# ANALYSIS OF EQUIVALENT CIRCUITS FOR LINAC TANKS

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Fig. 1 shows two chains of N inductance-capacitance-resistance loops having inductive coupling (with coupling constant k) between nearest neighbours. Chain A is terminated with half loops, chain B with full loops. The resonant frequency, $\omega_0$, and the quality factor, Q, are the same for all loops in both chains. Such chains are useful in studying the response and the sensitivity to error of tanks of magnetically coupled cavities of the type that might be used in an 805 Mhz standing-wave proton linac. Work at Los Alamos[1] has shown that for many of its properties a linac tank may be represented by an equivalent circuit of the type shown. In this paper an analysis is presented of the steady-state response of both chains to a driving EMF, shown in the $p^{th}$ loop in the figure. The useful insensitivity of the loop currents to parameter error in the $\pi/2$ mode is shown to extend to the second order of perturbation theory.

The circuit equations are conveniently written in the matrix form

$$MI = V/z. \qquad (1)$$

I is a vector whose elements are the N loop currents, while V is the vector of driving EMF's; $z = j\omega kL$. M is the tri-diagonal matrix

$$M = \begin{bmatrix} \rho\alpha & 1 & & & & \\ 1 & \alpha & 1 & & & \\ & 1 & \alpha & 1 & & \\ & & & \ddots & & \\ & & & 1 & \alpha & 1 \\ & & & & 1 & \rho\alpha \end{bmatrix}; \qquad (2)$$

$\rho = \frac{1}{2}$ for chain A and 1 for chain B. Some elements of M are functions of the driving frequency, $\omega$, as well as the loop parameters through

$$\alpha \equiv \frac{2}{k}\left[1-\omega_o^2/\omega^2 - j\, Q^{-1}\, \omega_o/\omega\right]. \qquad (3)$$

To get the loop currents we need the inverse matrix $M^{-1}$, -

$$I = M^{-1}\, V/z. \qquad (4)$$

I have been able to find closed expressions for the elements of $M^{-1}$ in terms of a complex quantity $\phi$, defined as a function of $\omega$ through (3) and

$$\alpha = -2\cos\phi \qquad (4)$$

Let $\{M^{-1}\}_{rs} \equiv a_{rs}$.

For chain A and $r \leq s$

$$a_{rs} = \frac{\cos(r-1)\phi\cos(N-s)\phi}{\sin\phi\,\sin(N-1)\phi} \qquad (5a)$$

For chain B and $r \leq s$

$$a_{rs} = -\frac{\sin r\phi\,\sin(N+1-s)\phi}{\sin\phi\,\sin(N+1)\phi} \qquad (5b)$$

If $r > s$, use $a_{rs} = a_{sr}$.

The second factor in the denominator of each form of (5) becomes very small for N values of $\omega$ corresponding to the N natural frequencies of the freely oscillating chain. In accelerator design we are interested in the relative values of the loop currents when $\omega$ is equal to one of these resonant frequencies. We shall put $\omega$ equal to one of the resonant frequencies and then expand $a_{rs}$ in powers of

$$\Omega = \frac{1}{kQ},$$

which is approximately $10^{-3}$ for typical accelerator parameters.

$$a_{rs} = \frac{b_{rs}}{\Omega} + C_{rs} + d_{rs}\,\Omega + \cdots \qquad (6)$$

Simple analytic formulae are available for the coefficients $b_{rs}$ etc. The larger the value of Q the more dominant is the first term on the right in (6). The loop currents calculated from the first term are exactly proportional to the currents which would flow in free oscillation. The effect of the second term in (6) is to introduce a loop-to-loop phase-shift whose magnitude is proportional to $\Omega$. This phase-shift is just what is required to account for the propagation of energy along the chain for dissipation in the resistors.

The $\pi/2$ mode has the important property that $C_{rs} = 0$ whenever $b_{rs} = 0$. This property corresponds to the fact that the odd-numbered loops are all in phase when a chain is driven in the $\pi/2$ mode, regardless of the value of Q.

While absence of phase-shifts is an important reason for choosing the $\pi/2$ mode, I am more concerned here with sensitivity to errors in loop parameters. These errors may be included in the circuit equations by replacing M by M+m; m is a symmetric matrix whose connection to three kinds of parameter errors we shall next consider.

*i)* Errors in the resonant frequencies of individual loops affect diagonal elements of m only. If the $r^{th}$ loop is mistuned by $\Delta\omega_o$, then (for chain B)

$$m_{rr} \simeq - \frac{4}{k} \frac{\Delta\omega_o}{\omega_o} \qquad (7)$$

The multiplier 4/k is large, of the order of 80. For an error of 0.1 Mhz in 800 Mhz $m_{rr}$ is of the order of 0.01.

