THE ELECTROMAGNETIC FIELD DISTRIBUTION DUE TO A BEAM-EXCITED RE-ENTRANT CYLINDRICAL CAVITY MOUNTED COAXIALLY WITH A CYLINDRICAL VACUUM CHAMBER

J.K. Trickett

CERN, Geneva, Switzerland

Abstract

An efficient method of solving the complex boundary value problem is described using Green's functions and an integral equation approach. From the knowledge of the field inside the beam, the coupling impedance can be simply calculated. A low frequency equivalent circuit is derived theoretically.

1. Introduction

An exact analysis is made of the electromagnetic field distribution due to a re-entrant cylindrical cavity with finite or infinite conductivity, mounted coaxially with a cylindrical vacuum chamber (Fig. 1). The problem is identical in principle to the classical unsolved problem of the diffraction of a plane wave by a slit, to which extensive literature is devoted. For instance, the Wiener-Hopf method furnishes only an asymptotic approximation. Keil and Zotter have made an analysis by matrix methods of the present problem on the basis of periodic structures except that the cavities are not re-entrant, and found a rapidly converging solution for the coupling impedance[1]).

PARAMETER SET A :

R1 = 0.0203m R2 = 0.075m R3 = 0.125m D = 7.5×10^{-3} m l = 0.250m

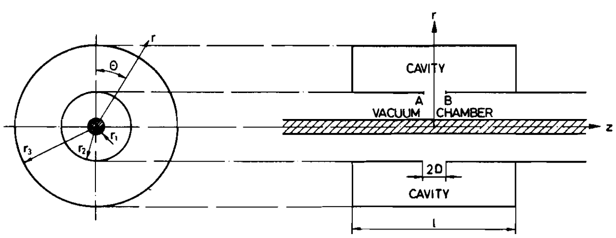

Fig. 1 Geometry

2. Outline of the method and of the attainable results

Here the complex boundary value problem is split into two simple boundary value problems by introducing as unknown the field in the gap $E_z(r_2,z)$ between the edges A and B of Fig. 1. By imposing continuity across the gap, an integral equation for $E_z(r_2,z)$ is obtained. Green's functions are used to obtain the field in each simple region. Inspection of these Green's functions shows that the integral equation, which is a Fredholm equation of the first kind, has a logarithmic singularity in the kernel. In order to have a well-behaved numerical problem, this singularity is subtracted out. The Chebychev integration formula is then used to evaluate the integral, which leads to a set of simultaneous equations, in $E_z(r_2,z)$ at the Chebychev nodes. By inverting the matrix represented by these equations, the solution is obtained. This method of solving the integral equation was chosen because it is simple to carry out, also the accuracy can be checked by varying the number of points taken in the Chebychev integration formula and noting the differences in the solutions (reduction method).

The solution for $E_z(r_2,z)$ is found to be oscillatory at the edges of the gap. However, by compensating for the singularities in the field at these points it is found that a smooth solution can be obtained that tends asymptotically to the electrostatic solution as the frequency is decreased. By substituting this result in the integral solution found for the vacuum chamber and again using Chebychev integration, the solution for any of the field components at any point in the vacuum chamber can be obtained (see Fig. 2).

Losses in the cavity are taken into account by using a first-order perturbation to the magnetic field due to the induced electric field on the walls. The additional coupling impedance is obtained by integrating the ratio of additional longitudinal electric field on the axis of the beam to the perturbed beam current (see Fig. 3). For low frequencies, an equivalent circuit with calculable parameters is derived for the combination of cavity and gap.

