Theory and phenomenology of neutrino oscillations

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A bit of history...

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I. Theory
Leptonic mixing

For \( m_\nu \neq 0 \) weak eigenstate neutrinos \( \nu_e, \nu_\mu, \nu_\tau \) do not coincide with mass eigenstate neutrinos \( \nu_1, \nu_2, \nu_3 \)

Diagonalization of leptonic mass matrices:

\[
e_L \rightarrow V_L e_L, \quad \nu_L \rightarrow U_L \nu_L \ldots \quad \Rightarrow
\]

\[-\mathcal{L}_{w+m} = \frac{g}{\sqrt{2}}(\bar{e}_L \gamma_\mu V_l^\dagger U_L \nu_L) W^\mu + \text{diag. mass terms}\]

Leptonic mixing matrix: \( U = V_l^\dagger U_L \)

\( \diamond \quad |\nu^a\rangle = \sum_i U^*_a i |\nu^\text{mass}_i\rangle \)
Oscillation probability in vacuum

For relativistic neutrinos: \[ E \approx p + \frac{m^2}{2p}, \quad L \approx t, \]

\[ P_{\nu_a \rightarrow \nu_b}(L) = \left| \sum_i U_{bi} e^{-i \frac{m_i^2}{2p} L} U_{ai}^* \right|^2 \]

– standard oscillation formula. For 2-flavor oscillations (good first approximation in many cases):

\[ |\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle \]
\[ |\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle \]

\[ P_{tr} = \sin^2 2\theta \sin^2 \left( \frac{\Delta m^2}{4E} L \right) \]
Modes of neutrinos oscillations

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- Dirac mass terms \( \bar{\nu}_L m_D N_R + h.c. \):
  - active - active oscillations \( \nu_{aL} \leftrightarrow \nu_{bL} \) \( (a, b = e, \mu, \tau) \)
  - Neutrinos are Dirac particles
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- **Majorana mass terms** \( \bar{\nu}_L m_L (\nu_L)^c + h.c. \):
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- **Dirac + Majorana mass terms**
  \[ \bar{\nu}_L m_D N_R + \bar{\nu}_L m_L (\nu_L)^c + \bar{N}_R M (N_R)^c + h.c. \]:
  - active - active oscillations \( \nu_aL \leftrightarrow \nu_bL \)
  - active - sterile oscillations \( \nu_aL \leftrightarrow (N_bR)^c \equiv (N_b^c)_L \)
  - Neutrinos are Majorana particles
Would observation of active - sterile $\nu$ oscillations mean that neutrinos are Majorana particles?

– Not necessarily!

In principle one can have active - sterile oscillations with only Dirac - type mass terms at the expense of introducing additional species of sterile neutrinos with opposite $L$. 
The MSW effect (Wolfenstein, 1978; Mikheyev & Smirnov, 1985)

Matter can change the pattern of neutrino oscillations drastically
Neutrino oscillations in matter

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Coherent forward scattering on the particles in matter

\[ V_{CC}^{e} \equiv V = \sqrt{2} G_F N_e \]

2f neutrino evolution equation:

\[
i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} -\frac{\Delta m^2}{4E} \cos 2\theta + V \\ \frac{\Delta m^2}{4E} \sin 2\theta \\ \frac{\Delta m^2}{4E} \sin 2\theta \\ \frac{\Delta m^2}{4E} \cos 2\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}
\]
Mixing in matter

\[ \sin^2 2\theta_m = \frac{\sin^2 2\theta \cdot (\frac{\Delta m^2}{2E})^2}{\left[ \frac{\Delta m^2}{2E} \cos 2\theta - \sqrt{2}G_F N_e \right]^2 + (\frac{\Delta m^2}{2E})^2 \sin^2 2\theta} \]
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Mikheyev - Smirnov - Wolfenstein (MSW) resonance:

