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Abstract

Perform a coordinate transformation so that the distribution function becomes almost stationary. Then expand the distribution function in Hermite polynomials, and use the expansion coefficients as dynamical variables. This method is efficient for computer calculations when the distribution function is roughly Gaussian.

1. Introduction

In many accelerator dynamics problems some sort of focusing keeps the particle together, and the density distribution function is an isolated island in phase space. The particles are not confined for example by metal walls, but are rather floating in a potential well. There is a certain number of particles on the bottom of this well, and everything else between  $-\infty$  and  $+\infty$  may be neglected. Very often the density distribution function has a more or less Gaussian shape. In a phase plane the particles travel around the origin on trajectories which are approximately ellipses. Usually we know quite a lot about the motion of the particles: we know for instance the no-space-charge solution, or we even know the solution for linearized space-charge forces.

Similar situations also exist in other branches of physics, for instance in the dynamics of a galaxy.

The method to be described tries to save time on the computer in two ways:

1. We tell the computer everything we already know about the problem,
2. We introduce an efficient description of shapes of distribution functions.

2. How to Tell the Computer what We Already Know

Let us assume that the distribution function is approximately a two-dimensional Gaussian in phase space:

$$\rho(t, q, p) \approx \exp\left(-[a(t)q^2 + 2b(t)qp + c(t)p^2]\right). \quad (1)$$

By performing a linear coordinate transformation we can transform the approximately elliptical equi-density contours into approximately circular ones:

$$\tilde{\rho}(t, \tilde{q}, \tilde{p}) \approx \exp\left(-[\tilde{q}^2 + \tilde{p}^2]\right) \quad (2)$$

with

$$\begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \quad (3)$$

A rotating ellipse in  $q, p$  appears to be a stationary circle in  $\tilde{q}, \tilde{p}$ .

An ellipse is defined by 3 quantities (i.e. area, eccentricity, tilt; or  $a, b, c$  of eq. (1)), and our matrix has 4 quantities. The matrix is therefore not uniquely defined. This is because an ellipse which has been transformed into a circle can be further transformed with no additional change, as we can rotate the circle through an arbitrary angle around its centre. We therefore have exactly one degree of freedom in our choice of transformation matrix. How can we use it?

The force between two particles depends only on their relative positions and not on their relative momenta (if their relative velocities are much less than the velocity of light). We therefore have to work out integrals of the type

$$\lambda(t, q) = \int_{-\infty}^{\infty} \rho(t, q, p) dp. \quad (4)$$

We notice that if we take  $\beta = 0$  in eq. (3) and then change  $p$ , only  $\tilde{p}$  will change but not  $\tilde{q}$ . With  $\beta = 0$ , integrals of the type indicated will transform into something like

$$\tilde{\lambda}(t, \tilde{q}) = \int_{-\infty}^{\infty} \tilde{\rho}(t, \tilde{q}, \tilde{p}) d\tilde{p}. \quad (5)$$

In the general case, with all matrix element non-zero, the integration path in  $\tilde{q}, \tilde{p}$  would not be parallel to any of the coordinate axes, and such 'skew' integrals would be much more difficult to work out. We therefore choose a transformation matrix

$$\begin{pmatrix} \tilde{q} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} \alpha(t) & 0 \\ \gamma(t) & \delta(t) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \quad (6)$$

We now put all our knowledge of the problem into the quantities  $\alpha, \gamma$  and  $\delta$ . If, for example, the particles travel around the origin in approximately circular orbits, the computer will only have to keep track of the deviation from such circles, while the gross motion - the travelling around - is described by  $\alpha, \gamma, \delta$ . The computer can then integrate the differential equations using larger time steps than would otherwise be possible.

3. Why Hermite Polynomials?

Hermite polynomials are orthogonal with a Gaussian as weight function:

$$\int_{-\infty}^{\infty} \exp(-z^2) H_r(z) H_s(z) dz = \delta_{rs} 2^r r! \sqrt{\pi} . \quad (7)$$

They therefore offer themselves to describe deviations from a basically Gaussian shape.

