LINAC CAVITY FIELD CALCULATIONS

G.E. Lee-Whiting
Chalk River Nuclear Laboratories, Ontario, Canada

Reports[1,2] at the last conference on calculations of fields in linac cells indicated that difficulties arose with Alvarez-type cells, especially at lower values of β. Convergence becomes slower as β is decreased, and there is some doubt about the accuracy of the results. In this paper three devices for improving the calculation are discussed. They are directed toward decreasing the number of mesh points, improving the starting values for the relaxation, and increasing the speed of convergence of the frequency-iteration.

Mesh & Series

Imagine a cylindrical surface concentric with the drift-tube and lying just outside it. Inside this surface we use a mesh-type of calculation essentially the same as the one described in reference (2). Outside the series is represented by a series of terms each one of which satisfies the differential equation as well as the boundary conditions on the outer cylindrical boundary and on the two planar boundaries of the cell. It is sufficient to describe the form of the one non-zero component of the magnetic field, H_ϕ. The solutions have the form

$$H_\phi = A_o Z_1(kr) + \sum_{n=1}^{N} A_n L_1(\zeta_n r)\cos(n\pi z/b) \quad (1)$$

Here k is the wave number, b is the cell half-length, and $\zeta_n = \{(n\pi/b)^2 - k^2\}^{\frac{1}{2}}$; z=0 is at the middle of the cell. $Z_1(kr)$ and $L_1(\zeta_n r)$ are linear combinations of the two solutions of Bessel's equation of order 1, chosen in such a way that E_Z vanishes on the outer cylindrical boundary.

The constants A_n are so far completely arbitrary. They are to be chosen to match the mesh solution in the inner region. Using the fact that H_ϕ does not change sign over the cell, one can show that the ratio of the n^{th} term in (1) to the zeroth is less than

$$C \exp\{-\zeta_n(r-r_1)\}; \quad (2)$$

C is a constant very roughly equal to unity and r_1 is the outer radius of the drift-tube. Now for large n

$$\zeta_n \sim kn/\beta. \quad (3)$$

Thus the terms in (1) fall off exponentially with n. Convergence is more rapid as β is decreased, a fortunate occurrence since the low-β cells are in other ways the more difficult to handle. The contribution of each term in (1) beyond the tenth can be reduced below 10^{-3} of the zeroth if $(r-r_1)$ is kept larger than 1 or 2 cm.

Relaxation proceeds point by point starting with the row of mesh points nearest the axis. It stops after completion of the second last row, which lies a few centimetres beyond the drift-tube radius. Then the coefficients A_n are calculated by least-squares fitting to the values of H_ϕ on the second last (or sometimes an earlier) row of the mesh. The series is used to generate values of H_ϕ at an extra row of mesh points lying just beyond the last true row. The values of H_ϕ on the extra row of mesh points are used to complete the cycle of relaxation. Although convergence has not been proved, the method is found to converge in practice. An overconvergence factor of 1.3 is used; in one case divergence appeared to occur at an overconvergence factor of 1.4. The radial derivative of H_ϕ at the seam between the two regions appears to be continuous; for some low values of β it is necessary to move the seam outward a centimetre or so to get this continuity.

The use of the series in the outer region reduces the number of mesh points needed for a given spacing by roughly an order of magnitude. In one case in which a cell was treated by the purely mesh method as well, the time required to meet the same convergence test with the mesh-series method was 15-times smaller.

Starting Values

Before the relaxation can begin we must supply values of H_ϕ at each of the mesh points. The closer these starting values are to the true solution the faster will be the convergence. As in the preceding section we divide the cell into two regions, but this time the dividing cylinder has a radius somewhat smaller than the drift-tube radius; it should lie in the narrowest part of the gap between drift tubes. In the outer region a solution of the type (1) is used. In the inner region a similar form with different constants but containing Bessel functions regular at the origin is employed. The two sets of constants are to be chosen in such a way as to satisfy the boundary condition on

the drift-tube as well as possible, while maintaining adequate continuity on the seam between the two regions. This is done by minimizing the sum of the square of "errors" at points spread roughly uniformly over the drift-tube contour and the seam. At a point on the drift-tube the error is the tangential electric field; at a point on the seam the square of the error is the weighted sum of the squares of the discontinuities in tangential components of $\underline{E}$ and $\underline{H}$. The minimization is carried out subject to the constraint that E_Z at one point be fixed. The problem reduces to the solution of a set of linear equations. Once these equations are solved the RMS error, minimized with respect to the coefficients in the series, may be calculated for the particular choice of trial wave-number k.

