## LONGITUDINAL AND TRANSVERSE T AND S TRANSIT TIME COEFFICIENTS

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### Summary

Difference equations describing the motion of a proton through an accelerating gap have been given by P. Lapostolle at the Frascati Conference (1965) and by the author (Washington Conference, 1967). The dynamical coefficients contained in them are discussed here. There are two types, called T- and S-coefficients. T-coefficients belong to motion across the whole gap and have a simple form (transit time factor times a modified Bessel function). Difference equations through the first half of the gap needed to determine the mid-gap values of the particle coordinates, involve in addition S-coefficients. Exact expressions (series expansions) for the latter are given here assuming that the Fourier coefficients of $E_z(r=a)$, the longitudinal field along the gap circumference, are known. Approximations for the S-coefficients are discussed. Corrected Tables of non-relativistic and relativistic difference equations are given.

### 1. Introduction

At the Frascati Conference P. Lapostolle[1] gave a set of difference equations which correct and extend the so-called Panofsky equations[2] describing the phase change and energy gain along a linear accelerator. In this new version transverse motion too was taken into account. A new and improved derivation (including relativistic effects) was given by the author[3]. The difference equations for the change of kinetic energy, phase, radial slope and position across the whole gap involve only T-coefficients which essentially consist of the transit time factor times a Bessel function. However, these still depend upon the unknown mid-gap values of the particle coordinates. For the determination of the latter difference equations for the first half of the gap must be used. These contain besides the T-coefficients S-coefficients which are of a more intricate nature. The expressions for all these coefficients are given here and approximations for them are discussed. The method how to derive this difference equation is described elsewhere[3)4)5]. Only corrected versions of the difference equations are listed in Tables III to VI.

### 2. Field Representations

The field in the accelerating gap is supposed to be an axially symmetrical time-harmonic TM-field (angular frequency $\omega$ ) whose component $E_z$ is symmetrical about z = 0, the centre of the gap.

$$E_{z,r}(z,r,\varphi)=E_{z,r}(z,r)\cos(\varphi+\varphi_o)$$

$$E_z(z,r)=E_z(-z,r) \qquad E_r(z,r)=-E_r(-z,r) \tag{1}$$

$$H_\Theta(z,r,\varphi)=H_\Theta(z,r)\sin(\varphi+\varphi_o)$$

$$H_\Theta(-z,r)=H_\Theta(z,r)$$

$\varphi = \omega t$ is time-angle (= phase). Field components are expressed by Fourier-integrals:

$$E_z(z,r) = \frac{E_1}{2\pi}\int_{-\infty}^{\infty} \frac{b(k_z)}{J_o(\gamma a)} J_o(\gamma r)\, e^{ik_z z}\, dk_z$$

$$E_r(z,r) = -i\frac{E_1}{2\pi}\int_{-\infty}^{\infty} b(k_z)\frac{k_z J_1(\gamma r)}{\gamma J_o(\gamma a)}\, e^{ik_z z}\, dk_z$$

$$\mu H_\Theta(z,r) = -\frac{E_1}{2\pi}\frac{k_o}{c}\int_{-\infty}^{\infty} b(k_z)\frac{J_1(\gamma r)}{\gamma J_o(\gamma a)}\, e^{ik_z z}\, dk_z$$

with (2)

$$\gamma = \gamma_1 + i\gamma_2 = ik_r = (k_o^2 - k_z^2)^{1/2} \qquad \gamma_2 \geq 0$$

$a$ is the inner radius of the drift tube. The amplitude function $b(k_z)$ can be related to field $E_z(z,a)$ applied along the gap ($r = a$, $|z| \leq p/2$):

$$E_z(z,a) = E_1 \sum_{s=\infty}^{-\infty} B_s \cos(2\pi sz/p) \qquad (3)$$

$$b(k_z) = 2 \sum_{s=\infty}^{-\infty} B_s (-1)^s \sin(k_z p/2)(k_z - 2\pi s/p)^{-1} \qquad (4)$$

($B_o = 1$, $B_s = B_{-s}$) by a Green's function for a wave guide. $p = g + 2R_i$ where $g$ is the gap length (= minimum distance between drift tubes) and $R_i$ is the radius of curvature of the inner drift tube rim. The instantaneous peak "voltage" (at $\varphi = -\varphi_o$) along the line $r$ = const. is:

$$V(r) = \int_{-\infty}^{\infty} E_z(z,r)\, dz = V_o J_o(k_o r)$$
$$= E_1\, b(0)\, J_o(k_o r)/J_o(k_o a) \qquad (5)$$

