EFFECT OF NONLINEAR COUPLING ON RESONANT VERTICAL BEAM EXTRACTION

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Abstract

Nonlinearity of the magnets in a synchrotron limits the amplitude that can be reached in third-integral resonant beam extraction. This limit is calculated for the case of extraction out of the median plane when the principal magnet nonlinearity is symmetric in the median plane. The method used is the construction of the appropriate canonical invariants in four-dimensional phase space, using a version of the formalism of Moser, Hagedorn and Schoch.

The effectiveness of resonant beam extraction at a third-integral resonance from a circular accelerator is limited by distortions of the phase-space trajectory from the ideal shape it would have if the guide and focusing fields were purely linear except for the component exciting the desired resonance.

If extraction is to take place in the median plane, the nonlinearity needed to excite the resonance has median plane symmetry, as do the unwanted nonlinearities that distort the phase space. Then the details of the extraction process can be obtained by solving the equations of motion in the median plane only. This case has been analyzed previously [1].

In some cases, however, it may be desirable to extract the beam out of the median plane (vertically). For example, if the synchrotron magnets are superconducting, a preferred magnet design provides equal horizontal and vertical apertures, while the beam in a synchrotron usually requires more horizontal than vertical aperture. It is then advantageous to utilize the "excess" vertical aperture for extraction. This design philosophy is followed in the Cold Magnet Synchrotron (CMS) study [2] undertaken at Brookhaven National Laboratory.

To analyze the extraction process for this case, we must treat the coupled two-dimensional equations of motion. We consider a Hamiltonian of the form

$$H_o(p_x,p_y,x,y,\theta) = \tfrac{1}{2}\left[p_x{}^2 + p_y{}^2 + R^2K_1(\theta)x^2 + R^2K_2(\theta)y^2\right] + \tfrac{1}{3}\,E(\theta)(y^3 - 3x^2y) + \tfrac{1}{3}\,F(\theta)(x^3 - 3xy^2) + \tfrac{1}{4}\,D(\theta)(x^4 - 6x^2y^2 + y^4) \qquad (1)$$

where $2\pi R$ is the orbit circumference, $\theta = s/R$ (s = distance along the equilibrium orbit), K_1 and K_2 are the linear focusing functions, E and F represent the sextupole components of the field that are, respectively, antisymmetric and symmetric in the median plane, and D represents the octupole components.

We transform to angle and action variables J_1, J_2, ϕ_1, ϕ_2 defined in terms of the solutions (assumed known) of the linearized problem (i.e., with $E = F = D = 0$) :

$$x_i = (2J_i\beta_i/R)^{\frac{1}{2}} \cos(\phi_i + \psi_i(\theta)) \qquad (2)$$

$$p_i = -\,(2J_iR/\beta_i)^{\frac{1}{2}} \left[\sin(\phi_i + \psi_i) + \alpha_i \cos(\phi_i + \psi_i)\right] \qquad (3)$$

Here i = 1, 2 refer to x and y coordinates; β_i, α_i, ψ_i, are the Courant-Snyder parameters [3] of the linear solution which are in the form

$$x_i = \beta_i^{\frac{1}{2}} \exp \pm i \left[\nu_i \theta + \psi_i\right] \qquad (4)$$

where ν_i are the betatron tunes, and β_i, α_i, ψ_i are periodic functions of θ satisfying

$$\frac{d\beta_i}{d\theta} = -\,2R\alpha_i \quad , \quad \frac{d\psi_i}{d\theta} = \frac{R}{\beta_i} - \nu_i \qquad (5)$$

In terms of the new variables the motion is determined by the Hamiltonian

$$H(J,\phi,\theta) = \nu_1J_1 + \nu_2J_2 + S(J_1,\phi_1,J_2,\phi_2,\theta) + T(J_1,\phi_1,J_2,\phi_2,\theta) \qquad (6)$$

where S and T equal the terms in E, F and D from (1) expressed in terms of the new variables; S and T are, respectively, third and fourth degree polynomials in $J_1^{\frac{1}{2}}$ and $J_2^{\frac{1}{2}}$ and are periodic in the angles ϕ_1,ϕ_2 and θ.

We now follow a modification of the procedure by Moser [4] as elaborated by Hagedorn and Schoch [5]. The procedure is to find canonical transformations which reduce the Hamiltonian to a form which does not depend explicitly on θ; such a transformed Hamiltonian is constant and, when transformed back to the original variables, yields invariant trajectories in phase space. For resonant beam extraction one has to arrange matters so that one of these invariants corresponds to a rapidly increasing amplitude in the y direction.

Instead of carrying out the transformations explicitly, we may look for the invariants directly. We seek functions

$$G(J_1,\phi_1,J_2,\phi_2,\theta)$$

which are invariant when J_i, ϕ_i follow the equations of motion. This was done by us for the one-dimensional case in a previous paper [1]; here we shall use a slightly different formalism.