The minimized RMS error is then calculated by the method outlined for each of a number of values of k in the region of a suspected eigenvalue. Usually the RMS error when plotted against k gives a smooth curve concave upward. An example is shown in Fig. 1. There is no difficulty in locating the minimum of this curve, k_o, within 1/2%. The value of k_o is used as a first guess at the eigenvalue for the mesh-series calculation, and starting values of H_ϕ are calculated from the series with the corresponding sets of coefficients.

To test the method its results were compared with results obtained by MESSYMESH[3] at 10 roughly uniformly spaced values of β stretching from 0.0402 to 0.4016. For β greater than 0.1 the agreement was of the order of 1 Mhz or better. Below $\beta = 0.1$, where the width of the gap becomes small compared to the drift-tube radius, the calculation with the present form of the program becomes more unreliable, and the clear minimum shown in Fig. 1 may not occur; it is then necessary to adjust either the spacing of the points at which the error is calculated or the relative weighting of electric and magnetic field errors. The minimized RMS error is usually about 5% of the maximum electric field strength. There are always a few points at which the errors are much worse; in good cases they rise to about 20%, in bad cases to 40 or 50%. In all cases the starting values of H_ϕ calculated gave good results in the relaxation process.

Iteration on k

A method different from that of reference (2) has been used for choosing successive trial values of the eigenvalue, k. Suppose that for a fixed value of k the relaxation is carried on until the change in the fields becomes negligible. Then the Rayleigh-Ritz upper bound on the frequency - *see reference (2), for example* - is a perfectly definite function of k which I shall call F(k). The curve in both parts of Fig. 2 is $k^* = F(k)$ calculated for the cell of MESSYMESH run number 30635. The value of k corresponding to the minimum in the curve $k^* = F(k)$ must correspond to the eigenvalue for our partially discretized problem. Therefore at this minimum $k^* = k$. Hence the eigenvalue corresponds to the intersection of the curve $k^* = F(k)$ and the straight line $k^* = k$. Computationally this means that we are looking for a zero of the function $F(k)-k$, a much easier task than locating a minimum.

In our program k is varied by 2% steps and a zero of $F(k)-k$ is calculated. For the results given in the next section not more than three values of k were needed in the refining process, once a zero was bracketed. When we tried to use the usual method, as described in reference (2), many tens of values were required. The success of our method depends on making sure that the residuals are adequately small before the k-iteration begins, and on the use of the starting values of the preceding section. When the relaxation was begun from cruder starting values, it was very difficult to ensure that F(k) was unchanged by more relaxation.

Note that our method of finding the eigenvalue actually allows a lower as well as an upper bound to be calculated. The usual variational calculation gives an upper bound only, but here the parameter being varied is itself the trial eigenvalue.

Results

In Table I a comparison is made between the results obtained by MESSYMESH and by the method of this paper for three different cells; the MESSYMESH result is printed above ours. TTF stands for transit-time factor; ΔE_o is the percentage difference between the mean field along the axis calculated by the line-integral method and by the surface-integral method. The smaller mesh size and the smaller ΔE_o for our method suggest that it is the more reliable. Both TTF and ZT^2 are somewhat higher in our results than in those by MESSYMESH. The comparison in reference (1) for similar cells shows that MESSYMESH also gives lower results than JESSY for both these quantities.

The overall time required to treat each of the cells in the table was under 30 minutes on a computer about 5 times slower than an IBM 7094. The time depends very much on the smallness required of the

residuals before the k-iteration begins, and we don't yet know how small they need be. The middle case in the table was done in 13 minutes.

Acknowledgment

Most of the programming for this work was done by A. M. Malecki and Miss J. P. Nicholls. Some earlier work on purely mesh calculations by Osama Aboul-Atta was incorporated into the program. The author benefited from conversations with H. C. Hoyt of Los Alamos before beginning the work.