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At the resonance: \( \theta_m = 45^\circ \) (\( \sin^2 2\theta_m = 1 \)) – maximal mixing
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\[ |\nu_e\rangle = \cos \theta_m |\nu_{1m}\rangle + \sin \theta_m |\nu_{2m}\rangle \]

\[ |\nu_{\mu}\rangle = -\sin \theta_m |\nu_{1m}\rangle + \cos \theta_m |\nu_{2m}\rangle \]

\( |\nu_{1m}\rangle, |\nu_{2m}\rangle \) – eigenstates of \( H \) in matter (matter eigenstates)
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\[ \sin^2 2\theta_m = \frac{\sin^2 2\theta \cdot \left( \frac{\Delta m^2}{2E} \right)^2}{[\frac{\Delta m^2}{2E} \cos 2\theta - \sqrt{2}G_FN_e]^2 + \left( \frac{\Delta m^2}{2E} \right)^2 \sin^2 2\theta} \]

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\[ |\nu_e\rangle = \cos \theta_m |\nu_{1m}\rangle + \sin \theta_m |\nu_{2m}\rangle \]

\[ |\nu_\mu\rangle = -\sin \theta_m |\nu_{1m}\rangle + \cos \theta_m |\nu_{2m}\rangle \]

\( N_e \gg (N_e)_{\text{res}} : \theta_m \approx 90^\circ \)

\( N_e = (N_e)_{\text{res}} : \theta_m = 45^\circ \)

\( N_e \ll (N_e)_{\text{res}} : \theta_m \approx \theta \)

\( |\nu_{1m}\rangle, |\nu_{2m}\rangle \) – eigenstates of \( H \) in matter (matter eigenstates)
Adiabatic flavour conversion

Adiabaticity: slow density change along the neutrino path

$$\frac{\sin^2 2\theta}{\cos 2\theta} \frac{\Delta m^2}{2E} L_\rho \gg 1$$

$L_\rho$ – electron density scale height:

$$L_\rho = \left| \frac{1}{N_e} \frac{dN_e}{dx} \right|^{-1}$$
Simple and useful formula for 2f conversion probability averaged over production/detection positions (or small energy intervals) (Parke, 1986):

\[ P_{tr} = \frac{1}{2} - \frac{1}{2} \cos 2\theta_i \cos 2\theta_f (1 - 2P') \]

\( \theta_i, \theta_f \) — mixing angles in matter in the initial and final points,
\( P' \) — hopping probability.

\( P' : \left\{ \begin{array}{ll} \ll 1 & \text{in adiab. regime} \\ \sin^2(\theta_i - \theta_f) & \text{in extreme non - adiab. regime} \end{array} \right. \)
Analogy: Spin precession in a magnetic field

\[
\frac{d\vec{S}}{dt} = 2(\vec{B} \times \vec{S})
\]

\[
\vec{S} = \{ \text{Re}(\nu_e^* \nu_\mu) , \text{Im}(\nu_e^* \nu_\mu) , \nu_e^* \nu_e - 1/2 \}
\]

\[
\vec{B} = \{(\Delta m^2 / 4E) \sin 2\theta_m , \ 0 , \ V/2 - (\Delta m^2 / 4E) \cos 2\theta_m \} 
\]
Analogy: Two coupled pendula

Mechanical model of the MSW effect
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Mechanical model of the MSW effect
Evidence for the MSW effect

Matter Interaction Effect: LMA

Current Data for $\nu_e$ Survival

$$V(x) \Rightarrow a_{\text{MSW}} V(x); \quad a_{\text{MSW}} = 1 \text{ strongly favoured}$$

(Fogli et al. 2003, 2004; Fogli & Lisi 2004)

More on MSW effect: talk of A. Friedland
II. Phenomenology
All current $\nu$ data except LSND can be explained in terms of oscillations between the 3 known neutrino species ($\nu_e, \nu_\mu, \nu_\tau$).
$3\nu$ vs $N_{\nu} \geq 4$ oscillation schemes

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LSND: most likely would require $\geq 1$ light sterile neutrinos $\nu_s$

(though some exotic scenarios exist: CPT violation, violation of Lorentz invariance, MaVaN, shortcuts in extra dimensions, decaying $\nu_s$, ...)
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MiniBooNE to confirm or refute the LSND result – an answer expected very soon!
3ν vs \( N_\nu \geq 4 \) oscillation schemes

All current \( \nu \) data except LSND can be explained in terms of oscillations between the 3 known neutrino species \( (\nu_e, \nu_\mu, \nu_\tau) \).

