DETERMINATION OF AN EQUIPMENT DESIGN CRITERION FOR THE CERN PS BOOSTER FROM STABILITY CONSIDERATIONS OF COHERENT SYNCHROTRON MOTION

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Abstract

The synchrotron motion in the presence of self-fields and resonant structures in the accelerator ring is analysed by means of a Hamiltonian and a non-Hamiltonian model. An equipment design criterion for the CERN Proton Synchrotron booster is derived from this analysis.

1. Introduction

A synchrotron can be considered as an inverse travelling-wave amplifier, since in both cases there is energy exchange between an RF field and the kinetic energy of the particles, the direction of the desired over-all energy flow being inversed in the case of the synchrotron. It is well known that all amplifying systems may develop self-excited oscillations. A synchrotron will be said to possess an instability of the coherent synchrotron motion if the interaction of the longitudinal beam motion with some surrounding RF structure satisfies the conditions of incipient self-excited oscillations. Such conditions are satisfied when infinitesimal perturbations of an equilibrium state turn out to be cumulative, i.e. if they lead to a self-sustained growth of the beam-induced voltage.

The possible beam-equipment interactions being extremely numerous, only one of the simplest cases will be considered here: interaction via discrete Fourier components of the bunched beam. Since the main features of this type of bunched beam interaction have been known for quite some time [see, for example, Lebedev[1] for a Hamiltonian model, and other authors[2-4] for a non-Hamiltonian one], this paper will concentrate on the more quantitative aspects, and especially on the dependence of the instability growth rates on the dissipative components of the electric field. The computed growth rates will be compared to the observations made in the CPS, when the ring of the latter contains, in addition to the ferrite-tuned cavities, an idling high-Q third-harmonic cavity, as the CPS is then in principle similar to the PSB. The PSB equipment design criterion will be chosen on the basis of this comparison.

2. A Hamiltonian Model

The basic feature of this model is the postulated existence of a single-valued, integrable, and differentiable particle distribution function $f(\Delta p,\phi,t)$, where $\Delta p,\phi$ designate the longitudinal momentum deviation and phase, respectively. Consider first the equilibrium state, which will be identified by the subscript 0. Subject to the above postulate one has for h identical bunches[1,5]:

number of particles

$$N = h\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} f_0(\Delta p,\varphi)\, d(\Delta p)\, d\varphi\,, \qquad (1)$$

linear charge density

$$\rho_0(\varphi) = \frac{e}{R}\int_{-\infty}^{+\infty} f_0(\Delta p,\varphi)\, d(\Delta p) = \sum_{k=-\infty}^{+\infty} \rho(k)\, e^{-ik\varphi}\,, \qquad (2)$$

induced field

$$E_{z0}(\varphi) = \frac{\beta c}{2\pi R}\sum_{k=-\infty}^{+\infty} Z(k)\,\rho(k)\, e^{-ik\varphi}\,, \qquad (3)$$

where R is the machine radius, βc the particle velocity, and Z(k) describes the interacting RF structures. It should be noted that in general $E_{z0}(\phi)$ possesses both a dynamic (Hamiltonian) and a dissipative (non-Hamiltonian) part.

Assuming that the static dissipative part of formula (3) is compensated externally, i.e. the existence of a beam orbit of constant radius R is assured, the synchrotron equation (below transition) for $v_{RF}(t) = V_{RF}\sin\omega_{RF}t$ is

$$\frac{RE_0}{hec^2}\frac{d}{dt}\left(\frac{\gamma}{\alpha\gamma^2-1}\frac{d\varphi}{dt}\right) = \frac{V_{RF}}{2\pi R}(\sin\varphi - \sin\varphi_s) + \mathrm{Im}\left\{E_{z0}(\varphi-\varphi_s)\right\}. \qquad (4)$$