*ii)* If the coupling constant between the $r^{th}$ and the (r+1) loops is $k(1+\varepsilon)$ instead of k, then

$$m_{r\ r+1} = m_{r+1\ r} = \varepsilon \qquad (8)$$

*iii)* Let the coupling constant between second neighbours be $k_2$. We shall treat all of this coupling as a perturbation.

$$m_{r\ r+2} = m_{r+2\ r} = k_2/k \simeq - .1 \qquad (9)$$

In the presence of perturbations we replace I in the circuit equations by (I+i); the elements of the vector i represent the changes in the loop currents resulting from the errors in circuit parameters. Ordinary first-order perturbation theory gives

$$i \simeq - M^{-1}\, mI. \qquad (10)$$

Here also the inverse matrix $M^{-1}$ plays an important role. Using expansions of the type (6) one can show that the relative change in current in the $n^{th}$ loop has the form

$$i_n/I_n = A\Omega^{-1} + B_n + C_n\Omega + \cdots . \qquad (11)$$

The coefficients A, $B_n$ etc. are linear combinations of the elements of m. The first term in (11) is large, but is independent of n; thus it does not affect the relative values of loop currents. The second term corresponds to a variation of current amplitude along the chain. In general it is not zero.

From this point onward it is convenient to confine our attention to the $\pi/2$ mode. Formulae for $i_n/I_n$ are to be used for odd-numbered loops only. The odd -numbered loops correspond to accelerating cells, the even-numbered loops to coupling cells. The great insensitivity of the $\pi/2$ mode (with either termination) to perturbations results from the vanishing of $B_n$ in various circumstances. The contribution of an element $m_{rs}$ to $B_n$ is zero if r and s are both odd or both even. Thus frequency errors in either type of cell have no effect on $B_n$. Furthermore, coupling between like cells of either type also has no effect on $B_n$; in particular, second-neighbour coupling does not affect the loop currents through $B_n$. However, couplings between unlike cells do have an effect. In particular, if the coupling between the $r^{th}$ and the $(r+1)^{th}$ loops is $k(1+\varepsilon)$ instead of k and r is even, then the value of $i_n/I_n$ (n odd) drops by $-\varepsilon$ as n passes over r.

For frequency errors the largest n-dependent contributions to $i_n/I_n$ come from the third term in (11). Because of the factor $\Omega$ they are small, but they are worth calculating. Assume an error in the $r^{th}$ loop only, giving $m_{rr}$ as calculated in (7). The perturbation of the current, a phase-shift, is shown in Fig. 2. The n-dependence of the phase-shift depends on whether r is even or odd. The abscissa is loop number (confined to odd integer values) and running from 1 to N. The driving EMF is assumed to be in the $p^{th}$ loop. When a coupling cell is mistuned (i.e. r even), the phase-shift is uniform except for a jump as n passes over r. The size of the jump is $r\Omega m_{rr}$ for chain B and $(r-1)\Omega m_{rr}$ for chain A; this is one of the few places where the type of termination has any effect for a $\pi/2$ mode. Note that the size of the jump is larger for mistuned loops near the driven loop. When the tuning error is in an accelerating cell (i.e. r odd), the phase-shift varies linearly between the mistuned loop and the driven loop and is uniform outside this interval. The difference between the two uniform sections is $(p-r)\Omega m_{rr}$ for both terminations. Note that this difference is largest when the mistuned cell is near an end of the accelerator. For both types of errors the form of the effect on the current distribution depends strongly on the position of the driven cell.

The phase-shifts we have been discussing are of the order of $10^{-4}$ radians. They are so small that one wonders whether second-order terms - i.e. terms proportional to $m_{r\rho}\, m_{s\sigma}$ - might not be more important. I have been able to show that $i_n$ may always be expressed as the ratio of two polynomials of order N in the elements of m. A second-order theory is obtained by discarding all terms of order higher than two in both polynomials. With

$$H \equiv M^{-1} m \qquad (12)$$

and

$$D \doteq 1 + \sum_r H_{rr}, \qquad (13)$$

$$\frac{i_n}{I_n} = -D^{-1} \sum_r \frac{a_{rp}}{a_{np}} [H_{nr} + \sum_s (H_{ss} H_{nr} - H_{ns} H_{sr})] \qquad (14)$$

The second term in D is proportional to the shift in resonant frequency induced by the perturbation and to Q. The denominator D and the first term in (11) together ensure that the loop currents become very small when the perturbations move the resonant frequency away from the driven frequency. In the rest of this paper it is assumed that the perturbations have been adjusted so that $D \simeq 1$.