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(though some exotic scenarios exist: CPT violation, violation of Lorentz invariance, MaVaN, shortcuts in extra dimensions, decaying \( \nu_s \), ...)

MiniBooNE to confirm or refute the LSND result – an answer expected very soon!

But: even if the LSND result is not confirmed, this would not exclude the possibility of light sterile neutrinos and \( \nu_a \leftrightarrow \nu_s \) oscillations – an intriguing possibility with implications to particle physics, astrophysics and cosmology

More on sterile neutrinos: talk of A. Kusenko
For 3 neutrino species: mixing matrix $\tilde{U}$ depends on $\theta_{12}, \theta_{23}, \theta_{13}, \delta_{CP}, \sigma_{1,2}$. Majorana-type $CP$ phases can be factored out in the mixing matrix:

$$\tilde{U} = UK , \quad K = \text{diag}(1, e^{i\sigma_1}, e^{i\sigma_2})$$

$\Rightarrow$ Majorana-type phases do not affect neutrino oscillations.

The relevant part of the mixing matrix:

$$U = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13} e^{-i\delta_{CP}} \\
0 & 1 & 0 \\
-s_{13} e^{i\delta_{CP}} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}$$

$$= O_{23} \left( \Gamma_{\delta} O_{13} \Gamma_{\delta}^\dagger \right) O_{12} , \quad \Gamma_{\delta} \equiv \text{diag}(1, 1, e^{i\delta_{CP}})$$
$U = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\
-s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\
s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23}
\end{pmatrix}$

Normal hierarchy:

Inverted hierarchy:
2f and effective 2f approximations

2f description: A good 1st approximation in most cases.
Reasons:

- Hierarchy of $\Delta m^2$: $\Delta m^2_{\text{sol}} \ll \Delta m^2_{\text{atm}}$
- Smallness of $|U_{e3}|$.

Exceptions: $P(\nu_\mu \leftrightarrow \nu_\tau)$, $P(\nu_\mu \rightarrow \nu_\mu)$ and $P(\nu_\tau \rightarrow \nu_\tau)$ when oscillations due to the solar frequency ($\sim \Delta m^2_{\text{sol}}$) are not frozen.

In any case, corrections due to 3-flavorness can reach $\sim 10\%$ -- cannot be ignored at present

Also: a number of pure 3f effects exist  ⇒

◊ 3f analyses are a must!
Effective 2f approximations

For oscillations driven by $\Delta m^2_{\text{sol}} \nu_3$ essentially decouples. Still a “memory” of $\nu_3$ through unitarity $\Rightarrow$ powers of $c_{13}$. Examples:

Survival probability of solar $\nu_e$ (Lim, 1987)

(the same for reactor $\bar{\nu}_e$ in KamLAND):

$$
\diamond \quad P(\nu_e \rightarrow \nu_e) \simeq c_{13}^4 P_{2ee}(\Delta m^2_{21}, \theta_{12}, c_{13}^2 V) + s_{13}^4,
$$

3f effects for Day-Night effect for solar $\nu_e$:

While $P_D(\nu_e) \propto c_{13}^4$,

$$
P_N(\nu_e) - P_D(\nu_e) \propto c_{13}^6
$$

(Blennow, Ohlsson & Snellman, 2004; E.A., Tortola & Valle, 2004)