($b(0) = p$). $V(a) = E_1 p$, $V_o = V(0) = E_1 p/J_o(k_o a)$. $E_1$ is the average field strength across the gap, $V_o$ is the voltage along the axis (of one cell). Evaluation of the integral representations (2) with the amplitude function (4) gives series expansions of the field. These still contain the unknown Fourier coefficients $E_1$, $B_n$. A finite number of them may be extracted from cavity fields calculated by numerical methods (mesh calculations[6]) which usually give numerical values for $U=-rH_\Theta(z,r)$. The series for $U = -rH(z,r)$ (Table I) is used for $r = a$ to set up a system of n equations of n Fourier-coefficients (n is the number of mesh points within the gap where values of U are given).

## 3. Beam Dynamics Coefficients

Beam dynamics coefficients are Fourier transforms (in z) of the field components. There are two types of them, T- and S-coefficients. T-coefficients are integrals over across the whole interval ($-\infty \leq z \leq \infty$). They appear in difference equations where the particle crosses the whole gap cell (or gap). Their expressions are simple. The S-coefficients are defined as integrals over $0 \leq z \leq \infty$ and are needed in difference equations along half of the cell. Their expressions are complicated.

The longitudinal, transverse and magnetic T-coefficient are defined as:

$$V_o T(k,r) \equiv \int_{-\infty}^{\infty} E_z(z,r) \cos(kz)\, dz = V_o T_o(k)\, I_o(k_r r)$$

$$V_o T_r(k,r) \equiv \int_{-\infty}^{\infty} E_r(z,r) \sin(kz)\, dz = V_o T_o(k)\, k\, I_1(k_r r)/k_r \qquad (6)$$

$$V_o T_m(k,r) \equiv -\int_{-\infty}^{\infty} \frac{c}{k_o} \mu H_\Theta(z,r) \cos(kz)\, dz = V_o T_o(k)\, I_1(k_r r)/k_r$$

with

$$k_r = (k^2 - k_o^2)^{1/2} \qquad (7)$$

They are known as soon as the transit time factor:

$$T_o(k) \equiv \int_{-\infty}^{\infty} E_z(z,0) \cos(kz)\, dz = (b(k)/b(0)) \times J_o(k_o a)/I_o(k_r a) \qquad (8)$$

is given and are nothing more than convenient abbreviations. $T_o(k)$ is a measure of the distribution of $E_z$ in the longitudinal direction. $T_o(k)$, its derivatives $T_o'(k)$, $T_o''(k)$ (and the above T-coefficients) are needed for the single value $k = \omega/\dot{z}_o$ ($\dot{z}_o$ = longitudinal velocity of the particle at the gap centre). They may be easily obtained from fields given numerically by numerical integration of the integral defined in (8) (and from the formulae (6)). If the cell is regarded as a closed cavity, the limits of integration $\pm\infty$ in (6), (8) and (9) must be replaced by $\pm L/2$ (L is cell length) and k by $2\pi/L$. This hardly changes the numerical values of these quantities. This method may be more convenient than to express $T_o(k)$ by the Fourier coefficients $B_n$ introduced in Chapter 2:

$$T_o(k) = T_{oo}(k)\,(1 + Y)$$

$$T_{oo}(k) = \frac{J_o(k_o a)}{I_o(k_r a)} \frac{\sin(kp/2)}{(kp/2)} \qquad (8a)$$

$$Y = -\sum_{n=1}^{\infty} \frac{2(-1)^n B_n (kp)^2}{(2\pi n)^2 - (kp)^2}$$

The longitudinal, transverse and magnetic S-coefficient are defined as:

$$V_o S_1(k,r) = 2 \int_0^\infty E_z(z,r)\sin(kz)\,dz$$

$$V_o S_r(k,r) = 2 \int_0^\infty E_r(z,r)\cos(kr)\,dz \qquad (9)$$

$$V_o S_m(k,r) = -\frac{2c}{k_o} \int_0^\infty \mu H_\theta(z,r)\sin(kz)\,dz$$

$S_1$ and $S_r$ (as well as $T_1$ and $T_r$) are dimensionless while $S_m$ (and $T_m$) have the dimension of a length. This may be inconvenient in numerics: $T_1$, $T_r$, $S_1$, $S_r$ are somewhat smaller than unity, $S_m$ and $T_m$ are smaller than these by a factor 10 to 100. Defining $kT_m$ and $kS_m$ in place of the expressions given in (6) and (8) brings disadvantages for the writing of difference equations. For numerical work it is suggested to normalize $T_m$ and $S_m$ by multiplying them by $2\pi/L$ and to modify dynamical formulae in Tables V and VI accordingly. (In all these formulae $S_m$ (or $dS_m/dk$, $d^2S_m/dk^2$) is preceded by k, so the normalization can be done in the following manner $kS_m = (kL/2\pi)x(2\pi S_m/L)=(kL/2\pi)xS_{m\ norm}$, $kL/2\pi \approx 1$).

The S-coefficients are related to each other by the following equations:

$$V_o\,\partial S_1(k,r)/\partial r = 2\omega\mu H_\theta(0,r)/k - V_o\,k_r^2\,S_r(k,r)/k$$

$$V_o\,\partial S_r(k,r)/\partial r = 2\,E_z(0,r) - V_o\,S_r(k,r)/r - k\,V_o S_1(k,r) \qquad (10)$$

$$V_o\quad S_m(k,r) = \frac{2}{k}\frac{c}{k_o}\,\mu H_\theta(0,r) - \frac{1}{k}\,V_o S_r(k,r)$$

These probably will not be exactly valid, if the limits of integration in (8) are $\pm$ L/2, but may represent a good approximation. They could be useful for the calculation of radial derivatives of S-coefficients since they permit to circumvent the need of radial derivatives of the field components. With the help of series (4) for $b(k_z)$, series expansions of the S-coefficients may be found. They are listed in Table II. Unfortunately they are complicated. $S_1$ and $S_r$ are more important than $S_m$ which only appears in relativistic formula where the need for the half gap difference equations is less severe.

Fourier coefficients $B_n$ and thereafter $S_r(k,r)$, $S_1(k,r)$ (with $k = \omega/\dot{z}_o$), $0 \le r \le a$, have been calculated from the potentials $U = -rH_\theta$ of various Alvarez cavities (0.6 - 8 MeV). There emerge the following results:

a) $S_1$ may be approximated by the first two terms and by the first (eventually second) term of the simple series in n. The contribution of the double series is negligible. The first term is small at low energy, appreciable at higher energies. This can be explained as follows: $kg/2 \approx \pi/2$. In low energy cavities $kp/2 = k(g/2 + R_i)$ comes near to $\pi$ and the cotangent is small, while at higher energies $R_i$ is small compared with g and kp/2 is nearer to $\pi/2$ and the cotangent is larger. The term proportional to $J_o(k_o r)$ is the most important one, it is always greater than $S_1(k,r)$; it is almost constant with r. The terms with n = 1 (2) contribute 30% (1%) at .6 MeV, 10% (2%) at 8 MeV at r = a, for smaller r the situation is better. Therefore at low energy the formula for $S_1$[1] derived for homogeneous field $E_z(z,a) = E_1$ ($|z| \le p/2$) appears not a very good approximation. $S_1$ decreases slightly with increasing energy.

b) $S_r$ may be approximated by the single term and by the first two (eventually three) terms of the series in n while the double series can be neglected. The absolute value of the term with n = 1 (2, the rest of the sum) contribute 110%, (10%, 2%) of $S_r$ at 0.6 MeV, 50% (25%, 37%) of $S_r$ at 8 MeV. At higher energy terms with greater n become important, but the error introduced by their neglection is less harmful since in the difference equations the $S_r$ are multiplied by $eV_o/2W$ and the kinetic energy W increases.

Derivatives of S-coefficients are under investigation and results will be given in ref.[5].

## 4. Difference Equations

Tables III to VI contain difference equations for particle dynamics in an accelerating gap. They have been slightly corrected in comparison with those published earlier[3]. Increments as needed in the thin lens approximation are listed, e.g. $\Delta\bar{\varphi} = \Delta(\varphi - z\,d\varphi/dz)$, $\Delta(r - z\,dr/dz) = \Delta\bar{r}$. For example, the trajectory of the entering non-relativistic (relativistic) particle is extended for free motion up to the centre; there is the thin lens, the coordinates are incremented by $\Delta W$, $\Delta\bar{\varphi}$, $\Delta r'$, $\Delta\bar{r}$ of Table III (Table IV) and then sets in free motion with these new values. The difference equations of these Tables (III and IV) contain the still unknown mid-gap values W, $\varphi$, $r'_o$, $r_o$. The difference equations of Table V and VI for the motion through the first half of the gap may be used to determine them. As a matter of experience, the corrections involving S-coefficients become less and less important with increasing particle energy.

