THE USE OF RF-KNOCKOUT FOR DETERMINATION OF THE CHARACTERISTICS OF THE TRANSVERSE COHERENT INSTABILITY OF AN INTENSE BEAM*

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Abstract

The response of an intense beam of interacting particles to a deflecting rf-signal is computed theoretically and shown to be closely related to transverse coherent beam stability. It is shown that the beam response to sinusoidal excitation provides a direct measure of the stability of beam modes for given machine conditions (beam intensity, octupole current, sextupole current, momentum spread, etc.). This measurement includes the properties of the beam surroundings as well as the frequency spread effective for Landau damping. Since it is generally difficult to evaluate theoretically the wall and beam properties that enter into stability calculations, the information which can be obtained from rf excitation experiments should be very valuable; especially in devising practical procedures for reducing the severity of coherent transverse instabilities.

Introduction

Transverse coherent beam instabilities[1] have been observed in virtually all high intensity accelerators and storage rings. The theory of these instabilities is well established. However, it is generally difficult to make an accurate estimate of the wall and beam properties that enter into the theory. This difficulty results from the complex nature of the beam environment.

The present note gives an analysis of a technique by which transverse stability as a function of beam and wall properties can be measured. For simplicity we restrict the treatment to the case of dipole oscillations of a single-species beam. The basic idea is to observe the response of a beam to a deflecting rf field. This technique was first used by the MURA group[2] several years before the detailed nature of the instability was understood. More recently, similar experiments have been performed on the Bevatron[3]. The MURA group[2] also gave a simplified analysis of the method, based on a single particle dynamics.

Although the model considered by the MURA group explains some of the important features of the instability, it does not include the interaction of the particles through both local and wake fields, nor does it give a quantitative description of the effect of Landau damping. The present note gives an analysis of beam response to a driving force in the presence of both self-field interaction and frequency spread. We find that a single analytic function (of complex frequency) enters into the dispersion relation that determines beam stability and into the response function that describes a driven beam oscillation. Thus analytic continuation permits, at least in principle, the determination from measured data on beam response to a driving force, of all the relevant parameters describing beam stability.

Alternatively, one can regard a beam as a harmonic oscillator with a frequency-dependent "damping constant". This damping constant, which results from Landau damping and wake-field anti-damping, when evaluated at the frequency of a natural mode of the system, is a measure of stability. The response of a beam to a driving force provides a measure of the damping constant at neighboring frequencies and by analytic continuation the damping at the mode-frequency can be obtained. Similarly, other information such as the stability coefficients U, V and the frequency spread ΔS, [see Ref. (1)], can be deduced from the response function.

1. Equations of Motion

The equation of motion of the ith particle may, in linear approximation, be written as

$$d^2x_i/dt^2 + Q_i^2\Omega_i^2x_i + A\,x_i = -B\,\bar{x} + G\,\exp(-i\omega_{rf}t) \tag{1.1}$$

where x_i is the position of the ith beam particle and $\bar{x} = (\Sigma\, x_i)/N$ is the position of the beam center of mass.

Space-charge forces acting between the particles are described by A and B. Only linear space-charge forces are included, so that higher order terms in x_i and $\bar{x}$ are neglected. Actually, the term $B\bar{x}$ contains both the local space-charge field as well as wake fields left by particles which are located at a different azimuthal position in the beam. However, for the coherent oscillation, $\bar{x}$ is the same at every azimuthal position except for a phase factor. We take the influence of this phase factor to be included in B. The quantity $Q_i^2\,\Omega_i^2\,x_i$ represents the external focusing (Ω_i is the ith particle revolution frequency). The action of the knock-out electrode on particle i is described by the G term [see Ref. (4)]. Only the harmonic of the electrode field with

$$\omega_{rf} \approx (n \pm Q_o)\Omega_o$$

is retained (Ω_o is the average revolution frequency and Q_o is the small-amplitude tune of a particle of average energy). The time derivative

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occurring in (1.1) is the "hydrodynamic derivative"

$$d/dt = \partial/\partial t + \Omega_i \, \partial/\partial\theta. \qquad (1.2)$$

The coefficients A and B can be interpreted in terms of the familiar coherent and incoherent frequency shifts[5], and also in terms of the stability coefficient[1] $U + V + iV$. The relevant relations, which were first obtained by L. J. Laslett (private communication), are derived in Ref. (4) and summarized in Table 1. We note that the "single particle frequency shift" A is real, whereas the "coherent shift" B includes contributions from resistive walls and, in general, is a complex quantity and different for different modes.