Deviations from 2f results: $(1 - c_{13}^4) \leq 0.1$, $(1 - c_{13}^6) \leq 0.13$
Reactor $\bar{\nu}_e$ oscillations

$\bar{\nu}_e$ survival probability:

\[
P_{\bar{\nu}_e \bar{\nu}_e} \approx 1 - \sin^2 2\theta_{13} \cdot \sin^2 \left( \frac{\Delta m^2_{31}}{4E} L \right) - c^4_{13} \sin^2 2\theta_{12} \cdot \sin^2 \left( \frac{\Delta m^2_{21}}{4E} L \right)
\]

- CHOOZ, Palo Verde, Double CHOOZ, ... ($L \lesssim 1$ km)

\[
\bar{E} \sim 4 \text{ MeV} ; \quad \frac{\Delta m_{31}^2}{4E} L \sim 1 ; \quad \frac{\Delta m_{21}^2}{4E} L \ll 1
\]

One mass scale dominance (2f) approximation:

\[
P(\bar{\nu}_e \rightarrow \bar{\nu}_e; L) = 1 - \sin^2 2\theta_{13} \cdot \sin^2 \left( \frac{\Delta m_{31}^2}{4E} L \right)
\]

(Note: Term $\sim \sin^2 2\theta_{12}$ cannot be neglected if $\theta_{13} \lesssim 0.03$, which is about the reach of currently discussed future reactor experiments)
Reactor $\bar{\nu}_e$ oscillations – contd.

- KamLAND ($\bar{L} \simeq 170$ km): $\frac{\Delta m^2_{21}}{4E} L \gtrsim 1$; $\frac{\Delta m^2_{31}}{4E} L \gg 1$

- $P(\bar{\nu}_e \rightarrow \bar{\nu}_e) \simeq c_{13}^4 P_{2\bar{e}e}(\Delta m^2_{21}, \theta_{12})$

N.B.: Matter effects a few % – can be comparable with effects of $\theta_{13} \neq 0$!
Genuine 3f effects
\( \mathcal{CP} \) and \( \mathcal{T} \) in \( \nu \) oscillations in vacuum

- \( \mathcal{CP} : \ P(\nu_a \to \nu_b) \neq P(\bar{\nu}_a \to \bar{\nu}_b) \)
- \( \mathcal{T} : \ P(\nu_a \to \nu_b) \neq P(\nu_b \to \nu_a) \)

CPT invariance: \( \diamond \ P(\nu_a \to \nu_b) \to P(\bar{\nu}_b \to \bar{\nu}_a) \)

\[ \mathcal{CP} \Leftrightarrow \mathcal{T} \text{ – consequence of CPT} \]

Measures of \( \mathcal{CP} \) and \( \mathcal{T} \) – probability differences:

\[ \Delta P_{ab}^{\mathcal{CP}} \equiv P(\nu_a \to \nu_b) - P(\bar{\nu}_a \to \bar{\nu}_b) \]

\[ \Delta P_{ab}^{\mathcal{T}} \equiv P(\nu_a \to \nu_b) - P(\nu_b \to \nu_a) \]

From CPT:

\[ \diamond \quad \Delta P_{ab}^{\mathcal{CP}} = \Delta P_{ab}^{\mathcal{T}}; \quad \Delta P_{aa}^{\mathcal{CP}} = 0 \]
3f case

One Dirac-type phase $\delta_{\text{CP}} \Rightarrow$ one $\mathcal{CP}$ and $\mathcal{T}$ observable:

\[ \Delta P_{e\mu}^{\text{CP}} = \Delta P_{\mu\tau}^{\text{CP}} = \Delta P_{\tau e}^{\text{CP}} \equiv \Delta P \]
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\[ \Delta P_{e\mu}^{\text{CP}} = \Delta P_{\mu\tau}^{\text{CP}} = \Delta P_{\tau e}^{\text{CP}} \equiv \Delta P \]

