THE PHASE MOTION STABILITY OF THE BUNCH IN A STORAGE RING DEPENDING ON THE FORM OF SYNCHROTRON FREQUENCY DISTRIBUTION FUNCTION

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Abstract

The conditions of the phase stability, taking into account the non-linearity of synchrotron oscillations in quasi-linear approximation, is considered. In this approximation for nonmonotonous distribution function of synchrotron frequencies, the appearance of phase instability at "good" (according to the known criteria) tuning of resonator is possible.

Up to now it is considered that the spread of synchrotron frequencies in a bunch leads only to additional damping (Landau damping) and thus improves the phase stability[1 - 4]. More careful investigation shows that under certain conditions the spread of synchrotron frequencies may cause the phase instability in the cases when it must not occur according to the known criteria[1].

The equations of the phase oscillations at the accelerating voltage $V\sin q\omega_0 t$ and perturbation in the form of longitudinal electric field $\mathcal{E}(x,t)$ may be written in a canonical form with Hamiltonian

$$H(x,p,t) = \frac{p^2}{2M} - \frac{eV}{2\pi q}\cos\frac{qx}{R} - e\int \mathcal{E}(x,t)\,dx \qquad (1)$$

where x is linear longitudinal coordinate in the rest system of an equilibrium particle, p is momentum canonically conjugated with x, $M=\left(R^2\omega_0\frac{d\omega_0}{dE_s}\right)^{-1}$ is "mass" of synchrotron motion, ω_0 is angular revolution frequency of an equilibrium particle.

In variables "action-angle" Hamiltonian of perturbed motion is

$$H = H_0(\mathcal{J}) + \tilde{H}(x,t) = H_0(\mathcal{J}) - e\int \mathcal{E}(x,t)\,dx\,; \qquad (2)$$

here $x=x(\psi,\mathcal{J})$, and the equations of motion are

$$\dot{\psi} = \frac{\partial H}{\partial \mathcal{J}} = \Omega_e(\mathcal{J}) + \frac{\partial \tilde{H}}{\partial \mathcal{J}}\,; \qquad \dot{\mathcal{J}} = -\frac{\partial \tilde{H}}{\partial \psi}\,. \qquad (3)$$

Here $\Omega_e(\mathcal{J})$ is angular frequency of unperturbed oscillations. Note, that for the linear system x and p are connected with ψ and $\mathcal{J}$ by relations

$$x = \sqrt{\frac{2\mathcal{J}}{M\Omega_e}}\sin\psi\,, \qquad p = \sqrt{2M\Omega_e\mathcal{J}}\cos\psi \qquad (4)$$

In our case perturbing field is formed by a beam itself; the latter may be considered as a linear current

$$i(\ell,t) = \sum_{m=-\infty}^{\infty} I_m(t)\, e^{jm(\omega_0 t - \frac{\ell}{R})} \qquad (5)$$

where ℓ is a longitudinal coordinate in laboratory system, $I_m(t)$ are complex amplitudes of harmonics. Performing Laplace transformation upon (5) we obtain

$$i(\ell,s) = \sum_{m=-\infty}^{\infty} I_m(s-jm\omega_0)\cdot e^{-j\frac{m\ell}{R}}. \qquad (6)$$

The electric field acting on the particles may be expressed through the impedance of external system

$$\mathcal{E}(\ell,s) = -\frac{1}{2\pi R}\sum_{m=-\infty}^{\infty} Z(s)\cdot I_m(s-jm\omega_0)\, e^{-j\frac{m\ell}{R}}. \qquad (7)$$

Performing the inverse Laplace transformation and going over to the rest system of an equilibrium particle by means of transformation $\ell = x + \omega_0 R t$ we obtain

$$\mathcal{E}(x,t) = -\frac{1}{2\pi R}\sum_{m=-\infty}^{\infty} e^{-j\frac{mx}{R}}\cdot L^{-1}\left\{Z(s+jm\omega_0)\cdot I_m(s)\right\}. \qquad (8)$$

For the evaluation of the complex amplitudes $I_m(t)$ it is necessary to use the distribution function $f(x,p,t) = f_0(x,p) + \tilde{f}(x,p,t)$ where $f_0(x,p)$ is equilibrium distribution, $\tilde{f}(x,p,t)$ is the perturbation. The perturbation of the linear charge density may be expressed through $\tilde{f}(x,p,t)$:

$$\rho(x,t) = eN\int \tilde{f}(x,p,t)\,dp$$

where N is the number of particles in the bunch. As ρ is the function periodic by x then

$$\rho(x,t) = \sum_{m=-\infty}^{\infty} \rho_m(t)\cdot e^{j\frac{mx}{R}} \qquad (9)$$

and

$$\rho_m(t) = \frac{1}{2\pi R}\int_{-\pi R}^{\pi R} \rho(x,t)\, e^{-j\frac{mx}{R}}dx = \frac{eN}{2\pi R}\iint \tilde{f}(x,p,t)\, e^{-j\frac{mx}{R}}dx\,dp. \qquad (10)$$