TABLE 1

Relation between the quantities A and B [Eq. (1.1)]; the Laslett Q-shifts ΔQ_{ic}, ΔQ_c; and the LNS-coefficients U and V.

$U + V + iV = - B/2\Omega_o Q_o$:	LNS-stability coefficient.
$\Delta Q = - (A + B)/2\Omega_o^2 Q_o$:	Coherent betatron frequency depression due to space-charge (coherent Laslett Q-shift).
$\Delta Q_{ic} = - A/2\Omega_o^2 Q_o$:	Incoherent Q-shift due to space-charge (single-particle Q-shift).

By combining these relations,

$$U + V + iV = \Omega_o(\Delta Q_c - \Delta Q_{ic}).$$

The above ΔQ_c includes in-phase (resistive) components of the self-field.

2. Solution of the Basic Equation

In solving (1.1) we must take the effect of frequency spread into account. The case in which this spread is due to momentum spread is simple and will be considered first. The response of the l.h.s. of (1.1) to a driving force $G \exp(-i\omega t)$ is simply

$$x_i = \frac{G \exp(- i\omega t)}{Q_i^2\Omega_i^2 + A - (\omega - n\Omega_i)^2} . \qquad (2.1)$$

Therefore, following the procedure first outlined by E. Courant[6], we insert a trial solution

$$x_i = G\, f_i \exp(-i\omega_{rf}t) , \qquad (2.2)$$

$$\bar{x} = G\, F \exp(-i\omega_{rf}t) .$$

The complex functions $f_i(\omega_{rf})$ and $F(\omega_{rf})$ include the phases of the oscillation and we assume that A and B are independent of frequency, for frequencies near the mode frequency.

Using the fact that

$$\bar{x} = (\Sigma\, x_i)/N \simeq \int n(p)x(p)dp , \qquad (2.3)$$

we find the beam response

$$F(\omega_{rf}) = (B + 1/I_p)^{-1} \qquad (2.4)$$

where

$$I_p(\omega_{rf}) = \int_0^\infty \frac{n(p)dp}{(Q\Omega)^2 + A - (\omega_{rf} - n\Omega)^2} . \qquad (2.5)$$

The function $n(p)$ is the energy distribution function of the particles, and

$$\int_0^\infty n(p)dp = 1.$$

Both Q and Ω are functions of p.

Equations (2.2), (2.4), and (2.5) describe the response of the beam. In addition to the particular solution (2.2) the oscillation of a particle contains the free betatron oscillation which is of random phase and therefore does not contribute to the average motion (2.3). If the beam is unstable there will be growing collective oscillations at the frequencies of the beam normal modes. These terms don't contribute to the response at frequency ω_{rf}, although they usually would preclude observation of beam response to the rf. This point is elaborated upon in Sec. 5 .

Next, we proceed to include the frequency spread due to nonlinearities in the external focusing. It was pointed out by Hereward[7] that in this case the response of the l.h.s. of (1.1) to a driving force $G \exp(-i\omega t)$ is

$$x_i = \frac{1 - K}{(Q\Omega)^2 + A - (\omega - n\Omega)^2}$$

$$- K \frac{(Q\Omega)^2 + A + (\omega - n\Omega)^2}{[(Q\Omega)^2 + A - (\omega - n\Omega)^2]^2} \; G \exp(-i\omega_{rf} t). \tag{2.6}$$

Here $K = \frac{1}{2} \frac{a}{Q\Omega} \frac{d(Q\Omega)}{da}$ is determined by the amplitude-dependence of the external betatron frequencies $Q\Omega$. Equation (2.6) is correct to first order in K and G.

