TRANSIENT BEAM LOADING IN STANDING WAVE LINACS*

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Introduction

Steady-state cavity excitation and beam loading in standing wave linacs have been analyzed by a number of authors.[1-6] More recently, Nishikawa[7-9] extended the analysis to include transient behavior of these phenomena. In all of the analyses to date, constant beam velocity and identical cavity cells are assumed. These assumptions, however, become less valid for low energy linac cavities such as those at the beginning of the Conversion linac. In this paper, the amplitude of the beam induced field is calculated. The analysis takes into account the changes in the beam velocity and cavity cell length along the linac. The method of analysis follows closely that of Nishikawa given in Ref. 7.

Under the approximation that the beam velocity varies linearly with the distance along the accelerator cavity, the value of the beam induced field is proportional to a constant $\bar{N}$, which is given by $V(L) - V(0)/V(L_o) - V(0)$, where V is the beam velocity, L_o is the length of the first cavity cell, and L is the length of the cavity tank. The expression for the beam induced field differs from that for an identical cell cavity by a constant multiplying factor K, which is given by $\bar{N}\, V(0)/N\, V_b$, where N is the number of cavity cells and V_b is the constant beam velocity assumed.

Method of Analysis

The field in a cavity, in general, can be taken to be

$$\vec{E}(r,\theta,z,t) = \vec{E}_e(r,\theta,z,t) + \vec{E}_b(r,\theta,z,t)$$

where $\vec{E}_e$ is the field due to cavity excitation and $\vec{E}_b$ is the field due to beam loading. In terms of the normal mode fields of the cavity

$$\vec{E}_e(r,\theta,z,t) = \sum_n V_{en}(t)\, \vec{E}_n(r,\theta,z)$$

and

$$\vec{E}_b(r,\theta,z,t) = \sum_n V_{bn}(t)\, \vec{E}_n(r,\theta,z) \quad .$$

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The normal amplitudes $V_{en}(t)$ and $V_{bn}(t)$ satisfy the differential equation[7]:

$$(\pounds_n) \begin{pmatrix} V_{en} \\ V_{bn} \end{pmatrix} = \begin{pmatrix} f_{en}(t) \\ f_{bn}(t) \end{pmatrix} \tag{2.1}$$

where

$$\pounds_n = \frac{d^2}{dt^2} + \left(\frac{1}{Q_n} + \frac{j}{Q_{on}} \right) \omega_n \frac{d}{dt} + \omega_n^2$$

Q_n = loaded Q

Q_{on} = unloaded Q

ω_n = n^{th} mode resonance frequency

f_{en} = source function due to cavity excitation

f_{bn} = source function due to beam loading.

The effects of beam loading can be characterized by the beam interaction integral

$$J_n(t) = \int_{cavity} \vec{J} \cdot \vec{E}_n \, dv \tag{2.2}$$

which is related to the source function by

$$f_{bn}(t) = - \frac{1}{\epsilon} \frac{d}{dt} J_n(t)$$

where $\vec{J}$ is the beam current density.

Let $\vec{E}_k$ be the resonant mode of the cavity and consider

$$\vec{E}_k(r,\theta,z) = \sum_s a_s \, \vec{h}_s(r,\theta,z)$$

where $\{h_s\}$ a set of orthogonal functions (modified space harmonics) which has the following properties: h_p is synchronous with the beam for some p and $\{h_s\}$ reduces to the space harmonics in the limit of identical cells.

Since coupling between the beam and the cavity field comes largely from interaction of the beam with the synchronous space harmonics,

$$J_k(t) \approx \sum_{\substack{\text{synchronous} \\ \text{harmonics} \\ (p)}} a_p \int \vec{J} \cdot \vec{h}_p \, dv \quad .$$

Under these conditions, the beam induced field is given by

$$\vec{E}_b(r,\theta,z,t) \approx V_{bk}(t)\ \vec{E}_k(r,\theta,z) \qquad (2.3)$$

with

$$\mathcal{L}_k\ V_{bk}(t) = -\frac{1}{\epsilon}\frac{d}{dt}\sum_p a_p \int \vec{J}\cdot\vec{h}_p\ dv \quad . \qquad (2.4)$$

In particular, for the Conversion linac, k = 0 and p = ± 1.

Beam Induced Field in a Linac Cavity

For a periodic structure, the normal mode field can be expanded[7] as

$$E_n(z) = \sum_{s=-\infty}^{\infty} a_{ns} \cos\left[\left(\frac{n\pi}{N} + 2\pi s\right) u\right]$$

where

$u = \frac{z}{L_o}$

N = number of cells in the cavity

L_o = length of each cavity cell.