Next we re-arrange (14) to exhibit the dependence on the elements of m, for frequency perturbations only.

$$\frac{i_n}{I_n} \simeq D^{-1} \left\{ \sum_r C_r(n) m_{rr} + \sum_r \sum_{s>r} C_{rs}(n) m_{rr} m_{ss} \right\} \qquad (15)$$

The first-order terms in (15) have already been considered, in Fig. 2. The second-order coefficients may also be expanded in powers of $\Omega$. They contain no $\Omega^{-1}$ terms; we shall retain only the terms independent of $\Omega$. Then the coefficient $C_{rs}(n)$ is independent of n if r and s are both even or both odd; that is, there are no second-order variations in loop current resulting from frequency errors in like cells. If one of r and s is even (say r) and the other odd there is an effect, but only if r lies between s and p. In other words, there is an effect only if the mistuned coupling cell lies between the mistuned accelerating cell and the driven cell. Then there is a jump in the current amplitude of magnitude $- m_{rr} m_{ss}$ as n passes over r toward p. If $m_{rr} m_{ss}$ is negative, the picture is similar to the upper half of Fig. 2, except that amplitude is to be plotted instead of phase.

The analysis may be extended to include the contributions from off-diagonal elements of m; terms of the form $m_{r\rho} m_{s\sigma}$ appear in (15). It has been shown that there is no n-dependence unless two of the four indices $r, \rho, s$, and $\sigma$ are odd and two even. Thus second-neighbour coupling between cells of one kind has no effect on relative currents, even in second-order perturbation theory. Nor are there any cross-terms between the second-neighbour coupling of accelerating cells and frequency errors in accelerating cells. The only other important terms arise from the combination of a frequency error in a coupling cell and the coupling between a pair of accelerating cells. This happens only if the mistuned coupling cell separates at least one of the accelerating cells from the driven cell. Again there is a jump in amplitude as n passes over the mistuned coupling cell, this time of magnitude $2m_{rr} m_{s\ s+2}$; if the coupling cell separates the two accelerating cells, the 2 is lacking.

In summary one can say that very definite statements have been made about the effects on relative currents of errors in the parameters of individual loops. These should be useful in further studies of the validity of the equivalent circuits. The magnitude of the effects resulting from parameter error depends on the distribution of the errors. For frequency errors of the order of 0.1 Mhz the effect on the currents will be of the order of 0.1%. Because errors in coupling-cell frequencies combine with both errors in accelerating-cell frequencies and accelerating-cell coupling, there is some reason to make the tolerance on frequency tighter for coupling cells than for the others.

## References

1. Nagle, D.E., Knapp, E.A., and Knapp, B.C. Rev. Sci. Instr. vol. 38, p. 1583, 1967.

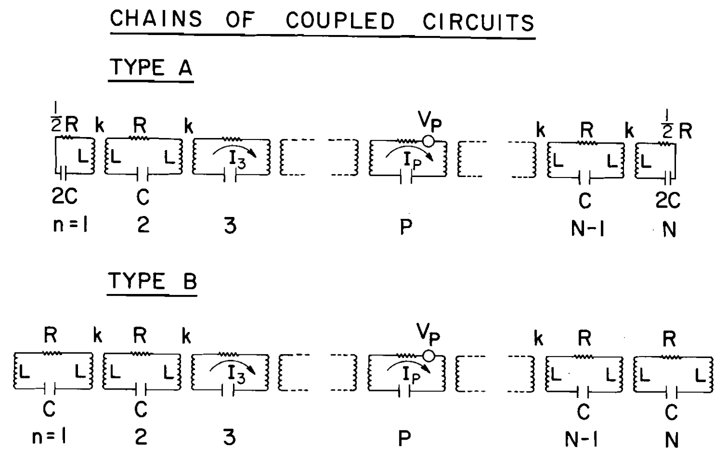


Fig. 1 Chains of resonant circuits representing linac tanks with different types of terminal cells. In each case there are N loops in all and the drive is in the $p^{th}$ loop.

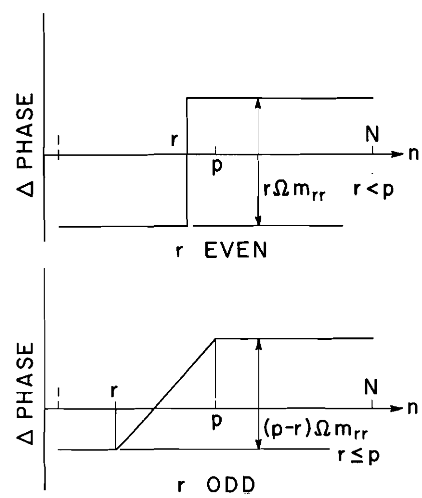


Fig. 2 Variation of phase with cell number n calculated in first-order perturbation theory for a frequency error in the $r^{th}$ cell for a $\pi/2$ mode with termination B. It is to be understood that n takes odd integral values only. The drive is in the $p^{th}$ loop, p odd. For termination A the only change is that the step for r even becomes $(r-1)\Omega m_{rr}$.