In the transformed frame we describe the distribution function by a series expansion in Hermite polynomials :

$$\tilde{\rho}(t, \tilde{q}, \tilde{p}) = \frac{1}{\pi} \exp\left(-(\tilde{q}^2 + \tilde{p}^2)\right) \sum_{i=0}^M \sum_{j=0}^{M-i} a_{i,j}(t) H_i(\tilde{q}) H_j(\tilde{p}), \quad (8)$$

where

$$a_{i,j}(t) = \frac{1}{2^i i! \sqrt{\pi}} \frac{1}{2^j j! \sqrt{\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\rho}(t, \tilde{q}, \tilde{p}) H_i(\tilde{q}) H_j(\tilde{p}) d\tilde{q} d\tilde{p}. \quad (9)$$

The expansion coefficients  $a_{i,j}(t)$  will then be our dynamical variables. The more the distribution function deviates from a Gaussian shape, the more coefficients will be needed. But usually the necessary number of coefficients will be much smaller than the number of superparticles necessary in a superparticle approach, or much smaller than the necessary number of mesh points in a finite-difference method.

#### 4. The Recipe

If the single-particle equations of motion are :

$$\frac{dq}{dt} = X(t, q, p) \quad (10)$$

$$\frac{dp}{dt} = Y(t, q, p) , \quad (11)$$

then the equation of motion for the distribution function (the Vlasov equation) is :

$$\frac{\partial \rho}{\partial t} = -X(t, q, p) \frac{\partial \rho}{\partial q} - Y(t, q, p) \frac{\partial \rho}{\partial p} . \quad (12)$$

We assume that X and Y in eqs. (10) and (11) can be derived from a Hamiltonian; this gives a Vlasov equation (12) that is Liouvillian.

First perform the coordinate transformation (6) and put

$$\tilde{\rho}(t, \tilde{q}, \tilde{p}) = \rho(t, q, p) . \quad (13)$$

By substitution we now find the equation of motion for the transformed distribution function :

$$\frac{\partial \tilde{\rho}}{\partial t} = -\tilde{X}(t, \tilde{q}, \tilde{p}) \frac{\partial \tilde{\rho}}{\partial \tilde{q}} - \tilde{Y}(t, \tilde{q}, \tilde{p}) \frac{\partial \tilde{\rho}}{\partial \tilde{p}} . \quad (14)$$

Usually  $\tilde{X}$  and  $\tilde{Y}$  will look more complicated than their untransformed counterparts, but the computer will prefer them in any case, as they yield a  $\rho$  that moves less than  $\rho$ . We include here the possibility that the transformation matrix in (6) has non-constant determinant. If so, the transformation is non-canonical and the transformed Vlasov equation (14) is non-Liouvillian. This is useful if we want to describe a situation with emittance growth in an efficient way. How it can be done in practice, see Sec.6.

Then substitute the expansion (8) into the transformed Vlasov equation (14). The left hand side is simply :

$$\frac{1}{\pi} \exp\left(-(\tilde{q}^2 + \tilde{p}^2)\right) \sum_{i=0}^M \sum_{j=0}^{M-i} \frac{da_{i,j}(t)}{dt} H_i(\tilde{q}) H_j(\tilde{p}) . \quad (15)$$

The forces are essentially described by Y and will therefore contain the integral  $\lambda$  shown in eq. (4). Similarly,  $\tilde{Y}$  will contain the integral  $\tilde{\lambda}$ . But this integral is easily worked out and is :

$$\tilde{\lambda}(t, \tilde{q}) = \frac{1}{\sqrt{\pi}} \exp(-\tilde{q}^2) \sum_{k=0}^M a_{k,0}(t) H_k(\tilde{q}) . \quad (16)$$

We also need  $\partial \tilde{\rho} / \partial \tilde{q}$  and  $\partial \tilde{\rho} / \partial \tilde{p}$ . They are easily found from one of the many useful\* recurrence relations for Hermite polynomials :

$$\frac{\partial \tilde{\rho}}{\partial \tilde{p}} = \frac{1}{\pi} \exp\left(-(\tilde{q}^2 + \tilde{p}^2)\right) \sum_{\ell=0}^M \sum_{m=1}^{M+1-\ell} (-1) a_{\ell, m-1}(t) H_{\ell}(\tilde{q}) H_m(\tilde{p}) \quad (17)$$

and similarly for  $\partial \tilde{\rho} / \partial \tilde{q}$ .