It may be shown that $I_m(t) = \beta c\,\rho_m(t)$; further we assume $\beta \cong 1$. Thus

$$I_m(t) = \frac{ecN}{2\pi R}\iint \tilde{f}(x,p,t)\, e^{j\frac{mx}{R}}dx\,dp. \qquad (11)$$

Going over to variables $\psi,\mathcal{J}$ and taking into account that $dx\,dp = d\psi\,d\mathcal{J}$ we obtain

$$I_m(t) = \frac{ecN}{2\pi R}\iint \tilde{f}(\psi,\mathcal{J},t)\, e^{j\frac{mx}{R}}d\psi\,d\mathcal{J}. \qquad (12)$$

The kinetic equation for $\tilde{f}(\psi,\mathcal{J},t)$ in the linear approximation may be written as

$$\frac{\partial \tilde{f}}{\partial t} + \Omega_e\frac{\partial \tilde{f}}{\partial \mathcal{J}} = \frac{\partial \tilde{H}}{\partial \psi}\cdot\frac{df_0}{d\mathcal{J}}\,. \qquad (13)$$

By Laplace transformation upon (13) we have

$$s\cdot F(\psi,\mathcal{J},s) - \tilde{f}_0(\psi,\mathcal{J}) + \Omega_e\frac{\partial F}{\partial \psi} = \frac{\partial \tilde{H}}{\partial \psi}\cdot\frac{df_0}{d\mathcal{J}} \qquad (14)$$

where

$$F(\psi,\mathcal{J},s) = \int_0^{\infty}\tilde{f}(\psi,\mathcal{J},t)\, e^{-st}dt\,; \quad \tilde{f}_0(\psi,\mathcal{J}) = \tilde{f}(\psi,\mathcal{J},0). \qquad (15)$$

The periodic by ψ solution of eq.(14) with the account of (8) may be written as

$$F(\psi,\mathcal{J},s) = \sum_{n=-\infty}^{\infty} F_n(\mathcal{J},s)\, e^{jn\psi} \qquad (16)$$

here

$$F_n(\mathcal{J},s) = \frac{\tilde{f}_{0n}}{s+jn\Omega_e} - \frac{\frac{en}{2\pi}\cdot\frac{df_0}{d\mathcal{J}}}{s+jn\Omega_e}\sum_{m=-\infty}^{\infty} A_{mn}\frac{Z(s+jm\omega_0)}{m} I_m(s)\,, \qquad (17)$$

$$\tilde{f}_{0n} = \frac{1}{2\pi}\int_0^{2\pi}\tilde{f}_0(\mathcal{J},\psi)\, e^{-jn\psi}d\psi\,, \quad A_{mn} = \frac{1}{2\pi}\int_0^{2\pi} e^{-j\left(\frac{mx}{R}+n\psi\right)}d\psi. \qquad (18)$$

The symbol $\sum'$ means the summation by m except $m = 0$.

Transforming (11) by Laplace and substituting k for m we have

$$I_k(s) = \frac{ecN}{2\pi R}\iint F(\psi,\mathcal{J},s)\, e^{j\frac{kx}{R}}\, d\psi\, d\mathcal{J} \qquad (19)$$

Putting $F(\psi,\mathcal{J},s)$ from (16) and (17) the system of linear equations for complex amplitudes of harmonics of a beam current may be obtained

$$I_k(s) + eI_{av}\sum_{m=-\infty}^{\infty}{}' \frac{Z(s+jm\omega_0)}{m} I_m(s) \sum_{n=-\infty}^{\infty}\int \frac{n A_{mn}A^*_{kn}\frac{df_0}{d\mathcal{J}}}{s+jn\Omega_e}\, d\mathcal{J} = 2\pi I_{av}\sum_{n=-\infty}^{\infty}\int \frac{\tilde{f}_{on}\, A^*_{kn}}{s+jn\Omega_e}\, d\mathcal{J} \qquad (20)$$

where $I_{av} = \frac{ecN}{2\pi R}$ is the average beam current, $k = \pm 1,2,3,\ldots$

The stability of solutions of the obtained system is defined by the location of the system determinant zeros in a plane of complex variables s. Now we introduce the following simplifying assumptions:

1. It is clear that oscillations may occur only with frequencies near to $n\Omega_e$; since denominators in (20) are of resonant nature in the sums by n only two terms($\pm n$) may be retained.

