ON SYNTHESIS OF AIR-CORE MAGNETIC SYSTEMS FOR SUPERCONDUCTING ACCELERATORS

V.S. Voronin

P.N.Lebedev Physical Institute, Academy of Sciences, Moscow, USSR

Abstract

A method is described to find configurations of surface currents, creating a given magnetic field. The method was used to find winding configurations for superconducting magnetic systems of various accelerators.

1. Introduction.

One of the key problems in designing air-core magnetic systems for accelerators is the choice of winding configuration, creating a given magnetic field in a certain working region (aperture). The general method for solving this problem on a computer is step-by-step refinement of some initial current configuration, Biot-Savart's formula being used for field calculations[1]. This method requires a great amount of computer time and strongly depends on the choice of initial approximation.

Below there is described a simplified method of synthesizing a current configuration for a given magnetic field requiring much less computer time and in some cases enables one to obtain an analytical solution. In any case, this method may be used for a preliminary choice of appropriate configuration. To simplify the problem, we use the approximation of an infinitely thin current layer (current surface) which is justified by the possibility of realizing relatively thin superconducting windings having large current density.

2. Synthesis of surface current configurations.

Let us consider a current surface S, surrounding some region, in which the given magnetic field is to be created. For any harmonic internal field B, on the surface there exists, in principle, a unique current distribution creating this field. Any distribution of surface current may be described in terms of magnetic moment distribution m, the current density being expressed by the formula

$$\bar{i}(P) = c\,\mathrm{Rot}\; m(P),\quad P \in S. \tag{1}$$

The meaning of (1) is simple: current lines are lines on which m = const. The magnetic field in the internal region, where current is absent, may be expressed in terms of a scalar potential: $\bar{B} = \mathrm{grad}V$. The density of magnetic moment distribution m is related to the boundary values of the internal field potential by the expression

$$-2\pi m(P) + \int_S \frac{\cos(\bar{r}_{PQ},\bar{n}_Q)}{r_{PQ}^2}\, m(Q)\,dS_Q = V(P). \tag{2}$$

This expression may be considered as an equation from which one can find a current configuration creating the magnetic field with given potential V. This integral equation, two-dimensional in the general case, may be reduced to one- dimensional equations for current surfaces of axial or cylindrical symmetry.

3. Axial symmetry.

In many cases, for magnetic systems of accelerators, we may take a current surface having rotational symmetry (for example, a spheroid or toroid). Then, using the Fourier expansion about ϕ (r,ϕ,z - cylindrical coordinate system)

$$V(r,\phi,z)=\sum V_N(r,z)e^{iN\phi},\quad m=\sum m_N e^{iN\phi} \tag{3}$$

we can obtain from (2) one-dimensional integral equations for the amplitudes of the magnetic moment harmonics m_N:

$$\int_L \left[\frac{(r-\tilde{r})n_{\tilde{r}}+(z-\tilde{z})n_{\tilde{z}}}{(r-\tilde{r})^2+(z-\tilde{z})^2}2F_N-\frac{rn_{\tilde{r}}(2F_N-F_{N+1}-F_{N-1})}{v^2((r+\tilde{r})^2+(z-\tilde{z})^2)}\right]\times \tilde{r}^{1/2}m_N(\tilde{r},\tilde{z})d\tilde{s}\; -2\pi r^{1/2}m_N(r,z)= V_N(r,z)r^{1/2}. \tag{4}$$

The contour L ($r=r(s), z=z(s)$) is the cross-section of the surface by a plane ϕ =const. The functions $F_N(v)$ appearing in equations (4) are elliptic-type integrals, expressed by

$$F_N(v)=(1-v^2)^{1/2}\int_0^{\pi/2}\frac{(-1)^N v^2\cos(2Nt)\,dt}{(1-(1-v^2)\sin^2 t)^{3/2}} \tag{5}$$

where

$$v^2 = \frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{(r+\tilde{r})^2+(z-\tilde{z})^2} \tag{6}$$

(On the calculation of these functions see Appendix). The case N = 0 corresponds to an axial-symmetry field.

4. Modulated two-dimensional approximation.

The second analogous case is that of a cylindrical current surface, the generating line of which is along the y-axis of the cartesian coordinate system. Such a surface may be considered as the limit of a toroid whose larger radius increases to infinity, while the shape of its cross-section remains unaltered. In this case for every harmonic component having the wave number k,

$$V = V_k(x,z)\exp(iky), \quad m = m_k\exp(iky),$$

one can also obtain a one-dimensional equation

$$2\int_L \frac{(x-\tilde{x})n_{\tilde{x}}+(z-\tilde{z})n_{\tilde{z}}}{(x-\tilde{x})^2+(z-\tilde{z})^2}\, k_1(ku)m_k(\tilde{x},\tilde{z})d\tilde{s} - -2\pi m_k = V_k. \qquad (7)$$

Here the contour L ($x=x(t), z=z(t)$) is the cross-section of the surface by a plane $y=\text{const}$; $u^2=(x-\tilde{x})^2+(z-\tilde{z})^2$; $k_1(x) = xK_1(x)$ (K_1 is the modified Bessel function). The case $k=0$ corresponds to a two-dimensional field. The general case may be described as a modulated two-dimensional field.