All these difference equations have been found by first order perturbation theory as described elsewhere[3),4),5)]. The action of the gap field on the otherwise freely moving particle is treated as a perturbation. Solutions of the equations of motion are expanded into powers of $\varkappa = eV_1 / (m\omega p\dot{z}_o)$ (= impact of the field during one period/free particle momentum) and terms linear in $\varkappa$ lead to the difference equations. In the non-relativistic equation only the electrical field has been retained. In the relativistic treatment are considered the magnetic field and the mass variation too. The integral representations (2) of the field are inserted into the first order equations and z and r in $eE_z(z,r,\varphi)$ equal the zero order solution $\varphi^{(o)} = \varphi/k$ and $r^{(o)} = \dot{r}_o\ \varphi/\omega + r_o$. The equations of motion can then be solved by quadratures. Afterwards the integrals in $k_z$ are evaluated by Cauchy's residue theorem. For integrals across the whole gap there are only dynamical poles at $k_z = \pm k$ (poles at $J_o(a\gamma) = 0$ corresponding to the evanescent wave guide modes can be neglected) rendering the terms in Tables III and IV. They cut out from the continuous spectrum of waves in (2) the partial waves whose phase velocity equals the particle velocity. For the half gap difference equations the trajectory ends at the gap centre. In the $k_z$-plane appear additional poles at $k_z = \pm 2\pi n/p$, n = 0,1,2,.. They describe standing waves as they exist within the gap and yield the simple series in n contained in the S-coefficients.


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Table I - Series Expansion of the Gap Field

$$|z| \le p/2 : \qquad U(z,r) = - r H_\theta(z,r) = 2 E_1 \varepsilon\omega r p \times$$

$$\times \left[ \frac{2}{k_o p} \frac{J_1(k_o r)}{J_o(k_o a)} + \sum_{n=1}^{\infty} B_n \cos(2\pi n \tfrac{z}{p}) \frac{1}{\mu_n} \frac{I_1(\mu_n r/p)}{I_o(\mu_n a/p)} - 2 \frac{p}{a} \sum_{n=0}^{\infty} \frac{B_n(-1)^n}{1+\delta_{no}} \sum_{\nu=1}^{\infty} \frac{J_1(j_\nu r/a)}{J_1(j_\nu)} \frac{\cosh(\eta_\nu z)\, e^{-\eta_\nu p/2}}{(p\eta_\nu)^2+(2\pi n)^2} \right]$$

$$E_z(z,r) = (\varepsilon\omega r)^{-1} \, \partial U(z,r)/\partial r \qquad\qquad E_r(z,r) = -(\varepsilon\omega r)^{-1} \, \partial U(z,r)/\partial z$$

Table II - Expressions for T- and S-coefficients

$$T_l(k,r) = T_o(k)\, I_1(k_r r) \qquad\qquad T_m(k,r) = T_o(k)\, I_1(k_r r)/k_r$$

$$T_r(k,r) = T_o(k)\, k\, I_1(k_r r)/k_r \qquad\qquad k_r = (k^2 - k_o^2)^{1/2}$$

$$S_l(k,r) = -\operatorname{ctg}(kp/2)\, T_o\, I_o(k_r r) + \frac{J_o(k_o r)}{pk/2} - 4 J_o(k_o a) \sum_{n=1}^{\infty} \frac{pk\, B_n}{(2\pi n)^2 - (kp)^2} \frac{I_o(\mu_n r/p)}{I_o(\mu_n a/p)}$$

$$- 8 J_o(k_o a)\, (kp) \left(\frac{p}{a}\right)^2 \sum_{n=1}^{\infty} \frac{B_n(-1)^n}{1+\delta_{no}} \sum_{\nu=1}^{\infty} \frac{J_o(j_\nu r/a)}{J_1(j_\nu)} \frac{\exp(-\eta_\nu p/2)}{(kp)^2+(\eta_\nu p)^2} \frac{j_\nu}{(2\pi n)^2+(\eta_\nu p)^2}$$