But now comes a problem which is not quite so simple. The right hand side contains products of two series, one of type (16) and one of type (17). To be able to find equations of motion for the  $a_{i,j}$ 's, we must rewrite the right hand side into a straightforward series like :

$$\frac{1}{\pi} \exp\left(-(\tilde{q}^2 + \tilde{p}^2)\right) \sum_{i=0}^M \sum_{j=0}^{M-i} b_{i,j}(t) H_i(\tilde{q}) H_j(\tilde{p}) . \quad (18)$$

Once this has been done, the equation of motion for the  $a_{i,j}$ 's is found by identifying term by term :

$$\frac{da_{i,j}(t)}{dt} = b_{i,j}(t) . \quad (19)$$

#### 5. How To Multiply Two Expansions

So, the problem is to find the  $b_{i,j}$ 's. The right hand side contains more than just the product of the two series (16) and (17), but let us perform this multiplication just to illustrate the procedure. Let us call the coefficients of this product  $c_{i,j}$  to distinguish them from  $b_{i,j}$  which contain slight-ly more.

\* The recurrence relation which is useful here is :

$$\frac{d}{dz} \exp(-z^2) H_n(z) = -\exp(-z^2) H_{n+1}(z)$$

In Morse and Feshbach it is misprinted with an extra factor 2, which makes the formula somewhat less useful.

In analogy with eq. (9) we write

$$c_{i,j} = \frac{1}{2^i i! \sqrt{\pi}} \frac{1}{2^j j! \sqrt{\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi \lambda \frac{\partial \rho}{\partial \tilde{p}} H_i(\tilde{q}) H_j(\tilde{p}) d\tilde{q} d\tilde{p} \quad (20)$$

The integration over  $\tilde{p}$  leads to expressions such as

$$\int_{-\infty}^{\infty} \exp(-\tilde{p}^2) H_m(\tilde{p}) H_j(\tilde{p}) d\tilde{p} \quad (21)$$

which is just the orthogonality integral, eq. (7).

But the integral over  $\tilde{q}$  is worse and gives

$$\int_{-\infty}^{\infty} \exp(-2\tilde{q}^2) H_i(\tilde{q}) H_k(\tilde{q}) H_\ell(\tilde{q}) d\tilde{q} \quad (22)$$

This looks frightening, but Gradshteyn and Ryzhik give this integral and it even has a nice analytical form which we denote here by  $GR_{i,k,\ell}$  for shortness. Putting these pieces of mathematics together, we have

$$c_{i,j} = \frac{-1}{2^i i! \pi} \sum_{k=0}^M \sum_{\ell=0}^{M+1-j} a_{k,0} a_{\ell,j-1} GR_{i,k,\ell} \quad (23)$$

We remark in passing that though we have to work out a number of coefficients  $\sim M^2$ , each of which contains double sums up to  $M$ , it is possible to store away certain subsums and use them again later so that the work involved is only proportional to  $M^3$ , rather than  $M^4$ .

### 6. A Small Refinement

To be able to get high accuracy with only a small number of expansion coefficients, it is essen-

tial that the solution around which we expand is as close to the exact solution as possible. That is, we have to make a good choice for  $\alpha, \gamma, \delta$ . Normally, these will be given in terms of a set of coupled differential equations, for instance those of the corresponding linearized theory. But we can do even better : it is possible to modify these differential equations so that the ellipse described by  $\alpha, \gamma, \delta$  automatically gives the correct second-order moments. This amounts to arranging that  $\dot{a}_{20} = \dot{a}_{11} = \dot{a}_{02} = 0$  if the initial condition is  $a_{20} = a_{11} = a_{02} = 0$ . The differential equation for  $\alpha, \gamma$  and  $\delta$  will be somewhat more complicated than in the linear model, but the resulting complication is rather moderate.

### 7. Conclusion

This method is suited to the calculation of non-linear space-charge problems where the particle distribution is roughly Gaussian. (However, one could also use other weight functions than Gaussians; this would require the use of a function family different from the Hermite polynomials). The method is being tried in order to explain the observed blow-up of bunches in the CERN PS whilst crossing transition.

### 8. Acknowledgments

While this method was developed, many people have been disturbed in their work by a certain person entering their office with a lot of unclear ideas, and leaving their office with somewhat less unclear ideas. Many thanks to all those who have suffered with patience and given their helpful advice. I am also grateful to the MPS Division for giving me the necessary time to work on this project.