According to the principles of Hamiltonian mechanics, the condition for invariance of G is

$$\frac{dG}{d\theta} = \frac{\partial G}{\partial \theta} + [G,H] = 0 \qquad (7)$$

where $[G,H]$ is the Poisson bracket, defined for any two functions U and V by

$$[U,V] = \sum_i \left[\frac{\partial U}{\partial \phi_i}\frac{\partial V}{\partial J_i} - \frac{\partial U}{\partial J_i}\frac{\partial V}{\partial \phi_i}\right]$$

We now write G in the form

$$G = \alpha J_1 + \beta J_2 + A + B + C \ldots \qquad (8)$$

where α and β are constants, and A, B, C, ... are, respectively, polynomials of degree 3, 4, 5, ... in $J_1^{\frac{1}{2}}$ and $J_2^{\frac{1}{2}}$ combined, and are periodic in the angles θ, ϕ_i.

By choosing $\alpha = 0$ or $\beta = 0$ in (8) we expect to find two different invariants (reducing to J_2 and J_1 in the absence of nonlinearities). We write S, T, A, B, C, ... as power series in $J_1^{\frac{1}{2}}$ and $J_2^{\frac{1}{2}}$ and Fourier series in the angles with appropriately labeled coefficients; for example

$$S^{n,m}_{k,r,s}$$

is the coefficient of $J_1^{m/2} J_2^{n/2} e^{i(k\theta + r\phi_1 + s\phi_2)}$ in S. In the third degree functions S and A, n + m = 3; in B and T, n + m = 4; in C, n + m = 5. We shall omit the superscript m in what follows.

Equation (7) then separates into the sequence

$$A_\theta + \nu_1 A_{\phi_1} + \nu_2 A_{\phi_2} = \alpha S_{\phi_1} + \beta S_{\phi_2} \qquad (9a)$$

$$B_\theta + \nu_1 B_{\phi_1} + \nu_2 B_{\phi_2} = [S,A] + \alpha T_{\phi_1} + \beta T_{\phi_2} \qquad (9b)$$

$$C_\theta + \nu_1 C_{\phi_1} + \nu_2 C_{\phi_2} = [S,B] + [T,A] \qquad (9c)$$

where subscripts denote partial differentiation.

We require solutions A, B, C, ... having the following properties :

1. A, B, C, ... are periodic functions of the three angles.

2. Because of the physical meaning of J and ϕ, a term of order $J_1^{p/2} J_2^{q/2}$ contains only Fourier components in ϕ_1, ϕ_2 with $|r| \leqslant p$ and $|s| \leqslant q$, with r even or odd as p is even or odd, and s even or odd as q is even or odd. (S and T also have this property.)

3. As the resonance $\nu_2 \to M/3$ is approached A, B, C ... must not become infinite. This, as we shall see, imposes a condition on β and on certain constants of integration.

The solutions are obtained in an obvious way by equating Fourier coefficients. In particular, the solution of (9a) is

$$(k + \nu_1 r + \nu_2 s)\, A^n_{krs} = (\alpha r + \beta s)\, S^n_{krs} \qquad (10)$$

If we put $\alpha = 0$ in (10), i.e., we look for the invariant responsible for extraction, we note that with $k = \pm M$, $r = 0$, $s = \mp 3$ the coefficient of A_{krs} goes to zero as the resonance $\nu_2 = M/3$ is approached. To keep A finite (condition 3 above) we must make β go to zero with $\varepsilon = \nu_2 - M/3$. We choose $\beta = \varepsilon$. Since the only coefficients in S with $s = \pm 3$ have $n = 3$ (condition 2), this gives

$$A = J_2^{3/2}\left[S^3_{M,0,-3} e^{i(M\theta - 3\phi_2)} + \text{comp. conj.}\right] + \varepsilon A' \qquad (11)$$

where the last term goes to zero as $\varepsilon \to 0$.

If we stop at this point, we obtain the well-known extraction invariant leading to a triangular separatrix in (y, p_y) phase space (if the small term $\varepsilon A'$ is neglected). The value of the coefficient $S^3_{m,0,-3}$ is related to the rate at which the amplitude of oscillations increases during extraction; it is easily verified that, if phase space distortion is neglected,

$$E_M = 2\left[S_{M,0,-3}\right] = \frac{(2\beta_2)^{\frac{1}{2}}}{9\pi}\ \frac{\Delta y}{y_e{}^2} \qquad (12)$$

where β_2 is the value at the extraction azimuth, y_e is the vertical position of the extraction septum, and Δy is the increase in amplitude per three revolutions at amplitude y_e.

To obtain the phase space distortions due to nonlinearities we must solve (9b). We make a few simplifying assumptions : (a) the median plane symmetric terms F are large compared to the symmetry-breaking terms E; thus coefficients S^n_{krs} with odd n are small compared to terms with even n. (b) during the extraction process the vertical amplitude becomes large compared to the horizontal amplitude, i.e., $J_1 \ll J_2$. (c) we are close to the resonance (ε small); therefore, in calculating B we confine ourselves to the resonant case $\varepsilon = 0$, $\nu_2 = M/3$.