Since the existence of $f_0(\Delta p,\phi)$ requires the existence of a time-independent Hamiltonian, it is necessary to assume that γ is not a function of t. In such a case formula (4) reduces to a canonical system

$$\frac{d\varphi}{dt} = \frac{\Delta p}{RM}\,,$$

$$\frac{d(\Delta p)}{dt} = \frac{eV_{RF}}{2\pi R}(\sin\varphi - \sin\varphi_s) + e\,\mathrm{Im}\left\{E_{z0}(\varphi-\varphi_s)\right\} = E_0(\varphi)\,, \qquad (5)$$

derivable from

$$H_0(\Delta p,\varphi) = \frac{1}{2RM}(\Delta p)^2 + V_0(\varphi)\,, \qquad M = \frac{E_0}{hc^2}\frac{\gamma^3}{\alpha\gamma^2-1}\,, \qquad (6)$$

where one has postulated the existence of a "potential" $V_0(\phi)$ defined by

$$E_0(\varphi) = -\frac{dV_0(\varphi)}{d\varphi}\,. \qquad (7)$$

Introducing a new set of canonical variables $\Delta p = \Delta p(\varepsilon,\tau)$, $\phi = \phi(\varepsilon,\tau)$, with

$$\varepsilon = \varepsilon(\Delta p,\varphi) = H_o(\Delta p,\varphi), \quad \tau = \tau(\Delta p,\varphi) = MR\int_{\varphi_{2e}}^{\varphi} \frac{d\varphi}{\Delta p(\varepsilon,\varphi)}, \tag{8}$$

where

$$\varphi_{2e} = \varphi(\varepsilon,0) > \varphi_s, \ \tau = 0 \quad \text{for } \Delta p = 0, \ \varphi = \varphi_{2e},$$

one has

$$\frac{d\varepsilon}{dt} = 0, \ \frac{d\tau}{dt} = 1, \tag{9}$$

$$\frac{\partial f_o(\varepsilon,\tau)}{\partial\tau} = 0 . \tag{10}$$

The general solution of the Vlasov equation (10) is $f_0(\varepsilon,\tau) = f_0(\varepsilon)$.

Let us now consider perturbations of an equilibrium state, described by a given $f_0(\varepsilon)$, of the special form

$$f_p(\Delta p,\varphi,\nu)\, e^{i\nu t}, \tag{11}$$

ν = complex frequency to be determined. Assuming again the existence of Fourier decompositions one has then

$$f(\Delta p,\varphi,t) = f_o(\Delta p,\varphi) + f_p(\Delta p,\varphi,\nu)\, e^{i\nu t}, \tag{12}$$

$$\begin{aligned}\rho(\varphi,\nu,t) &= \frac{e}{R}\, e^{i\nu t}\int_{-\infty}^{+\infty} f_p(\Delta p,\varphi,\nu)\, d(\Delta p) \\ &= \sum_{k=-\infty}^{+\infty} \rho(k,\nu)\, e^{i(\nu t - k\varphi)},\end{aligned} \tag{13}$$

$$E_z(\varphi,\nu,t) = \frac{\beta c}{2\pi R}\sum_{k=-\infty}^{+\infty} Z(k,\nu)\,\rho(k,\nu)\, e^{i(\nu t - k\varphi)}, \tag{14}$$

$$\frac{d\varphi}{dt} = \frac{\Delta p}{RM}, \quad \frac{d(\Delta p)}{dt} = E_o(\varphi) + e\,\mathrm{Im}\left\{E_z(\varphi-\varphi_s,\nu,t)\right\}, \tag{15}$$

where Eqs. (13) and (14) can be interpreted as "slowly varying" travelling waves. If Eq. (15) is to be a canonical system, it is necessary to postulate the existence of a "potential function" $V(\phi,\nu,t)$, defined by

$$e\,\mathrm{Im}\left\{E_z(\varphi,\nu,t)\right\} = -\frac{d}{d\varphi} V(\varphi,\nu,t) . \tag{16}$$

From Eqs. (16) and (8) follows

$$H(\Delta p,\varphi,\nu,t) = H_o(\Delta p,\varphi) + V(\varphi,\nu,t), \tag{17}$$

$$\frac{d\varepsilon}{dt} = \frac{e}{MR}\frac{E_z}{1+V'_\varepsilon}\Delta p(\varepsilon,\tau), \quad \frac{d\tau}{dt} = 1 + V'_\varepsilon ; \tag{18}$$

$$\frac{e}{MR}\frac{\Delta p . E_z}{1+V'_\varepsilon}\left(\frac{\partial f_o}{\partial\varepsilon} + \frac{\partial f_p}{\partial\varepsilon}\right) + (1+V'_\varepsilon)\frac{\partial f_p}{\partial\tau} + i\nu f_p = 0 . \tag{19}$$

