\cosh K \frac{h}{2}} \tag{12}$$

where $r_o = \frac{1}{4\pi\varepsilon_o} \frac{e^2}{mc^2}$. (13)

4 - Numerical evaluation for Saturne. Conclusions.

Using the distribution :

$$G(z) = \begin{cases} \frac{\pi}{2\tau} \cos \pi \frac{z}{\tau} & \text{for } |z| < \frac{\tau}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{14}$$

and $\omega \sim (n - \nu_x)\, \omega_o$ (15)

it is easy to compute U and V.

The parameters of Saturne, corresponding to the energy at which radial coherent instabilities are observed,are the following :

$$\begin{array}{ll} \beta = 0.478 & (E - E_o \sim 130 \text{ MeV}) \\ \nu_x \sim 0.72 & (\text{field index } n_o \sim 0.6) \\ R/\rho \sim 1.3 & \\ n = 1 & \\ h \sim \Delta \sim 0.1 \text{ m} & \\ \tau/h \sim 1/2 & \\ W/R \sim 1/25 & \\ h/W \sim 1/5 & \end{array} \tag{16}$$

One obtains with these parameters :

$$\frac{|U|}{N} \sim 1.44\ 10^{-9} s^{-1} \quad *$$

$$\frac{V}{N} \sim 4.1\ 10^{-12} s^{-1} \tag{17}$$

The LNS criterion of stability being :

$$\Delta S \gtrsim U \tag{18}$$

$$\text{with} \begin{cases} S = (n - \nu_x)\, \Omega \\ \Delta S \sim \omega_o \frac{\partial \nu_x}{\partial x^2} \left(\frac{\Delta}{2}\right)^2 = \frac{1}{64} \frac{\beta c \Delta^2}{\rho \nu_x} \frac{d^2 n_o}{dx^2} \end{cases} \tag{19}$$

the stability threshold is :

$$N_{th} \sim 2.5\ 10^{13}\ n''_o \tag{20}$$

The octupolar component of the field n''_o is deduced [3]) from $n_o(x,t)$ beam measurements which gave :

$$n''_o \sim 20\ m^{-2} \text{(without extra octupoles)} \tag{21}$$

$$\text{so : } N_{th} \sim 5\ 10^{14} \text{protons.} \tag{22}$$

This limit is far larger than the threshold for coherent vertical instabilities [(3-26a) from LNS gives $N_{th} \sim 2 \times 10^{12}$ protons which is about 5-6 times the observed threshold]. Thus, with approximations used in this linear theory (no coupling, azimuthally uniform pipe with no electrodes, uniform density in the x-y direction, etc) the coherent radial instabilities observed at $\sim 10^{12}$ particles seem unrelated to the finite resistivity of the horizontal walls. The influence of the vertical walls, though quite remote from the beam, remains to be checked.

* This provides $\frac{|U_r|/N_r}{|U_v|/N_v} \sim \frac{1}{60}$. H.G. Hereward pointed out in a private discussion during the Conference that this ratio should be $\sim \frac{1}{2}$. So this figure has to be checked again.

References

1. L.J. Laslett, V.K. Neil, A.M. Sessler, "Transverse resistive instabilities of intense coasting beams in particle accelerators" R. S. I., Vol 36,4, 436-448, April 1965.
2. R.A. Beck, G. Gendreau, J. -L. Laclare, G. Leleux, "Instabilité radiale due à la résistivité des parois horizontales. Application à Saturne". Internal report, DSS. SOC 71-03, September 1971.
3. D. Boireau, J. Faure, G. Gendreau, G. Leleux, "Expérimentation du 10 décembre 1970. Mesure de $\nu_x(x,t)$" - Internal report SEFS/ TD 80, SOC/70-07.