Solving (9b) by Fourier analysis gives

$$B^n_{krs} = \frac{3}{2}\left[\frac{n+s+2}{k+\nu_1 r+\nu_2 s} S^{n-1}_{k-M,r,s+3} S^3_{M,0,-3} - \frac{n-s+2}{k+\nu_1 r+\nu_2 s} S^{n-1}_{k+M,r,s-3} S^3_{-M,0,3}\right] \qquad (13)$$

Because of assumption (a) above, the terms with n = 3 predominate, and B is of the order $F E_M J_1^{\frac{1}{2}} J_2^{\frac{3}{2}}$.

The angle-independent terms in B (k = r = s = 0) are left indeterminate by this procedure. To find these we must go to equation (9c) for C and invoke condition 3, namely that C remains finite for $\varepsilon = 0$. Thus for (krs) = ± (M, 0, -3) the right hand side of (9c) must vanish. This condition gives, after considerable labor,

$$B^4_{000} = T^4_{000} - \sum_k \left\{ \frac{\left[S^2_{k,1,0}\right]^2}{k + \nu_1} + \frac{\left[S^2_{k,1,2}\right]^2}{k + \nu_1 + 2\nu_2} - \frac{\left[S^2_{k,-1,2}\right]^2}{k - \nu_1 + 2\nu_2} \right\} \qquad (14)$$

and a similar expression for B^2_{000}. Note that these terms do not involve the resonance-exciting term E_M.

The invariant, to this order, is thus of the form

$$G = \varepsilon J_2 + E_M J_2^{\frac{3}{2}} \cos(M\theta - 3\phi_2 - \delta) + \varepsilon A' + b(\phi_1, \phi_2, \theta)\, J_1^{\frac{1}{2}} J_2^{\frac{3}{2}} + B^2_{000} J_1 J_2 + B^4_{000} J_2^2 \qquad (15)$$

where δ is a phase angle.

The other invariant is obtained in the same way by solving (9) with $\beta = 0$, $\alpha = 1$. Since there are no resonance problems here, the invariant is just equal to J_1 plus terms of degree at least $\frac{3}{2}$ in J_1 and J_2 combined; therefore, if we limit ourselves to terms of up to the second degree we may regard J_1 as a constant parameter as far as the invariant (15) is concerned. Furthermore, because of assumption (a) the term in $b(\phi_1, \phi_2, \theta)$ in (15) is small compared to the B^4_{000} term.

Therefore, to this order, the constancy of (15) means that there is an approximate invariant curve in vertical phase space, of the form

$$G = \varepsilon' J_2 + E_M J_2^{\frac{3}{2}} \cos(M\theta - 3\phi_2 - \delta) + B^4_{000} J_2^2 \qquad (16)$$

with $\varepsilon' = \varepsilon + B^2_{000} J_1$; thus the primary effect of J_1 is a shift in the resonance frequency.

This is formally just like the one-dimensional situation, but here the principal distortion term $B^4_{000} J_2^2$ arises from the nonlinear coupling of horizontal and vertical oscillations. It is just of the same order of magnitude as the corresponding distortion term in the one-dimensional (median plane extraction) case, but depends on a different detailed combination of Fourier components.

For the purposes of beam extraction, the phase space distortions must not prevent the amplitude from reaching the position of the extraction septum. Using the expression (12) for E_M and noting that $J_2 = y_{max}^2/2\beta_2$, we find that the peak amplitude of the extracted beam at resonance is

$$y_{max} = \frac{2\beta_2 \Delta y}{9\pi y_e^2 B^4_{000}} \qquad (17)$$

This must be small compared to y_e; thus we require :

$$B^4_{000} < \frac{2\beta_2 \Delta y}{9\pi y_e^3}$$

To achieve this in an accelerator whose magnets possess appreciable nonlinearity, one may build in a set of compensating sextupole magnets so as to cancel the combination of sextupole Fourier components entering into (14). However, there is a conflicting requirement for compensating sextupoles: it is necessary to control the momentum dependence of ν_1 and ν_2 (chromaticity) in order to ensure that no part of a beam with momentum spread is too close to undesirable resonances, but at the same time to retain a large enough spread in the ν's to provide Landau damping of coherent instabilities. This means that the two integrals

$$\int_0^{2\pi} F(\theta)\beta_i(\theta)\eta(\theta)d\theta; \quad i = 1,2 \qquad (18)$$

should have certain prescribed values, where $\eta(\theta)$ is the "momentum compaction function". This is accomplished with two sets of sextupoles, periodically spaced around the orbit, one set located at azimuths where β_1 is large and the other where β_2 is large. If these are appropriately energized their distribution will possess many high-order Fourier harmonics, and B^4_{000} will still be appreciably different from zero. To make B^4_{000} small enough, we can then energize octupoles. By (14), only the azimuthal mean

$$\int_0^{2\pi} \beta_2^2\, T(\theta)d\theta$$

has to be controlled; thus (at least to the degree of approximation treated here) the azimuthal distribution of octupoles may be chosen on the basis of design convenience.

We see thus that in an accelerator with nonlinear magnets, sextupoles may be introduced so as to produce the desired dependence of betatron frequencies on momentum, and then vertical resonant beam extraction can be accomplished if octupoles are energized with a mean strength so as to make the quantity B^4_{000} (eq. 14) small enough.

References

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