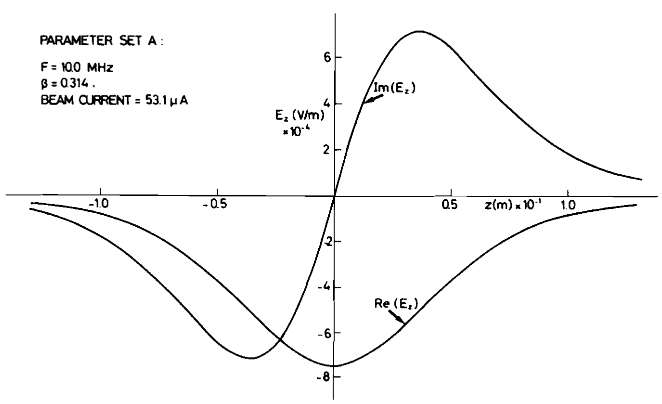

Fig. 2 Longitudinal electric field on beam axis

3. Construction of the solution

The beam has a uniform charge density upon which is superimposed a given travelling wave charge perturbation. The solution for the vacuum chamber is split into two parts:

i) the beam with charge perturbation in a uniform vacuum chamber (boundary condition: $E_z = 0$ on the chamber wall);

ii) an infinite set of TM modes in an empty vacuum chamber [boundary condition: $E_z = E_z(r_2,z)$ in the gap, $E_z = 0$ on the chamber wall elsewhere].

The sum of these two parts gives the desired solution for the vacuum chamber.

Solution (i) is well known to be (the factor e^{jwt} is omitted for conciseness):

$$H_\theta(r,z) = \rho\, r_1 \beta\, I_0'(\bar{\alpha}_0 r) \times$$
$$\times \{I_0'(\bar{\alpha}_0 r_1)\cdot K_0(\bar{\alpha}_0 r_2) - K_0'(\bar{\alpha}_0 r_1)\cdot I_0(\bar{\alpha}_0 r_2)\} \times$$
$$\times\, e^{-jk_z\cdot z}/z_0\varepsilon_0 I_0(\bar{\alpha}_0 r_2) \tag{1}$$

$$H_\theta(r,z) = \rho\, r_1 \beta\, I_0'(\bar{\alpha}_0 r_1) \times$$
$$\times \{I_0'(\bar{\alpha}_0 r)\cdot K_0(\bar{\alpha}_0 r_2) - K_0'(\bar{\alpha}_0 r)\cdot I_0(\bar{\alpha}_0 r_2)\} \times$$
$$\times\, e^{-jk_z\cdot z}/z_0\varepsilon_0 I_0(\bar{\alpha}_0 r_2)$$

where $0 \le r \le r_1$ and $r_1 \le r \le r_2$, respectively; z_0 is the impedance of free space; $\bar{\alpha}_0 = k/\beta\gamma$, $k_z = k/\beta$; and a prime denotes differentiation of the function with respect to its argument.

Solution (ii) is carried out using a Green's function, and we have:

$$H_\theta(r',z') = \int_{-D}^{+D} G_v(r',z,z')\, E_z(r_2,z)\cdot dz;$$
$$G_v(r',z,z') = \frac{jk}{r_2 z_0} \times$$
$$\times \sum_{m=1}^{\infty} \frac{J_1(\alpha_m r'/r_2)}{K_m J_1(\alpha_m)}\cdot e^{-K_m|z-z'|}\,, \tag{2}$$

$k_m = \sqrt{(\alpha_m^2/r_2^2) - k^2}$, where α_m is the m^{th} zero of $J_0(\alpha)$.

The solution for the cavity is:

$$H_\theta(r',z') = \int_{-D}^{+D} G_c(r',z,z')\, E_z(r_2,z)\cdot dz\ ;$$
$$G_c(r',z,z') =$$
$$= \frac{jk}{z_0}\left[\frac{F(k,z,z')}{r'\ \ln(r_3/r_2)k\ \sin k\ell} - \sum_{n=1}^{\infty} \frac{2F(k_n,z,z')\ \phi_2(y_n r',\, y_n r_2)}{\pi\, T_n\, y_n\, k_n \sin k_n\cdot \ell}\right];$$