Using (2.6) we obtain in a similar fashion as was used to derive (2.4), [see Ref.(7)]:

$$F(\omega_{rf}) = (B + 1/I_a)^{-1} \tag{2.4a}$$

where

$$I_a(\omega_{rf}) \approx - \int_0^\infty \frac{g'(a)\, a^2\, Q\Omega\, da}{(Q\Omega)^2 + A - (\omega_{rf} - n\Omega)^2}, \tag{2.5a}$$

Here, $g(a)$ is the amplitude distribution function of the particles, and we have normalized $g(a)$ such that

$$\int_0^\infty g(a)\, d(a^2\, Q\Omega) = 1.$$

Both Q and Ω are functions of a.

Finally we consider the combined effect of momentum spread and nonlinearity. We first consider a group of particles with the same momentum but of different betatron amplitude a_i ; and thereafter we sum over all groups. If there is no correlation between betatron amplitude and momentum of the particles we may write

$$F(\omega_{rf}) = (B + 1/I)^{-1} \tag{2.4b}$$

where now

$$I(\omega_{rf}) \approx - \int_0^\infty \frac{g'(a)\, a^2\, Q\, \Omega\, n(p)\, da\, dp}{(Q\Omega)^2 + A - (\omega_{rf} - n\Omega)^2} \tag{2.5b}$$

$$\int_0^\infty n(p) dp = 1$$

$$\int_0^\infty g(a)\, d(a^2\, Q\Omega) = 1.$$

Now Q and Ω are functions of a and p.

The observation which forms the basis for this paper is that $I(\omega_{rf})$, Eq. (2.5b), is the same dispersion integral that appears in the LNS theory[1] and is used there to determine the complex mode-frequency ω_c from the relation*

$$1/I(\omega_c) = 2\, Q_o \Omega_o (U + V + iV). \tag{2.6}$$

In the present notation, we write (2.6) as

$$B + 1/I(\omega_c) \equiv 1/F(\omega_c) = 0. \tag{2.7}$$

3. Measurement

3.1 Sinusoidal Excitation

The fact that beam response (2.4b) and mode stability (2.7) are governed by the same function $F(\omega)$ suggests determining the mode-frequency ω_c by analytic continuation of the function $F(\omega_{rf})^c$ as measured for real frequencies (ω_{rf}).

To elucidate the procedure, let us introduce the inverse of $F(\omega)$:

$$X(\omega) = 1/F(\omega).$$

Now, because $X(\omega)$ [as defined by (2.4b), (2.5b), and (3.1)] is analytic, we can expand around some frequency ω_1 to obtain $X(\omega_c)$:

$$X(\omega_c) = X(\omega_1) + \sum_{n=1}^{\infty} \frac{X^{(n)}(\omega_1)}{n!} (\omega_c - \omega_1)^n. \tag{3.2}$$

The quantities $X(\omega_1)$ and $X^{(n)}(\omega_1)$ can be determined for ω_1 real from the measured response curve $X(\omega)$ and hence ω_c can be determined from (2.7) and (3.2).

If ω_1 is close to ω_c we can neglect higher order terms in (3.2) and obtain, from (2.7),

$$(\omega_c - \omega_1) = - X(\omega_1)/X'(\omega_1). \tag{3.3}$$

Of great interest is the imaginary part $\mathrm{Im}(\omega_c)$ which is a direct measure of the mode stability. This quantity can, for example, be deduced from the phase response

$$\alpha(\omega) = \tan^{-1} \left\{ \frac{\mathrm{Re}[X(\omega)]}{\mathrm{Im}[X(\omega)]} \right\}. \tag{3.4}$$

Thus, let us assume that we measure the slope of the quantity $\tan \alpha$

$$s = d(\tan \alpha)/d\omega \tag{3.5}$$

at a frequency ω_a where $d\, \mathrm{Im}[X(\omega)]/d\omega = 0$. Then we have, by virtue of (3.3), (3.4), and (3.5),

$$s(\omega_a) = \mathrm{Re}[X'(\omega_a)]/\mathrm{Im}[X(\omega_a)] \tag{3.6}$$

and

$$\mathrm{Im}(\omega_c) \approx - 1/s(\omega_a). \tag{3.7}$$

In other words, the slope $s(\omega_a)$ is a direct measure of beam stability.