If $V(\phi,\nu,t)$ is known, the Vlasov equation (19) can be solved by successive iterations to any order of accuracy[5]. For small perturbations and to first order one finds, without any knowledge of $V(\phi,\nu,t)$,

$$\rho(k,\nu) = \frac{-e^2}{2\pi MR}\int_0^{T_0}\int_{-\infty}^{0} e^{ik\varphi_o(\tau,\varepsilon)} . e^{-i\nu\tau} . \frac{\partial f_o}{\partial\varepsilon}\int^{\tau} e^{i\nu t} . \Delta p(\varepsilon,t) . E_z\big(\varphi(\varepsilon,t)-\varphi_s,\nu\big)\, dt\, d\varepsilon\, d\tau , \tag{20}$$

where T_0 is the synchrotron period of the equilibrium trajectory passing through $\Delta p = 0$, $\phi = \phi_{2e}$. Inserting Eq. (14) into Eq. (20) one is led after some manipulation to the linear eigenvalue problem

$$\rho(k,\nu) = \sum_{m=-\infty}^{+\infty} I(k,m,\nu)\, Z(m,\nu)\,\rho(m,\nu),$$

$$k = 0, \pm 1, \pm 2, \ldots , \tag{21}$$

with known but rather cumbersome expressions for the $I(k,m,\nu)$. The eigenvalue problem (21) defines a discrete set ν_n of "slow waves" (13) and (14). The resulting synchrotron equation is

$$\begin{aligned}&\frac{RM}{e}\frac{d^2\varphi}{dt^2} \\ &= \frac{V_{RF}}{2\pi R}(\sin\varphi - \sin\varphi_s) + \mathrm{Im}\left\{E_{z_o}(\varphi-\varphi_s)\right\} + \mathrm{Im}\left\{\sum_{n=-\infty}^{+\infty} E_z(\varphi-\varphi_s,\nu_n)\, e^{i\nu_n t}\right\}.\end{aligned} \tag{22}$$

The quantity $-\mathrm{Im}\,\nu_n$ describes the instability growth rate of the "mode" ν_n. For an RF structure of (coupling) impedance $Z(\omega)$ one has

$$Z(k,\nu) = Z(\omega_k), \quad \omega_k = k\omega_{RF} - \nu . \tag{23}$$

In the case of the PSB and CPS it turns out that $|\nu_n| \approx |n| \cdot 2\pi/T_0)$, $n = \pm 1, \ldots, \pm 5$. These modes were computed[5] by solving Eq. (21) by means of the reduction method, the convergence of which was not always rapid. The growth rates $-\mathrm{Im}\,\nu_n$ were found to decrease with an increase of bunch length for the "rigid" bunch" mode ν_1; however, they were found to increase with the bunch length for the modes ν_2 and ν_3. For $|n| = 4$ and 5 one had $-\mathrm{Im}\,\nu_n \ll -\mathrm{Im}\,\nu_1$.

Before using the numerical values of $-\mathrm{Im}\,\nu_n$ for the design of PSB equipment, the precaution was taken to test the validity of the postulates (7) and (16), and to some extent also the necessity of (1). Such a test can readily be carried out for the rigid-bunch mode ν_1, because in this case the charge of the bunch can be thought to be concentrated at the bunch "centre", and Eqs. (1), (10), and (19) are no longer required.

3. A Non-Hamiltonian Model

The basic feature of this model is the postulated existence of an equilibrium travelling-wave beam current spectrum[2-4]

$$i_b(t) = \beta c\sum_{k=-\infty}^{+\infty} \rho(k)\, e^{-i\frac{k}{h}(\varphi-\varphi_s)} . e^{i\theta_k}, \tag{24}$$

where $\rho(k)$ and θ_k are determined by measurement[5], instead of being deduced from an equation such as (2). Consider an impedance $Z(\omega)$ in the path of $i_b(t)$. The equilibrium voltage on $Z(\omega)$ is then of the form

$$V(\varphi) = \sum_{k=-\infty}^{+\infty} V_k \cdot e^{i\varphi_k} \cdot e^{-i\frac{k}{h}\varphi}, \tag{25}$$

where again, if one prefers, the constants V_k, ϕ_k can be assumed to be known experimentally. They can of course also be calculated from (24), and vice-versa.