Analogous to this expansion, for a nonperiodic structure, we take

$$u = \frac{1}{T}\int_o^z \frac{dz'}{V(z')}$$

where

$T = \frac{2\pi}{\omega}$

ω = rf frequency of the beam.

Let the time required for a beam to cross each cell be T, i.e.,

$$\int_{cell} \frac{dz'}{V(z')} = T \quad ,$$

so that

$$\int_o^L \frac{dz'}{V(z')} = NT \quad ,$$

and assume

$$v(z) = V(|z|) \quad \text{for } -L < z < 0$$

where L = length of the cavity tank.

Under these conditions, the set of modified space harmonics is given by

$$h_{ns} = \cos\left[\left(\frac{2\pi}{N} + 2\pi s\right)\int_o^t \frac{dz'}{TV(z')}\right]$$

which satisfies the orthogonal condition

$$\int_o^{2L} h_{ns}\ h_{ns'}\ \frac{dz}{V(z)} = NT\ \delta_{ss'} \quad .$$

Thus, for E_n an even function of z,

$$a_{ns} = \frac{2}{N}\int_o^N E_n \cos\left[\left(\frac{n\pi}{N} + 2\pi s\right) u\right] du \quad .$$

For the Conversion linac, the resonant normal mode field along the beam axis may be characterized[7] by

$$E_o = \begin{cases} \dfrac{\xi_o\ L_k}{\sqrt{2}\ g_k}\ , & z \text{ in the } k^{th} \text{ gap} \\ 0\ , & \text{elsewhere} \end{cases}$$

where

g_k = width of the k^{th} gap

L_k = length of the k^{th} cell

and $\xi_o(r)$ gives the radial dependence of the field. Furthermore, we assume that the time required for the beam to travel across each gap is the same. Then

$$E_o = \begin{cases} \dfrac{\xi_o\ L_o}{\sqrt{2}\ g_o}\ , & \left(k + \frac{1}{2} - \frac{\Delta u}{2}\right) < u < \left(k + \frac{1}{2} + \frac{\Delta u}{2}\right), \\ & \text{for } k = 0,1\ldots,N-1 \\ 0\ , & \text{elsewhere} \end{cases}$$

which can readily be Fourier analyzed to give

$$a_{os} = (-1)^s \frac{\xi_o \sin\left(\pi s \frac{g_o}{L_o}\right)}{\sqrt{2}\left(\pi s \frac{g_o}{L_o}\right)}$$

where $(T\Delta u)$ = time required to cross a gap, i.e., $\Delta u = g_o/L_o$. Figure 1 shows some graphs of the function E_o. Hence, the resonant mode field along

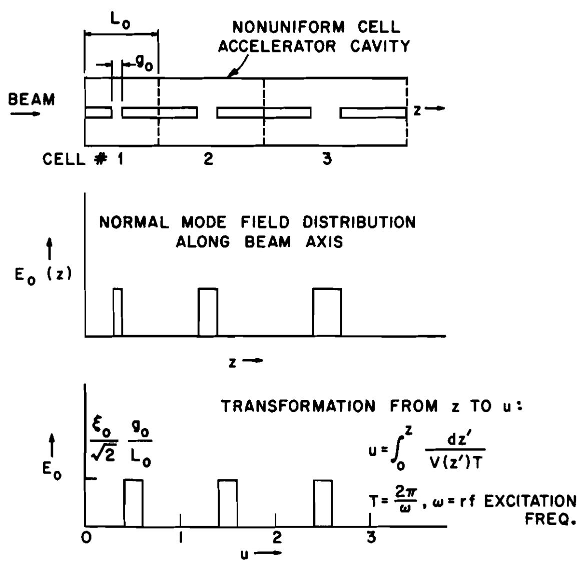

Fig. 1. Cavity normal mode field distribution along beam axis.

the beam axis for the Conversion linac cavities is given by

$$E_o(z) = \sum_{s=0}^{\infty} a_{os} \cos \left[\frac{2\pi s}{T} \int_o^z \frac{dz'}{V(z')} \right]$$

where a_{os} is given above.

For a particle of charge q' entering the cavity at $t = t_b$, the beam cavity interaction integral is given by

$$J_o(t,t_b) = \frac{q'}{T} \sum_{s=0}^{\infty} a_{os} \int_o^L \delta \left[\int_o^z \frac{dz'}{TV(z')} - \frac{t - t_b}{T} \right]$$

$$\cos \left[\frac{2\pi s}{T} \int_o^z \frac{dz'}{V(z')} \right] dz$$

or

$$J_o(t,t_b) = q' \sum_{s=0}^{\infty} a_{os} \int_o^N V \, \delta \left(u - \frac{t - t_b}{T} \right)$$

$$\cos (2\pi s \, u) \, du \qquad (3.1)$$

As an approximation to the beam velocity, we take

$$V(z) - V(0) = Az$$

where A is a constant (see Fig. 2).