2. The amplitudes of the particles oscillations in a bunch are small so that the latter may be considered sinusoidal according to (4).

3. Let us limit the sum by m by the terms with $m = \pm m_0$ so that $\frac{m_0}{R}\sqrt{\frac{2\mathcal{J}}{M\Omega_e}} \ll 1$.

Then,

$$A_{mn}\cdot A^*_{kn} \cong \frac{(mk)^n}{(n!)^2}\cdot\left(\frac{\mathcal{J}}{2R^2M\Omega_e}\right)^n \qquad (21)$$

Putting in (20) and remaining in the sum by n only two terms ($\pm n$) we write the system as follows

$$I_k(s) - \frac{2j\cdot eI_{av}\cdot k^n}{[(n-1)!]^2}\sum_{m=-m_0}^{m_0}{}' m^{n-1} I_m(s)\cdot Z(s+jm\omega_0)\int\left(\frac{\mathcal{J}}{2R^2M\Omega_e}\right)^n \frac{\Omega_e\frac{df_0}{d\mathcal{J}}}{s^2+n^2\Omega_e^2}\, d\mathcal{J} = 2\pi I_{av}\int A^*_{kn}\left(\frac{\tilde{f}_{on}}{s+jn\Omega_e}+\frac{\tilde{f}_{o,-n}}{s-jn\Omega_e}\right) d\mathcal{J}. \qquad (22)$$

Let us multiply the left- and right-hand side of (22) by $k^{n-1}\cdot Z(s+jk\omega_0)$ and summarize all the equations by k from $-m_0$ to $+m_0$ denoting

$$V_n(s) = \sum_{m=-m_0}^{m_0}{}' m^{n-1}\cdot I_m(s)\cdot Z(s+jm\omega_0). \qquad (23)$$

Then the system of equtions is reduced to one equation from which

$$V_n(s) = \frac{2\pi I_{av}\sum\limits_{k=-m_0}^{m_0}{}' k^{n-1} Z(s+jk\omega_0)\int A^*_{kn}\left(\frac{\tilde{f}_{o,n}}{s+jn\Omega_e}+\frac{\tilde{f}_{o,-n}}{s-jn\Omega_e}\right)d\mathcal{J}}{1-\frac{2j\,eI_{av}}{[(n-1)!]^2}\sum\limits_{k=-m_0}^{m_0}{}' k^{2n-1} Z(s+jk\omega_0)\int\left(\frac{\mathcal{J}}{2R^2M\Omega_e}\right)^n\frac{\Omega_e\frac{df_0}{d\mathcal{J}}}{s^2+n^2\Omega_e^2}\, d\mathcal{J}} \qquad (24)$$

The stability of this solution is defined by the location of denominator zeros [5]. The stability condition may be obtained with the help of Nyquist criterion which does not require the solving dispersion equation. The proof of Nyquist criterion applying to similar equations was given in [6,7].

Let us introduce the notations

$$K_n(s) = eI_{av}\cdot G_n(s)\sum_{k=-m_0}^{m_0}{}' k^{2n-1} Z(s+jk\omega_0), \qquad (25)$$

$$G_n(s) = \frac{2j}{[(n-1)!]^2}\int\left(\frac{\mathcal{J}}{2R^2M\Omega_e}\right)^n\frac{\Omega_e\frac{df_0}{d\mathcal{J}}}{s^2+n^2\Omega_e^2}\, d\mathcal{J}. \qquad (26)$$

The function $K_n(s)$ is defined for values s with the positive real part and at some limitations imposed on $\frac{df_0}{d\mathcal{J}}$ [7] is analitic in the right half-plane. The straight line $s = j\Omega+\sigma\ (\sigma>0)$ at σ = const is transformed in the plane of $K_n(s)$ in some contour which is called Nyquist diagram at $\sigma \to +0$. According to Nyquist criterion the denominator of (24) has no zeros in the right half-plane if Nyquist diagram does not encircle the point $K_n = 1$. Nyquist diagram is defined by equation

$$K_n(j\Omega) = eI_{av}\cdot\lim_{\sigma\to+0} G_n(j\Omega+\sigma)\sum_{k=-m_0}^{m_0}{}' k^{2n-1} Z(s+jk\omega_0). \qquad (27)$$