$$S_r(k,r) = \operatorname{ctg}(kp/2)\, T_o(k)\, k\, I_1(k_r r)/k_r + 4 J_o(k_o a) \sum_{n=1}^{\infty} \frac{B_n (2\pi n)^2}{(2\pi n)^2 - (kp)^2} \frac{1}{\mu_n} \frac{I_1(\mu_n r/p)}{I_o(\mu_n a/p)}$$

$$- 8 J_o(k_o a) \frac{p}{a} \sum_{n=1}^{\infty} \frac{B_n(-1)^n}{1+\delta_{no}} \sum_{\nu=1}^{\infty} \frac{J_1(j_\nu r/a)}{J_1(j_\nu)} \frac{\exp(-\eta_\nu p/2)}{(kp)^2+(\eta_\nu p)^2} \frac{(\eta_\nu p)^2}{(kp)^2+(\eta_\nu p)^2}$$

$$\frac{1}{p} S_m(k,r) = -\operatorname{ctg}(kp/2)\, T_o(k) \frac{1}{k_r p} I_1(k_r r) + \frac{2}{kp} \frac{J_o(k_o r)}{k_o p} - 4 J_o(k_o a) kp \sum_{n=1}^{\infty} \frac{B_n}{(2\pi n)^2-(kp)^2} \frac{1}{\mu_n} \frac{I_1(\mu_n r/p)}{I_o(\mu_n a/p)}$$

$$- 8 J_o(k_o a)\, kp \frac{p}{a} \sum_{n=1}^{\infty} \frac{B_n(-1)^n}{1+\delta_{no}} \sum_{\nu=1}^{\infty} \frac{J_1(j_\nu r/a)}{J_1(j_\nu)} \frac{\exp(-\eta_\nu p/2)}{(kp)^2+(\eta_\nu p)^2} \frac{1}{(2\pi n)^2+(\eta_\nu p)^2}$$

$$\mu_n = \left[ (2\pi n)^2 - (k_o p)^2 \right]^{1/2}$$

Table III - Nonrelativistic change of longitudinal kinetic energy, reduced phase, radial slope and reduced radial position across a gap

$$\Delta W = eV_o \; T_o I_o \; \cos\varphi + eV_o \; d/dk(T_o k_r I_1) \; r' \sin\varphi$$

$$\Delta\bar{\varphi} = \alpha k \; d/dk(T_o I_o) \; \sin\varphi - \alpha k \; d^2/dk^2(T_o k_r I_1) \; r' \cos\varphi$$

$$\Delta r' = -\alpha \; (T_o k I_1/k_r) \; \sin\varphi + \alpha \left[ d/dk(T_o k I_1') - T_o I_o \right] \; r' \cos\varphi$$

$$\Delta\bar{r} = -\alpha \; d/dk(T_o k I_1/k_r) \; \cos\varphi - \alpha \left[ d^2/dk^2(T_o k I_1') - d/dk(T_o I_o) \right] \; r' \sin\varphi$$

Table IV - Change of the Same Quantities in the Relativistic Case

$$\Delta W_r = \Delta W \qquad \left[ - eV_o \; (T_o k I_1/k_r) \; r' \sin\varphi \right]^{*}$$

$$\Delta\bar{\varphi}_r/(1-\beta_o^2)^{1/2} = \Delta\bar{\varphi} \; (1 - k_o^2/k^2) \qquad - \alpha k \; (k_o^2/k^2) \; (T_o I_1/k_r) \; r' \cos\varphi$$

$$\Delta r_r'/(1-\beta_o^2)^{1/2} = \Delta r' (1 - k_o^2/k^2) \qquad + \alpha \; (k_o^2/k^2) \; T_o \; (I_1' - I_o) \; r' \cos\varphi$$

$$\Delta\bar{r}_r/(1-\beta_o^2)^{1/2} = \Delta r \; (1 - k_o^2/k^2) - \alpha(k_o^2/k^2) \; (T_o I_1/k_r) \cos\varphi \; - \; \alpha \; (k_o^2/k^2) \; d/dk(2T_o I_1' - T_o I_o) \; r' \sin\varphi$$

* Adding the term in the square brackets gives the gain in total kinetic energy.