References

1. P. F. Dahl, K. Jellett, G. Parzen, S. Giordano, J. P. Hannwacker, and D. Young, 1966 Linear Accelerator Conference, page 115.

2. H. C. Hoyt, 1966 Linear Accelerator Conference, page 119.

3. B. Austin, T. W. Edwards, J. E. O'Meara, M. L. Palmer, D. A. Swenson, and D. E. Young, MURA report 713 (1965). The results in Table I were actually taken from a summary of computer outputs made for the Brookhaven National Laboratory (Part I) dated November 1967.

Note Added During the Conference

The paper by Martini and Warner (this conference, page 512) also uses

(a) a combination of a mesh in one part of the cell with a series in another part;

(b) analytical starting values, though the latter appear to be different in form.

Although work on these ideas appears to have been initiated at about the same time at CERN and in Chalk River, the former group has proceeded more rapidly and has treated a considerably larger number of cases.

TABLE I

Comparison with MESSYMESH Results

(result of this paper below MESSYMESH result)

MESSYMESH Run No.	L/2 cm	Frequency Mhz	TTF	Mesh Size cm	ΔE_o %	ZT^2 MΩ/m
20243	3	200.935 200.46	.641 .664	.2500 .1250	6.9 -2.6	26.7 28.9
30441	11	207.080 207.23	.832 .850	.5000 .2500	4.3 -0.3	52.8 55.8
30635	23.5	199.089 199.17	.802 .809	.5000 .2500	1.1 -0.3	40.4 41.4

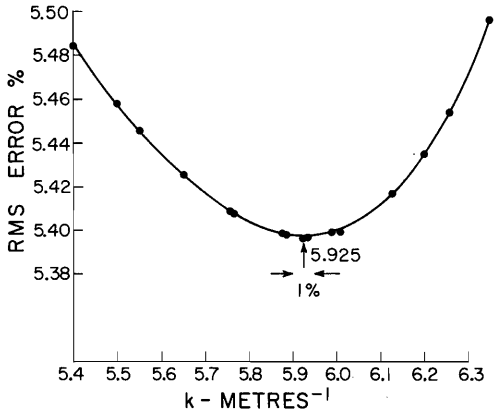

Fig. 1 RMS error in satisfying the boundary conditions, minimized with respect to the coefficients in the series representing the field, plotted against trial eigenvalue.

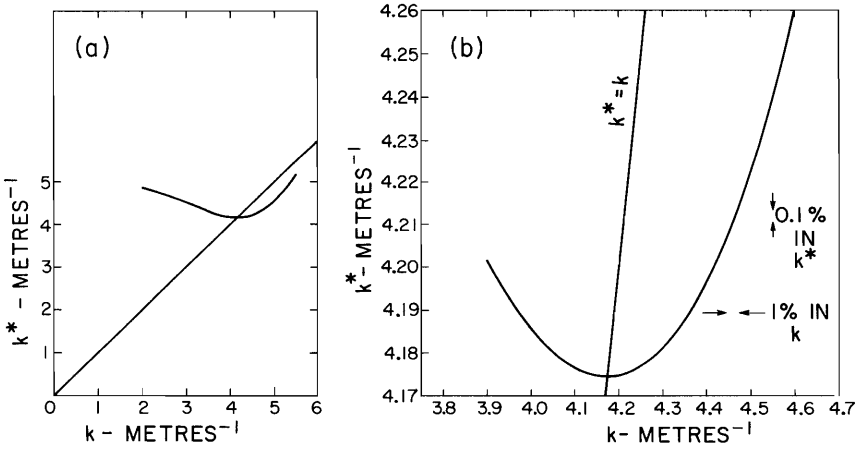

Fig. 2 The curve is the Rayleigh-Ritz upper bound on the wave-number calculated as a function of trial wave-number, k, with fields which have been relaxed until the residuals are negligible. The straight line k* = k goes through the minimum of the curve. Part (b) is an enlargement of a portion of (a) near the minimum. The cell corresponds to MESSYMESH run no. 30635.