The equations (4) and (7) can be solved on a computer by known methods. Computer time required is small, so one can evaluate many current surfaces. Below we will only consider some cases, when analytical solutions are possible.

5. Modulated two-dimensional field in circular aperture.

Let us take the contour L in the equation (7) in the form of a circle of radius a: $x = a\cdot\cos(t)$, $z = a\cdot\sin(t)$. Then its solution can be obtained in the form of multipolar expansion

$$-2\pi m_k(t) = V_k(t) - \sum_{n=1}^{\infty} c_n(ka)V_{kn}\sin(nt), \qquad (8)$$

where V_{kn} are the multipolar coefficients for boundary potential amplitudes of internal field,

$$V_k(t) = \sum_{n=1}^{\infty} V_{kn}\sin(nt),$$

and $c_n(x) = 1-(2I_n(x)(nK_n(x)+xK_{n-1}(x)))^{-1}$

(I_n and K_n are the modified Bessel functions). We have taken only median-plane fields for simplicity.

6. Finite-length multipole magnet.

When designing multipole magnets the common practice is to use a two-dimensional approximation in which the magnet is assumed to be of infinite length. A real multipole has finite length and the winding conductors are connected to each other in some way at its ends. Let us consider a finite-length multipole with a circular aperture. In an accelerator, the main characteristic of a short multipole is the mean value of the magnetic field along the y-axis. From formula (8) one can see that the dependence of the mean field on the transverse coordinates x,z is determined by the dependence on t of magnetic moment density, averaged over y:

$$m_o(t) = L_{eff}^{-1}\int_{-\infty}^{\infty} m(t,y)dy \qquad (9)$$

The dependence of $m_o(t)$ on t must be multipolar: $m_o \sim \sin(nt)$. This requirement may be satisfied by the following magnetic moment distribution

$$m(t,y) = m_n(y)\sin(nt) \qquad (10)$$

where $m_n(y)$ is the magnetic moment form factor. The formula (10) gives the coil configuration of a short multipole magnet having an undistorted mean field.

To calculate the magnetic field created by the distribution (10) we will use the integral in formula (4) since a circular cylinder is also a rotational surface. Using the symbols of par.3 we obtain that the distribution of magnetic moment on a cylinder $r = a$,

$$m(\phi,z) = m_N(z)\cos(N\phi), \qquad (11)$$

creates a magnetic field with the potential

$$V(r,\phi,z) = V_N(r,z)\cos(N\phi). \qquad (12)$$

For the region close to the multipole axis, calculation of the integral gives the result

$$V_N(r,z)_{r\ll a} = -f_N(z)(r/a)^N. \qquad (13)$$

Here the form factor of near-axial field is given by

$$f_N(z) = A\int_{-\infty}^{\infty} \frac{a^2(N+1)/N - (z-\tilde{z})^2}{(a^2+(z-\tilde{z})^2)^{N+3/2}}\, m_N(\tilde{z})d\tilde{z} \qquad (14)$$

where $A = \pi a^{2N}(2N-1)!!/(2N-2)!!$

Let us consider the example: multipole magnet with a linear decrease of magnetic moment amplitude at the edges, i,e. (Fig.1)

$$m_N(z) = (L_2-L_1)^{-1}\begin{cases} L_2+z, & -L_2<z<-L_1, \\ L_2-L_1, & -L_1<z<L_1, \\ L_2-z, & L_1<z<L_2. \end{cases}$$

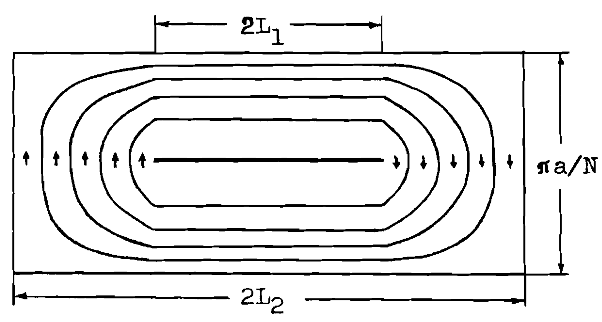

Fig.1. One-pole coil of a short multipole magnet.