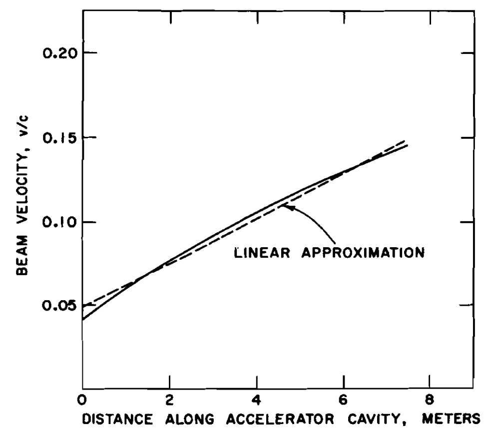

Fig. 2. Beam velocity versus distance along accelerator cavity (linac tank No. 1).

Then

$$u(z) = \frac{1}{AT} \ell n \frac{V(z)}{V(0)}$$

so that for $u = (t - t_b)/T$,

$$V = V(0) \, e^{A(t - t_b)} \, .$$

Using this result, the integral given by Eq. (3.1) may be evaluated, yielding

$$J_o(t,t_b) = q' \, V(0) \, e^{A(t-t_b)}$$

$$\sum_{s=0}^{\infty} a_{os} \cos \left(2\pi s \frac{t - t_b}{T} \right)$$

for $0 < t - t_b < NT$.

The beam-cavity interaction integral for a bunch can now be found by summing the interaction integral for each particle over all the particles in the bunch. For a short bunch of uniform charge distribution, we obtain

$$J_o(t,\bar{t}_b) \approx q\, V(0)\, e^{A(t - \bar{t}_b)} \sum_{s=0}^{\infty} a_{os} f_s \cos\left(2\pi s \frac{t - \bar{t}_b}{T}\right) ,$$

$$0 < t - \bar{t}_b < T \qquad (3.2)$$

where q is the total charge in the bunch, δt_b is the length of a bunch, $\bar{t}_b$ is the time the center of the bunch enters the cavity and

$$f_s = \frac{\sin\left(\pi s \frac{\delta t_b}{T}\right)}{\left(\pi s \frac{\delta t_b}{T}\right)} .$$

Terms of the order of AT ($\approx 1/N$) have been neglected in Eq. (3.2).

From the symmetry of the system, the beam interaction integral with the synchronous harmonic summed over all bunches inside the cavity has periodicity T. Thus, performing this summation, we find

$$J_o(t,\bar{t}_b) \approx a_{o1} f_1 q V(0) \sum_{k=0}^{N-1} e^{A(t-\bar{t}_b+kT)} \cos\left[\frac{2\pi}{T}(t - \bar{t}_b + kT)\right]$$

or

$$J_o(t,\bar{t}_b) \approx a_{o1} f_1 \bar{N}\, q\, V(0)\, e^{A(t-\bar{t}_b)} \cos\left(2\pi \frac{t-\bar{t}_b}{T}\right) \qquad (3.3)$$

where

$$\bar{N} = \frac{V(L) - V(0)}{V(L_o) - V(0)}$$

for $0 < t - \bar{t}_b < T$.

Furthermore, if the change in beam velocity across each cell is small, then

$$\frac{V(L_o)}{V(0)} = e^{AT} \approx 1$$

so that

$$J_o(t,\bar{t}_b) \approx -\frac{\xi_o}{\sqrt{2}} T_1 f_1 \bar{N}\, q\, V(0) \cos\left(2\pi \frac{t-\bar{t}_b}{T}\right) \qquad (3.2)$$

where

$$T_1 = \frac{\sin\left(\pi \frac{g_o}{L_o}\right)}{\left(\pi \frac{g_o}{L_o}\right)} = \text{transit time factor}$$

$$f_1 = \frac{\sin\left(\pi \frac{\delta t_b}{T}\right)}{\left(\pi \frac{\delta t_b}{T}\right)} = \text{bunch length form factor.}$$

Finally, the amplitude of the beam induced field can be found by solving Eq. (2.1) combined with Eq. (3.2). This solution yields

$$V_b(t) = \begin{cases} 0 , & t < t_o \\ -\dfrac{\xi_o T_1 f_1 \bar{N}\, q\, Q\, V(0)}{\sqrt{2}\, \epsilon\, \omega} \left[1 - e^{-\frac{\omega}{2Q}(t-t_o)}\right] e^{j\omega(t-\bar{t}_b)} , & t > t_o \end{cases} \qquad (3.3)$$

for the condition that there is no beam before $t = t_o$. The quantity Q is the loaded Q of the cavity. Derivation of this solution is given in the appendix. The beam induced field can be found from Eq. (2.3):

$$\vec{E}_b(r,\theta,z,t) \approx V_b(t)\, \vec{E}_o(r,\theta,z) . \qquad (3.4)$$