\( \Delta P = -4s_{12} c_{12} s_{13} c_{13}^2 s_{23} c_{23} \sin \delta_{\text{CP}} \)

\[ \times \left[ \sin \left( \frac{\Delta m_{12}^2}{2E} L \right) + \sin \left( \frac{\Delta m_{23}^2}{2E} L \right) + \sin \left( \frac{\Delta m_{31}^2}{2E} L \right) \right] \]
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Vanishes when

- At least one $\Delta m_{ij}^2 = 0$
- At least one $\theta_{ij} = 0$ or $90^\circ$
- $\delta_{\text{CP}} = 0$ or $180^\circ$
- In the averaging regime
- In the limit $L \to 0$ (as $L^3$)
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\begin{align*}
\diamond \quad \Delta P_{e\mu}^{CP} &= \Delta P_{\mu\tau}^{CP} = \Delta P_{\tau e}^{CP} \equiv \Delta P \\
\Delta P &= -4s_{12}c_{12}s_{13}c_{13}^2s_{23}c_{23}\sin\delta_{\text{CP}} \\
&\quad \times \left[ \sin \left( \frac{\Delta m_{12}^2}{2E} L \right) + \sin \left( \frac{\Delta m_{23}^2}{2E} L \right) + \sin \left( \frac{\Delta m_{31}^2}{2E} L \right) \right]
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Very difficult to observe!

See talk of O. Mena
Normal matter \([\text{(\# of particles)} \neq \text{(\# of anti-particles)}]\):

The very presence of matter violates C, CP and CPT

\(\Rightarrow\) Fake (extrinsic) CP. Exists even in 2f case. May complicate study of fundamental (intrinsic) CP.
CP and T in \( \nu \) oscillations in matter

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Matter with density profile symmetric w.r.t. midpoint of neutrino trajectory does not induce any fake T. Asymmetric profiles do, but only for \( N \geq 3 \) flavors – an interesting 3f effect.
\( CP \) and \( T \) in \( \nu \) oscillations in matter

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◊ May fake fundamental \( T \) and complicate its study (extraction of \( \delta_{CP} \) from experiment)
$\mathbb{C}P$ and $\mathcal{T}$ in $\nu$ oscillations in matter

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Matter with density profile symmetric w.r.t. midpoint of neutrino trajectory does not induce any fake $\mathcal{T}$. Asymmetric profiles do, but only for $N \geq 3$ flavors – an interesting 3f effect.

\[\diamond \quad \text{May fake fundamental } \mathcal{T}\text{ and complicate its study (extraction of } \delta_{CP} \text{ from experiment)}\]

Induced $\mathcal{T}$: absent when either $U_{e3} = 0$ or $\Delta m_{\text{sol}}^2 = 0$ (2f limits)

\[\Rightarrow \quad \text{Doubly suppressed by both these small parameters – effects in terrestrial experiments are small}\]
In 2f approximation: no matter effects on $\nu_\mu \leftrightarrow \nu_\tau$ oscillations

$[V(\nu_\mu) = V(\nu_\tau) \text{ modulo tiny rad. corrections}].$

Not true in the full 3f framework! (E.A., 2002; Gandhi et al., 2004)

$\Delta m^2_{31} = 2.5 \times 10^{-3} \text{ eV}^2$, $\sin^2 \theta_{13} = 0.026$, $\theta_{23} = \pi/4$, $\Delta m^2_{21} = 0$, $L = 9400 \text{ km}$

Red curves – w/ matter effects, green curves – w/o matter effects on $P_{\mu\tau}$
Another possible matter effect
Parametric resonance in oscillating systems with varying parameters: occurs when the rate of the parameter change is correlated in a certain way with the values of the parameters themselves.
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\[ \Omega \]
Parametric resonance in oscillating systems with varying parameters: occurs when the rate of the parameter change is correlated in a certain way with the values of the parameters themselves.