For $G_n(j\Omega+\sigma)$ we may obtain with the accuracy of high order

$$G_n(j\Omega+\sigma) = \frac{2Q_c}{[(n-1)!]^2\Omega_0}\int\left(\frac{\mathcal{J}}{2R^2M\Omega_e}\right)^n\frac{\frac{df_0}{d\mathcal{J}}}{1+jnQ_c(x_{on}-x_e)}\, d\mathcal{J} \qquad (28)$$

where

$$x_{on} = \frac{2\left(\frac{\Omega}{n}-\Omega_0\right)}{\Omega_0},\quad x_e = \frac{2(\Omega_e-\Omega_0)}{\Omega_0},\quad Q_c = \frac{\Omega_0}{2\sigma}, \qquad (29)$$

Ω_0 is synchrotron frequency of small oscillations. The dependence of the synchrotron frequency on amplitude when the latter is not so large may be written approximately as

$$\Omega_e^2 \cong \Omega_0^2\left(1-\varkappa\frac{a^2}{4R^2}\right) \qquad (30)$$

where a^2 is the square of the oscillations amplitude $a^2 = \frac{2\mathcal{J}}{M\Omega_e}$. Hence, we obtain

$$x_e \cong -\varkappa\frac{\mathcal{J}}{2R^2M\Omega_e}.$$

Note, that the distribution function $f_0(\mathcal{J})$ may be changed to variable x_e.

Then the frequency spread may be described by the relative width of distribution function $\delta \cong \frac{2\Delta\Omega_e}{\Omega_0}$.

Let us change the variable in (28)

$$u = -\frac{\varkappa}{\delta}\cdot\frac{\mathcal{J}}{2R^2M\Omega_e} \qquad (31)$$

and go to the new distribution function $W(u)$ so that

$$2\pi f_0(\mathcal{J})\, d\mathcal{J} = W(u)\, du. \qquad (32)$$

If impose the requirement that $W(u)>0$ then

$$f_0(\gamma)=\frac{1}{2\pi}W(u)\left|\frac{du}{d\gamma}\right|=\frac{\varkappa}{2\pi\delta}\cdot\frac{W(u)}{2R^2|M|\Omega_0}\ . \qquad (33)$$

For definitness hereafter we suppose the particle energy to be above the transition, i.e. $M<0$, $\gamma<0$. Then

$$G_n(j\Omega+\sigma)=\frac{\left(\frac{\delta}{\varkappa}\right)^{n-1}Q_c}{2\pi\Omega_0^2R^2|M|[(n-1)!]^2}\int\frac{(-u)^n\frac{dW}{du}du}{1+jn\delta Q_c(u_{on}-u)}\ . \qquad (34)$$

When $M>0, \gamma>0$ (34) will be of the opposite sign. Going over to the limit by $\sigma\to+0$ and denoting

$$A_n=\pi\cdot(-u_{on})^n\left.\frac{dW}{du}\right|_{u=u_{on}},\quad B_n=\int_{-\infty}^{\infty}\frac{(-u)^n\frac{dW}{du}}{u-u_{on}}du \qquad (35)$$

we obtain

$$G_n(j\Omega)=\frac{\left(\frac{\delta}{\varkappa}\right)^{n-1}\cdot\frac{1}{\delta}}{2\pi R^2\Omega_0^2|M|\cdot n\cdot[(n-1)!]^2}\cdot(A_n+jB_n)\ . \qquad (36)$$

Note that in (35) one means the principal value of the integral.

Now

$$K_n(j\Omega)=\frac{eI_{av}\left(\frac{\delta}{\varkappa}\right)^{n-1}\frac{1}{\delta}}{2\pi R^2\Omega_0^2|M|n[(n-1)!]^2}(A_n+jB_n)\sum_{k=-m_0}^{m_0}k^{2n-1}Z(j\Omega+jk\omega_0) \qquad (37)$$

The system impedance Z as it is seen from (37) depends on the frequency Ω. But as $\Omega\cong n\Omega_0$ and $\Omega_0\ll\omega_0$ in the expression for Ω we may suppose with a good accuracy $\Omega\simeq\Omega_0$.

Let us designate

$$Z(jk\omega_0+jn\Omega_0)=R^+_{kn}+jX^+_{kn}\,;\; Z(-jk\omega_0+jn\Omega_0)=Z^*(jk\omega_0-jn\Omega_0)=R^-_{kn}-jX^-_{kn} \qquad (38)$$

where $R^+_{kn}, R^-_{kn}, X^+_{kn}, X^-_{kn}$ are real and imaginary parts of the external system impedance at the frequencies $k\omega_0\pm n\Omega_0$ respectively.