The evaluation of measured data is simplified if the functional form of $X(\omega)$ is known; $X(\omega)$ in turn is determined by the dispersion integral (2.5b). This integral is evaluated for various distribution functions in [Refs. (1) and (8)]. Results from

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* The "single-particle" shift A does not appear explicitly in Ref. (1) but is incorporated into the ν-value. The function $h(a)$ of Ref. (1) is related to $g(a)$ used above by $h' = 2g'Q\Omega$ and the a- and p-dependence of $Q\Omega$ is neglected in the numerator of the dispersion integral.

some representative examples are presented in Ref. (4).

In addition to the effective damping (3.7) the unknown quantities A, Re(B), Im(B), and ΔS can be deduced from the measured response curve $F(\omega_{rf})$ if we can anticipate the shape of the distribution function. As an example, these four quantities can be determined if we measure ω_a, $X(\omega_a)$, a frequency ω_b where

$$\mathrm{Re}[X(\omega_b)] = 0 \quad \text{and} \quad X(\omega_b).$$

A detailed example may be found in Ref. (4).

3.2 Pulse Excitation

An alternative measuring technique which can be used is based on pulse exciting the beam and observing the transient behavior of the modes[2]. Excitation of a given mode may be accentuated by choice of the pulse waveform. The transient behavior of a mode is clearly

$$\bar{x} = \bar{x}(0)\exp(-i\omega_c t)$$

with ω_c the mode frequency determined by (2.7). The decay rate of the transient, $\mathrm{Im}(\omega_c)$, is a direct measure of the "effective damping".

4. Bunched Beams

The generalization to a machine with equally shaped, equally spaced and equally populated bunches[9] is straightforward. The same measuring techniques that were discussed for a coasting beam can be used in this case to measure the stability of "coherent bunch modes."

The other limiting case, where the bunch-to-bunch spread is large enough to decouple the bunches[9] will need a somewhat modified measuring method. Since the bunches are largely decoupled, each bunch will resonante at a slightly different frequency. By observing the response of a bunch in the neighborhood of its resonance we may measure the "single bunch stability." At the same time the bunch-to-bunch frequency spread can be detected.

5. Discussion

The quantities $1/s(\omega_a)$ or $-\mathrm{Im}(\omega_c)$, which can be measured as described in Sec. 3, are measures of the effective stability of the mode under consideration. Thus, by measuring these quantities as a function of relevant machine parameters, such as intensity, octupole current, energy spread and wall properties, one can predict threshold conditions $[1/s(\omega_a) \to 0,\ \mathrm{Im}(\omega_c) \to 0]$ and presumably thus devise procedures for reducing the instability.

These measurements can only be performed in an intensity range such that the machine is stable since otherwise the driven response will be masked by spontaneously growing (or stimulated) coherent modes. However, measured data can be extrapolated to the threshold. If the wall impedances are not strongly frequency-dependent, one can often make measurements near the stable modes and extrapolate from there to the frequencies of the unstable modes, a technique employed in both Refs. (2) and (3).

It is noted that the measurement of $1/s(\omega_a)$ or $\mathrm{Im}(\omega_c)$ will not give explicit information on the values of U and V, but rather a quantity related to $V - \Delta S$. However, U and V are only of interest for calculating the effective beam stability and this quantity is directly obtained from the measurement. If required, U, V, and ΔS can also be derived from the beam response curve, as described in Sec. 3.1.

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