$$F(k,z,z') = \cos k(z'-\ell/2)\cdot \cos k(z+\ell/2) \quad \text{for} \quad z \le z'\ ;$$
$$\cos k(z-\ell/2)\cdot \cos k(z'+\ell/2) \quad \text{for} \quad z \ge z'\ ;$$
$$\phi_2(y_n r', y_n r_2) = J_0'(y_n r')\, Y_0(y_n r_2) - Y_0'(y_n r')\, J_0(y_n r_2)\ ;$$
$$T_n = (r_3^2/2)\ \phi_2^2(y_n r_3, y_n r_2) - 2/\pi^2 y_n^2\ ;$$

y_n is the n^{th} zero of $J_0(yr_3)\, Y_0(yr_2) - Y_0(yr_3)\, J_0(yr_2)$;

$$k_n = \sqrt{k^2 - y_n^2}\ . \tag{3}$$

Imposing continuity in the gap of the azimuthal magnetic field we thus have, from results (1), (2), and (3):

$$\frac{\rho\, r_1 \beta}{z_0 \varepsilon_0}\cdot \frac{I_0'(\bar{\alpha}_0 r_1)}{I_0(\bar{\alpha}_0 r_2)}\cdot \frac{1}{\bar{\alpha}_0 r_2}\cdot e^{-jk_z\cdot z'} =$$
$$= \int_{-D}^{+D} \{[G_c(r_2,z,z') - G_v(r_2,z,z')] \times$$
$$\times\, E_z(r_2,z)\}\cdot dz \quad \text{for} \quad |z'| < D\ . \tag{4}$$

With asymptotic expressions for the Bessel functions and their zeros, it can be seen that the kernel of this integral equation (4) has a logarithmic singularity of the form $\ln|z-z'|$. This is subtracted out by re-writing Eq. (4) in the following fashion:

$$f(z') = \int_{-D}^{+D} [E_z(r_2,z) - E_z(r_2,z')]\ K(z,z')\cdot dz\ +$$
$$+\ E_z(r_2,z') \int_{-D}^{+D} K(z,z')\cdot dz\ . \tag{5}$$

Now, the electrostatic field between two flat plates is known to vary as $(1 - z^2/D^2)^{-\frac{1}{2}}$, which is the weight function in the Chebychev integration formula. Hence the singularities in $E_z(r_2,z)$ can be compensated and the oscillations in the numerical solution thereby removed by substituting $\bar{E}_z(r_2,z) = (1 - z^2/D^2)^{\frac{1}{2}} E_z(r_2,z)$ into the integral equation to give

$$f(z') = \int_{-D}^{+D} \{[\bar{E}_z(r_2,z) - \bar{E}_z(r_2,z')] \times$$
$$\times\, (1 - z^2/D^2)^{-\frac{1}{2}}\ [K_1(z,z') + \ln|z-z'|]\}\cdot dz\ +$$
$$+\ \bar{E}_z(r_2,z') \int_{-D}^{+D} \{(1 - z^2/D^2)^{-\frac{1}{2}} \times$$
$$\times\, [K_1(z,z') + \ln|z-z'|]\}\cdot dz\ , \tag{6}$$

where $K(z,z') = K_1(z,z') + \ln|z-z'|$ and we can use the known[2] integral

$$\int_{-D}^{+D} (1 - z^2/D^2)^{-\frac{1}{2}}\ \ln|z-z'|\cdot dz = \pi D\ \ln(D/2),\ |z'| \le D\ . \tag{7}$$

By equating the two sides of Eq. (6) at the n nodes of the Chebychev integration formula, a smooth solution for $\bar{E}_z(r_2,z)$ is now obtained. The difference in the solution for n = 48 and 36 was less than one per cent.