Equalizing the imaginary part K_n to zero we obtain the equation which allow to find u_{on}:

$$A_n\sum_{k=1}^{m_0}k^{2n-1}(X^+_{kn}+X^-_{kn})+B_n\sum_{k=1}^{m_0}k^{2n-1}(R^+_{kn}-R^-_{kn})=0. \qquad (39)$$

The system is stable if at u_{on} found from (39) the inequality $\mathrm{Re}\,K_n(u_{on})<1$ is valid or (putting $\Omega_0^2=\frac{eVq}{2\pi R^2|M|}$)

$$\frac{I_{av}\left(\frac{\delta}{\varkappa}\right)^{n-1}\cdot\frac{1}{\delta}}{Vqn[(n-1)!]^2}\cdot\sum_{k=1}^{m_0}k^{2n-1}\{A_n(R^+_{kn}-R^-_{kn})-B_n(X^+_{kn}+X^-_{kn})\}<1. \qquad (40)$$

If $\sum k^{2n-1}(R^+_{kn}-R^-_{kn})\neq0$ then from (39) and (40) inequality may be obtained which expresses the stability condition as the following:

$$\frac{I_{av}\left(\frac{\delta}{\varkappa}\right)^{n-1}\frac{1}{\delta}}{Vqn[(n-1)!]^2}\cdot\frac{A_n^2+B_n^2}{A_n}\sum_{k=1}^{m_0}k^{2n-1}(R^+_{kn}-R^-_{kn})<1\ . \qquad (41)$$

If the left-hand side of (41) is negative the system is stable at any current of the beam; the sign of the left-hand side is defined by the signs of A_n and of the resistance sum of the external system. At $n=1$ (41) has the same form as the stability condition obtained in[1] by the assumption that the frequency of synchrotron oscillations does not depend on amplitude. But there is a difference: in[1] A is a positive quantity which is equal to the value of distribution function at $x_e=x_o$. Here A_n is expressed through the derivative of the distribution function and may change the sign. It is necessary to note that the reason of the mentioned difference is the synchrotron frequency dependence on the oscillations amplitude.

With the essentially positive value A the absolute stability is determined only by the properties of external system. If A_n is the quantity that changes the sign the absolute stability will also depend on the beam properties.

To illustrate this situation let us consider some examples.

1. Normalized to unity distribution function is

$$W(u)=\begin{cases}e^u\,, & -\infty<u\le0,\\ 0\,, & u>0\,.\end{cases} \qquad (42)$$

Let us consider the stability of oscillations with $n=1$. If $\sum k(R^+_k-R^-_k)\neq0$ eq.(39) may be written as

$$-\frac{B}{A}=\frac{\sum k(X^+_k+X^-_k)}{\sum k(R^+_k-R^-_k)}=\Delta \qquad (43)$$

where Δ does not depend on u_o. Fig.1 shows schematically A and $-\frac{B}{A}$ against u_o.

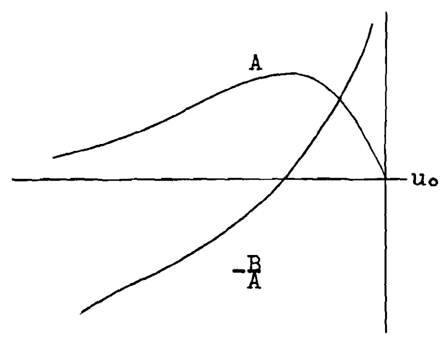

Fig.1

As it is seen from Fig.1 eq.(43) at any Δ has the single root and always $A>0$. In this case eq.(41) and that obtained in[1] practically coincide.

2. Now let us consider the distribution which differs from (42) by the fall near zero. This distribution may occur, for example, after the kick exciting the oscillations of the bunch if the damping time is much more than the decoherention time. The dependence of A and $-\frac{B}{A}$ on u_o for such distribution is represented by Fig.2.

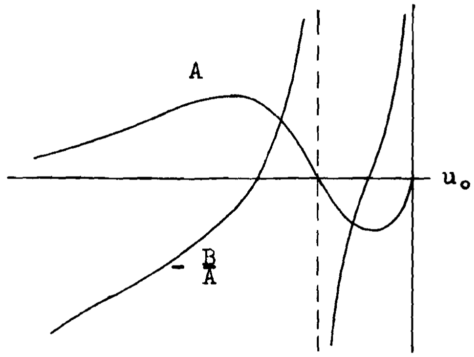

Fig.2

In this case eq.(43) at any Δ has two roots and signs of A for these roots are opposite. Therefore the system is always potentially unstable,i.e.it is excited by the beam current above some threshold.

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