Consider only a $Z(\omega)$ corresponding to a resonator of $Q \geq 10$ and let $K(t,\tau)$ be the Green function associated with $Z(\omega)$. If perturbations displace the rigid bunch from ϕ_s to $\phi(t)$, and $Z(\omega)$ is detuned by $\Delta\omega$ from $k\omega_r$, $\omega_r = k\beta c/R$, then for one component of (25) the induced perturbation field on the beam axis is[5]

$$2\pi R\, E_{zk}(\varphi) = -\operatorname{Im}\left\{ \frac{V_k \cdot e^{i\varphi_k} \cdot e^{i\frac{k}{h}\varphi(t)} \cdot e^{ik\omega_r t}}{\left(\int^t K'_t(t,\tau) e^{ik\omega_r\tau} \cdot d\tau\right)_{\Delta\omega=0}} \int^t K'_t(t,\tau)\, e^{ik\omega_r\tau} \cdot e^{-i\frac{k}{h}\varphi(\tau)}\, d\tau \right\}. \tag{26}$$

The synchrotron equation in the presence of this field becomes

$$\frac{RE_0}{hec^2}\frac{d}{dt}\left(\frac{\gamma^3}{\alpha\gamma^2-1}\frac{d\varphi}{dt}\right) = \frac{V_{RF}}{2\pi R}(\sin\varphi - \sin\varphi_s) + \sum_{k=-\infty}^{+\infty}\left(E_{zk}(\varphi) - E_{zk}(\varphi_s)\right), \tag{27}$$

where ϕ_s is "statically" adjusted by an external beam control.

Equation (27) is a non-linear integro-differential equation which is difficult to solve analytically. But contrary to Eqs. (5) and (18) it is also valid for $\gamma = \gamma(t)$ if V_k and ϕ_k are known as functions of γ. The linearized version of Eq. (27) is, however, readily solved, and leads to an eigenvalue problem of the form

$$s^2 + 1 + \frac{\delta}{V_{RF}\cos\varphi_s}\sum_{m=-\infty}^{+\infty} m V_m \left[\frac{\Delta\omega\cos\varphi_m + \delta\sin\varphi_m}{\delta^2 + (\Delta\omega)^2} - \frac{\Delta\omega\cos\varphi_m + (\delta+\Omega_0 s)\sin\varphi_m}{(\delta+\Omega_0 s)^2 + (\Delta\omega)^2}\right] = 0, \tag{28}$$

where $\Omega_0 = 2\pi T_0^{-1}$ and δ is the half-bandwidth of the resonator.

Comparing Eq. (28) to Eq. (21) one finds that $-\mathrm{Im}\,\nu_1$ corresponds to Re s. For the PSB it is verified that $|\nu_1| \approx |s|$, as expected, but rather surprisingly it was found that

$$\mathrm{Re}\, s \gtrsim 4(-\mathrm{Im}\,\nu_1), \tag{29}$$

i.e. the non-Hamiltonian model yields considerably faster instability growth rates. Landau damping is included in the values of V_k and ϕ_k[3].

4. Comparison with a CPS Experiment

In order to decide whether Eq. (21) or Eq. (28) constitutes a more realistic design constraint on PSB equipment, the computed growth rates were compared to the observations in a CPS machine development[6], when the CPS ring contained an idling high-Q cavity tuned to about 28 MHz. In such an operation the CPS becomes, in principle, similar to the PSB. An intense beam-cavity interaction was found to exist near $\gamma = 24$, when 60 ω_r sweeps across the resonant frequency of the cavity. Both the cavity voltage and the peak-detected voltage at a wide-band pick-up station were found to possess a pronounced maximum together with a strong sinusoidal amplitude modulation at about 1.5 Ω_0. In the absence of a beam-cavity interaction, the voltage at the wide-band pick-up station near $\gamma = 24$ has an almost constant envelope. From Eqs. (21) and (28) one finds max $(-\mathrm{Im}\,\nu_1) \approx 2\ \mathrm{sec}^{-1}$ and max (Re s) $\approx 30\ \mathrm{sec}^{-1}$, respectively. Because the interaction predicted by the Hamiltonian model is too weak to account for the observations, it was decided to use Eq. (28) for PSB equipment design purposes. The difference between $-\mathrm{Im}\,\nu_1$ and Re s is likely to be due to the insufficient accuracy of the postulates (7) and (16), which amount to a partial "dynamic" compensation of the dissipative part of the field E_z.

References

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