For small-ampl. osc.:

$$\Omega_{\text{res}} = \frac{2\omega}{n}$$

$$n = 1, 2, 3, ...$$
Different from MSW eff. – no level crossing!


Resonance condition:

\[ X_3 \equiv - (\sin \phi_1 \cos \phi_2 \cos 2\theta_{1m} + \cos \phi_1 \sin \phi_2 \cos 2\theta_{2m}) = 0 \]

\[ \phi_{1,2} \text{ – oscillation phases acquired in layers 1, 2} \]
Earth’s density profile (PREM model):
Earth’s density profile (PREM model):

![Graph showing Earth's density profile](image)
Fulfilled for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations of core-crossing $\nu$'s in the Earth for a wide range of energies and zenith angles!

Intermed. energies

$\cos \Theta = -0.89 \quad \sin^2 2\theta_{13} = 0.01$

(Liu, Smirnov, 1998; Petcov, 1998; EA 1998)

High energies, $\cos \Theta$ - dependence

(EA, Maltoni & Smirnov, 2005)
Parametric resonance of $\nu$ oscillations in the Earth:
can be observed in future atmospheric or accelerator experiments if $\theta_{13}$ is not much below its current upper limit
Some recent developments
Oscillations of low-$E$ neutrinos in matter

Equivalently: Oscillations in low-density matter ($V \ll \frac{\Delta m^2}{2E}$). Matter effects small – can be considered in perturbation theory. Implications: oscillations of solar and SN neutrinos in the Earth.

In 3f framework

$$P_{2e}^{+} - P_{2e}^{(0)} = \frac{1}{2} c_{13}^4 \sin^2 2\theta_{12} \int_0^L dx V(x) \sin \left[ 2 \int_x^L \omega(x') dx' \right]$$

where

$$\diamond \quad \omega(x) = \sqrt{[\cos 2\theta_{12} \delta - c_{13}^2 V(x)/2]^2 + \delta^2 \sin^2 2\theta_{12}} , \quad \delta = \frac{\Delta m^2_{21}}{4E}$$

2f case ($\theta_{13} = 0$): de Holanda, Liao & Smirnov, 2004; Ioannisian & Smirnov, 2004;

3f case: E.A., Tórtola & Valle, 2004
Attenuation effect

Perfect energy resolution

Finite energy resolution: effects of density variation far from detector suppressed. Attenuation length $d$:

$$d \simeq l_{osc} \frac{E}{\Delta E}$$

(de Holanda et al., 2004)
Oscillations above the MSW resonance

Equivalently: oscillations in dense matter \((V > \delta \equiv \frac{\Delta m^2}{4E})\)

Oscillation probability in matter of arbitrary density profile:

\[
P = \delta^2 \sin^2 2\theta \left| \int_0^L dx e^{-2i\phi(x)} \right|^2, \quad \phi(x) = \int_0^x dx' \omega(x') - \text{adiab.phase}
\]

E.A., Maltoni & Smirnov, 2005
Unsettled issues?

A number of issues in $\nu$ oscillation theory still being debated

- Equal energies or equal momenta?

- Evolution in space or in time?

  Claim: evolution in time is never observed.

- Is wave packet description necessary?
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- Is wave packet description necessary?
  - Yes, if one wants to rigorously justify the standard oscillation probability formula. Once done, can be forgotten unless the issues of coherence become important.
Unsettled issues?

Do charged leptons oscillate?
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– No, they don’t
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  – No, they don’t

• Is the standard oscillation formula correct?
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– Yes, it is. In particular, no extra factors of two in the oscillation phase. But: theoretically interesting and important to study the limits of applicability.
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A number of subtle issues of oscillation theory remain unsettled (e.g., rigorous wave packet treatment, limits of applicability of standard formula, oscillations of non-relativistic neutrinos, …). At present, this is (rightfully) of little concern for practitioners.
Future tasks

- Search for best strategies for measuring neutrino parameters
- Study of subleading effects and effects of non-standard neutrino interactions
- Study of the domains of applicability and limitations of the current theoretical framework

Future experimental results may bring some new surprises and pose more challenging problems!
General properties of $P_{ab}$

3 flavours $\Rightarrow 3 \times 3 = 9$ probabilities

$$P_{ab} = P(\nu_a \rightarrow \nu_b),$$

plus 9 probabilities for antineutrinos $P_{\bar{a}\bar{b}}$.