For a dipole (N=1), the form factor of near-axial field will be equal to

$$f_1(z) = (L_2-L_1)^{-1}(P(L_2+z)+P(L_2-z)-P(L_1+z)-$$
$$-P(L_1-z)) \text{ where } 2P(x)= x^2(a^2+x^2)^{-1/2}. \qquad (15)$$

It is evident that the same formulae give the 'bottoming' of the field in the region between two similar multipole magnets having large lengths and a linear decrease of magnetic moment at their ends. It is to be noted that integration over parameter a between the limits a and b (i.e. evaluation of $\int f_N(z,a)ada$) will give the form factor of near-axial field for a multipole magnet having a finite thickness of winding.

7. A thin current disk.

Let us take the current surface in the form of double-sheet disk, which we may consider as the limiting case for a spheroid, the vertical aperture of which approaches zero. If the current is divided equally between the sheets, the field between them (in a working region of zero height) will have only a vertical component. Equation (4) becomes meaningless in this case, but considering the limit of zero vertical aperture, we obtain the formula for calculating the magnetic moment distribution creating the magnetic field with amplitude $B_N(r)$ of N-th azimuthal harmonic:

$$-m_N(r)=\int_{r/a}^{1}\frac{a(r/a)^N t^{1-N}dt}{(t^2-r^2/a^2)^{1/2}}\int_0^1\frac{s^{1+N}B_N(ast)ds}{\pi^2(1-s^2)^{1/2}} \qquad (16)$$

For a field with amplitude

$$B_N(r) = B_o(r/a)^q \qquad (17)$$

(complex values of field index q correspond to spiral field) we obtain from (16)

$$-m_N(r) = Q(r/a)^N(a^2-r^2)^{1/2}F(\frac{N-q}{2},1;\frac{3}{2};1-r^2/a^2)$$
$$\text{where } Q = \frac{1}{2}B_o\pi^{-3/2}\,G(\frac{N+q+2}{2})/G(\frac{N+q+3}{2}) \qquad (18)$$

(F(a,b;c;x) is the hypergeometric function, G(x) is the gamma function).

The distribution (18) gives infinite current density at the edge of the disk. To eliminate this shortcoming, we will take a superposition of such distributions with the radius a of the disk varying between a and b; f(ã) is the weighting function. The field in the region $r < a$ will be again of the form (17); the region $a < r < b$ will play the role of a 'protective ring' providing finite current density. For $f(x) \sim x^{N-4}$, we obtain

$$-m_N(r)= D(r/a)^{N-3}\cdot\begin{cases}M(r/b)-M(r/a), & r < a,\\ M(r/b), & a < r < b,\end{cases} \qquad (19)$$

where $D = \frac{a}{3}Q(N-q-3)/((b/a)^{N-q-3}-1)$,

$$M(x) = (1-x^2)^{3/2}F(\frac{N-q}{2},1;\frac{5}{2};1-x^2).$$

In the case of a uniform field (N=q=0) with $f(x) \sim x^{-1}(b^2-x^2)^{-1/2}$, we obtain a current distribution with constant current density in the protective ring:

$$i_\phi(r)= \frac{2\pi B_o/c}{ch^{-1}(b/a)}\begin{cases}\frac{2}{\pi}\sin^{-1}\left[\frac{r}{a}\left(\frac{b^2-a^2}{b^2-r^2}\right)^{1/2}\right], & r < a,\\ 1, & a < r < b.\end{cases} \qquad (20)$$

Appendix

The functions $F_N(v)$ defined by (5) are expressed in terms of hypergeometric function:

$$F_N(v)= A\frac{(1-v^2)^{N+1/2}}{2^{2N+1}}F(N-1/2,N+1/2;2N+1;1-v^2) \qquad (21)$$

where $A = \pi(2N+1)!!/(2N)!!$

By applying the formulae for conversion and analytical continuation of the hypergeometric function, we obtain a series which is asymptotic with respect to N

$$F_N(v)= \frac{Av^{1/2}(1-v)^{N+1/2}}{2(1+v)^{N-1/2}}F(-\frac{1}{2},\frac{3}{2};N+1;-\frac{(1-v)^2}{4v}) \qquad (22)$$

and the series about the point v=0

$$F_N(v)=(1-v^2)^{N+1/2}(1-\sum_{n=0}^{\infty}D_n(h_n-2\ln(v))v^{2n+2}) \qquad (23)$$

where $D_n=(N-1/2)_{n+1}(N+1/2)_{n+1}/(n!(n+1)!)$,

$h_n= g(1+n)+g(2+n)-g(n+N+1/2)-g(n+N+3/2)$,

g(x) is the logarithmic derivative of the gamma function.

The series (22) is convenient for calculations in the region $v > 2.5/(N+5)$ where 9 terms give the accuracy of 10^{-5}; in the region $v < 2.5/(N+5)$ the series (23) must be applied, and no more than 11 terms are needed for an accuracy of 10^{-5}.

References

1. Proceedings of the 1968 Summer Study on Superconducting Devices and Accelerators, BNL, New York.