4. Losses

Using the Green's function for the cavity, an integral round the walls is obtained for the first-order perturbation to the magnetic field in the cavity due to losses in the walls. In this integral, terms due to each mode and to mode coupling can be identified. As the frequencies are such that we are below cut-off for the first TM mode in the cavity for this particular geometry, it is a good approximation to neglect mode coupling and modes higher than the TEM. This gives for the additional magnetic field due to losses:

$$H_\theta(r',z') = -\{R_s/[r' z_0^2 \sin^2 k\ell \log e^2 (r_3/r_2)]\} \times$$
$$\times \int_{-D}^{+D} I(z,z') \, E_z(r_2,z) \, dz ,$$

where $I(z,z')$ is tedious to write[3] but is not an unduly complicated function of z and z'; $R_S = (1 + j)/\sigma\delta$, σ is the conductivity of and δ the skin depth in the wall material. Because of linearity, this solution can be added to that already obtained for the cavity, thereby modifying the kernel in the integral equation. The effect of losses is to cause the real and imaginary parts of $E_Z(r_2,z)$ to be no longer purely odd and even, respectively, about the gap centre line, thereby introducing a resistive term into the impedance seen looking into the cavity.

5. Coupling impedance

The additional coupling impedance is calculated as below:

$$z_c = -\int_{-\infty}^{+\infty} [E_z(0,z)/(I_0 \, e^{-jk_z \cdot z})] \cdot dz \qquad (8)$$

where $E_Z(0,z)$ is the additional electric field on the beam axis, and I_0 is the perturbed beam current at $z = 0$. As $E_Z(0,z)$ decays, since we are below cut-off for the first TM mode in the vacuum chamber for this particular geometry, as:

$$\exp\{-[(\alpha_1^2/r_2^2) - k^2]^{\frac{1}{2}}|z|\} ,$$

this integral is performed numerically using the Laguerre integration formula (see Fig. 3).

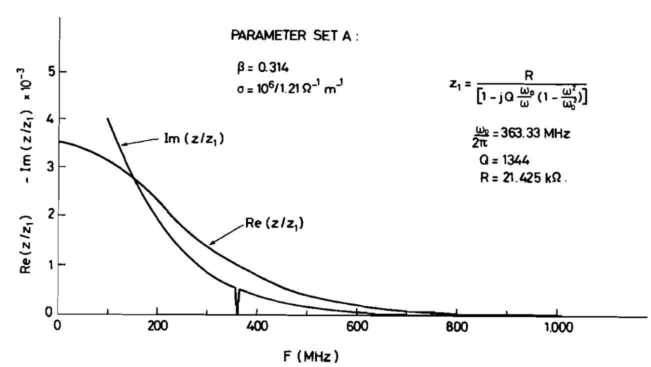

Fig. 3 Frequency variation of normalized coupling impedance

6. Low-frequency approximation to the circuit parameters

At low frequencies ($k\ell < \pi/4$), asymptotic expressions apply and the integral equation can be written in the form:

$$A + Bz' = \int_{-D}^{+D} (C + D \ln|z - z'|) \, E_z(r_2,z) \cdot dz , \qquad (9)$$

where A, B, C, and D are constants. Using the known integral (7), the solution is found to be the electrostatic one, and by identifying terms in this solution we can find, for an equivalent humped RLC resonator:

$$L = (\mu_0 \ell/2\pi) \log_e (r_3/r_2) ;$$

$$R = (1/2\pi\sigma\delta)[2 \log_e (r_3/r_2) + (\ell - 2D)/r_2 + \ell/r_3] ;$$

$$C = -2r_2 \varepsilon_0 \ln(\pi D/2r_2) - 2r_2 \varepsilon_0 \ln[\pi D/2(r_3 - r_2)] , \qquad (10)$$

for the combination of gap and cavity. We immediately recognize L and R as the low-frequency inductance and series resistance of the cavity, and C can be taken to be the capacitance of the gap.

7. Conclusions

An efficient method of splitting up a complex boundary value problem has been described. The rapid convergence -- tested by the reduction method -- of the method in this case has rested on the identification and compensation of the singularities in the solution. Work still to be done involves the extension of the method to other cases and a more complete treatment of the losses with mode coupling.

Acknowledgement

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