A comparison of the result obtained here with that given by Nishikawa[7] shows that they differ by a multiplicative factor,* K, given by

$$K = \frac{J_o[\text{Eq. (3.2)}]}{J_o[\text{Nishikawa}]} = \frac{\bar{N}\, V(0)}{N\, V_b}$$

*The values of K for cavity numbers 1 and 2 of the Conversion linac are $2\,\frac{V(0)}{V_b}$ and $1.4\,\frac{V(0)}{V_b}$, respectively.

where V_b is the constant beam velocity assumed by Nishikawa. In particular, if we take

$$V_b = \frac{L}{NT}$$

then

$$K = \frac{T\,V(0)}{L_o} \approx 1 \quad ,$$

and both results have the same value.

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Appendix

The amplitude of the beam induced field in the accelerator cavity satisfies the differential equation:

$$\left[\frac{d^2}{dt^2} + (1 + jk)\frac{\omega}{Q}\frac{d}{dt} + \omega^2 \right] =$$

$$-\frac{1}{\epsilon}\frac{d}{dt} J_o(t)\, H(t - t_o) \qquad (2.1)$$

under the condition that there is no beam before $t = t_o$. Here H is the Heaviside function and

$$k = \frac{\text{Loaded } Q}{\text{Unloaded } Q} = \frac{Q}{Q_o} \quad .$$

The value of J_o is given by Eq. (3.3), which can be expressed in complex notation

$$J_o = K\, e^{j\omega t}$$

where

$$K = -\frac{\xi_o}{\sqrt{2}} T_1 f_1 \bar{N}\, q\, V(0)\, e^{-j\omega \bar{t}_b} \quad .$$

The boundary conditions are taken to be $V(t_o)$ and $V'(t_o)$ equal zero.

By a change of variable $\tau = t - t_o$, Eq. (2.1) becomes

$$\left[\frac{d^2}{d\tau^2} + (1 + jk)\frac{\omega}{Q}\frac{d}{d\tau} + \omega^2 \right] V_b(\tau) =$$

$$j\,\frac{\omega}{\epsilon} K\, e^{j\omega t_o}\, e^{j\omega\tau}\, H(\tau) \qquad (A1)$$

and $V_b(0) = V_b'(0) = 0$. Taking the Laplace transform of Eq. (A1) and solving for the transform of $V_b(\tau)$ gives

$$V(s) = \frac{\bar{K}}{(1 + \eta s)\left(1 + \frac{2\rho}{\omega} s + \frac{s^2}{\omega^2}\right)} \qquad (A2)$$

where

$$\eta = -\frac{1}{j\omega}$$

$$\rho = (1 + jk)\frac{1}{2Q}$$

$$\bar{K} = -\frac{K}{\epsilon\omega^2} e^{j\omega t_o} \quad .$$

Inverting V(s) yields

$$V_b(\tau) = \frac{\bar{K}\,\eta\,\omega^2}{(1 - 2\rho\eta\omega + \eta^2\omega^2)} \left[e^{-\tau/\eta} - e^{-\rho\omega\tau} \cos \omega\sqrt{1 - \rho^2}\,\tau \right]$$

$$+ \frac{K\,\omega\,(1 - \eta\rho\omega)\, e^{-\rho\omega\tau} \sin\omega\sqrt{1-\rho^2}\,\tau}{\sqrt{1 - \rho^2}\,(1 - 2\rho\eta\omega + \eta^2\omega^2)} \quad . \qquad (A3)$$

Since the value of Q for an accelerator cavity is generally sufficiently high ($1/Q \ll 1$), we have $|\rho^2| \ll 1$ and

$$1 - 2\rho\eta\omega + \eta^2\omega^2 \approx \frac{1}{jQ} \quad .$$

Under these conditions

$$V_b(\tau) \approx -\bar{K}\omega Q \left[\left(1 - e^{-\frac{\omega}{2Q}\tau}\right) e^{j\omega\tau} - \frac{1}{2Q} e^{-\frac{\omega}{2Q}(1+jk)\tau} \sin \omega\tau \right] . \qquad (A4)$$

Neglecting the second term in this expression and changing the variable from τ back to t, we find

$$V_b(t) = -\frac{\xi_o T_1 f_1 \bar{N}\, q\, Q\, V(0)}{\sqrt{2}\,\epsilon\,\omega} \left[1 - e^{-\frac{\omega}{2Q}(t-t_o)} \right] e^{j\omega(t-\bar{t}_b)}\, H(t-t_o) \quad . \qquad (A5)$$

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