Unitarity conditions (probability conservation):

$$\sum_b P_{ab} = \sum_a P_{ab} = 1 \quad (a, b = e, \mu, \tau)$$

5 indep. conditions $\Rightarrow 9 - 5 = 4$ indep. probabilities left.

Additional symmetry: the matrix of matter-induced potentials $\text{diag}(V(t), 0, 0)$ commutes with $O_{23}$ $\Rightarrow$ additional relations between probabilities.
Define

$$\tilde{P}_{ab} = P_{ab}(s_{23}^2 \leftrightarrow c_{23}^2, \sin 2\theta_{23} \rightarrow -\sin 2\theta_{23})$$

(e.g., $\theta_{23} \rightarrow \theta_{23} + \pi/2$). Then

$$P_{e\tau} = \tilde{P}_{e\mu} \quad P_{\tau\mu} = \tilde{P}_{\mu\tau} \quad P_{\tau\tau} = \tilde{P}_{\mu\mu}$$

2 out of 3 conditions are independent $\Rightarrow 4 - 2 = 2$

indep. probabilities (e.g., $P_{e\mu}$ and $P_{\mu\tau}$) $\Rightarrow$

◊ **All 9 neutrino oscillation probabilities can be expressed through just two!** (E.A., Johansson, Ohlsson, Lindner & Schwetz, 2004)

$$P_{\bar{a}\bar{b}} = P_{ab}(\delta_{CP} \rightarrow -\delta_{CP}, V \rightarrow -V) \Rightarrow$$

◊ **All 18 $\nu$ and $\bar{\nu}$ probab. can be expressed through just two**
General dependence on $\delta_{\text{CP}}$

Another use of essentially the same symmetry: rotate by

$$O'_{23} = O_{23} \times \text{diag}(1, 1, e^{i\delta_{\text{CP}}})$$

From commutativity of $\text{diag}(V(t), 0, 0)$ with $O'_{23}$ ⇒

General dependence of probabilities on $\delta_{\text{CP}}$:

$$P_{e\mu} = A_{e\mu} \cos \delta_{\text{CP}} + B_{e\mu} \sin \delta_{\text{CP}} + C_{e\mu}$$

$$P_{\mu\tau} = A_{\mu\tau} \cos \delta_{\text{CP}} + B_{\mu\tau} \sin \delta_{\text{CP}} + C_{\mu\tau}$$

$$+ D_{\mu\tau} \cos 2\delta_{\text{CP}} + E_{\mu\tau} \sin 2\delta_{\text{CP}}$$

(Yokomakura, Kimura & Takamura, 2002)
3f effects in atm. $\nu$ oscillations

$\Delta m_{21}^2 \rightarrow 0$ (E.A., Dighe, Lipari & Smirnov, 1998): 

$$\frac{F_e - F_e^0}{F_e^0} = P_2(\Delta m_{31}^2, \theta_{13}, V_{CC}) \cdot (r s_{23}^2 - 1)$$

$s_{13} \rightarrow 0$ (Peres & Smirnov, 1999):

$$\frac{F_e - F_e^0}{F_e^0} = P_2(\Delta m_{21}^2, \theta_{12}, V_{CC}) \cdot (r c_{23}^2 - 1)$$

At low energies $r \equiv F_\mu^0/F_e^0 \sim 2$; also $s_{23}^2 \sim c_{23}^2 \sim 1/2$ – a conspiracy to hide oscillation effects on e-like events!