Table V - Change of Kinetic Energy, Reduced Phase, Radial Slope and Reduced Radial Position along the First Half of the Gap

$$\Delta w_1 = \Delta W/2 + (eV_o/2) \left[ S_1 \; \sin\varphi - d/dk \; \partial S_1/\partial r \; r' \cos\varphi \right]$$

$$\Delta\bar{\varphi}_1 = \Delta\bar{\varphi}/2 - (\alpha k/2) \left[ dS_1/dk \; \cos\varphi + d^2/dk^2 \; \partial S_1/\partial r \; r' \sin\varphi \right]$$

$$\Delta r_1' = \Delta r'/2 - (\alpha/2) \left[ S_r \; \cos\varphi + (d/dk \; \partial S_r/\partial r + S_1) \; r' \sin\varphi \right]$$

$$\Delta\bar{r}_1 = \Delta\bar{r}/2 + (\alpha/2) \left[ dS_r/dk \; \sin\varphi - (d^2/dk^2 \; \partial S_r/\partial r + dS_1/dk) \; r' \cos\varphi \right]$$

Table VI - Change of the Same Quantities in the Relativistic Case

$$\Delta w_{r1} = \Delta W_r/2 + (eV_o/2)\left\{ S_1 \sin\varphi - \left( \frac{\partial}{\partial r} \frac{dS_1}{dk} \left[ - S_r \right]^{*} \right) r' \cos\varphi \right\}$$

$$= \Delta w_1 + r' \left[ - T_r \sin\varphi - S_r \cos\varphi \right]^{*}$$

$$\Delta\bar{\varphi}_{r1} = \Delta\bar{\varphi}_r/2 - (1-\beta_o^2)^{1/2} \frac{\alpha k}{2}\left\{\left(1 - \frac{k_o^2}{k^2}\right)\left(\frac{dS_1}{dk}\cos\varphi + \frac{\partial}{\partial r}\frac{d^2S_1}{dk^2} r' \sin\varphi\right) - \frac{k_o^2}{k^2}\left(\frac{dS_r}{dk} + k\frac{dS_m}{dk}\right) r' \sin\varphi\right\}$$

$$= (1-\beta_o^2)^{1/2}\left\{\left(1 - \frac{k_o^2}{k^2}\right) \Delta\bar{\varphi}_1 - \frac{k_o^2}{k^2}\frac{\alpha k}{2} r'\left(\frac{T_o I}{k_r}\cos\varphi - \left[\frac{dS_r}{dk} + k\frac{dS_m}{dk}\right] r' \sin\varphi\right)\right\}$$

$$\Delta r'_{r1} = \Delta r'_r/2 - (1-\beta_o^2)^{1/2}\frac{\alpha}{2}\left\{\left(S_r + \frac{k_o^2}{k^2} k S_m\right)\cos\varphi + \left(\frac{\partial}{\partial r}\left[\frac{dS_r}{dk} + \frac{k_o^2}{k^2} k \frac{dS_m}{dk}\right] + S_1\right) r' \sin\varphi\right\}$$

$$\Delta\bar{r}_{r1} = \Delta\bar{r}_r/2 + (1-\beta_o^2)^{1/2}\frac{\alpha}{2}\left\{\left(\frac{dS_r}{dk} + \frac{k_o^2}{k^2} k\frac{dS_m}{dk}\right)\sin\varphi - \left(\frac{\partial}{\partial r}\left[\frac{d^2S_r}{dk^2} + \frac{k_o^2}{k^2} k \frac{d^2S_m}{dk^2}\right] + \frac{dS_1}{dk}\right) r' \cos\varphi\right\}$$

$^{*}$ Adding the term in the square bracket gives the gain in total kinetic energy.

Common to Tables III to VI

$\alpha = eV_o/(2W)$ $\qquad\qquad$ $k_o^2/k^2 = (dz/dt)_o^2/c^2 \approx \beta_o^2$

$W = m/2\ (dz/dt)_o^2 = (m/2)\ \dot{z}_o^2 \quad m = \text{rest mass}$ $\qquad\qquad$ $-k_o^2/k^2 = k_r^2/k^2 \approx 1 - \beta_o^2$

The argument $k = \omega/\dot{z}_o$ of $T_o(k)$, $k_r r_o$ of the modified Bessel functions $I_n(k_r r)$ and $k, r_o$ of $S_1(k,r_o)$, $S_r(k,r_o)$, $S_m(k,r_o)$ and the subscript $_o$ of $\varphi_o$, $r_o$ and $r'_o = (dr/dz)_o$ have been dropped. In all the expressions of Tables III to VI all these parameters refer to mid-gap values.