Reason: a peculiar flavour composition of the atmospheric $\nu$ flux.

(Because of $\theta_{23} \sim 45^\circ$, $P_{e\mu} \sim P_{e\tau}$; but the original $\nu_\mu$ flux is $\sim 2$ times larger than $\nu_e$ flux $\Rightarrow$ compensation of transitions from and to $\nu_e$ state).
Breaking the conspiracy – 3f effects

\[
\frac{F_e - F_e^0}{F_e^0} \approx P_2(\Delta m_{31}^2, \theta_{13}) \cdot (r s_{23}^2 - 1) \\
+ P_2(\Delta m_{21}^2, \theta_{12}) \cdot (r c_{23}^2 - 1) \\
- 2s_{13} s_{23} c_{23} r \text{Re}(\bar{A}_{ee} \bar{A}_{\mu e})
\]

Interference term not suppressed by the flavour composition of the $\nu_{\text{atm}}$ flux; may be (partly) responsible for observed excess of upward-going sub-GeV e-like events

Interf. term may not be sufficient to fully explain the excess of low-$E$ e-like events – a hint of $\theta_{23} \neq 45^\circ$? (Peres & Smirnov, 2004)
Evolution in the rotated basis

Evolution matrix $S(t, t_0)$: $\nu(t) = S(t, t_0) \nu(t_0)$. Satisfies

\[ i \frac{d}{dt} S(t, t_0) = H S(t, t_0) \quad \text{with} \quad S(t_0, t_0) = 1. \]

\[
H = (O_{23} \Gamma_\delta O_{13} \Gamma_\delta^+ O_{12}) \text{diag}(0, \delta, \Delta) (O_{12}^T \Gamma_\delta O_{13}^T \Gamma_\delta^+ O_{23}^T) + \text{diag}(V(t), 0, 0)
\]

\[
= (O_{23} \Gamma_\delta O_{13} O_{12}) \text{diag}(0, \delta, \Delta) (O_{12}^T O_{13}^T \Gamma_\delta^+ O_{23}^T) + \text{diag}(V(t), 0, 0)
\]

where

\[
\delta \equiv \frac{\Delta m_{21}^2}{2E}, \quad \Delta \equiv \frac{\Delta m_{31}^2}{2E}
\]

Oscillation probabilities:

\[
P_{ab} = |S_{ba}|^2
\]

Define

\[
O'_{23} = O_{23} \Gamma_\delta
\]

The matrix $\text{diag}(V(t), 0, 0)$ commutes with $O'_{23}$ $\Rightarrow$ go to the rotated basis
\[ \nu = O'_{23} \nu', \quad \text{or} \quad S(t, t_0) = O'_{23} S'(t, t_0) O'_{23}^\dagger, \]

In the rotated basis \( H' = O'_{23} H O'_{23}^\dagger \). Explicitly:

\[
H'(t) = \begin{pmatrix}
  s_{12}^2 c_{13}^2 \delta + s_{13}^2 \Delta + V(t) & s_{12} c_{12} c_{13} \delta & s_{13} c_{13} (\Delta - s_{12}^2 \delta) \\
  s_{12} c_{12} c_{13} \delta & c_{12}^2 \delta & -s_{12} c_{12} s_{13} \delta \\
  s_{13} c_{13} (\Delta - s_{12}^2 \delta) & -s_{12} c_{12} s_{13} \delta & c_{13}^2 \Delta + s_{12}^2 s_{13}^2 \delta
\end{pmatrix}
\]

Dependence on \( \theta_{23} \) and \( \delta_{\text{CP}} \) can be obtained in the general case by rotating back to the original flavour basis. Also: easy to apply PT approximations

- If \( \frac{\Delta m_{21}^2}{2E} L \ll 1 \) – neglect \( \delta = \frac{\Delta m_{21}^2}{2E} \)
- If \( \theta_{13} \) is very small – neglect \( s_{13} \)

or use expansion in these small parameters