LONGITUDINAL COUPLING IMPEDANCE OF PICKUP PLATES WITH TERMINATIONS AT BOTH ENDS

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The purpose of this note is to compute the longitudinal impedance of a pickup plate with terminations at both ends. The plate has length \( l \), characteristic impedance \( Z_0 \) and azimuthal half angle \( \phi_0 \) (Figure 1). Each termination carries an impedance \( Z_T \).

1. Transmission line equations.

When a linear charge disturbance

\[ \lambda_1 = \lambda_{10} e^{j(\omega t-kz)} \]  \hspace{1cm} (1)

develops in the beam, due to charge conservation, the disturbance current is

\[ I_1 = I_{10} e^{j(\omega t-kz)}, \]  \hspace{1cm} (2)

with

\[ I_{10} = \lambda_{10} B_W c \]  \hspace{1cm} (3)

where \( k = n/R \) defines the longitudinal mode number \( n \) and \( \omega \) the angular velocity of the disturbance; \( R \) is the radius of the machine. \( B_W c = \omega/k \) is the linear velocity of the disturbance. \( z \) is the direction of motion of the particles. This disturbance will induce a charge density \( \sigma_1 \) and current density \( J_1 \) on the pickup plate:

\[ \sigma_1 = -\frac{\lambda_1}{2\pi b} g_{\text{long}}, \]  \hspace{1cm} (4)

\[ J_1 = -\frac{\lambda_1}{2\pi b} g_{\text{long}}, \]  \hspace{1cm} (5)

where \( q = k \sqrt{1-B_W^2} \); \( a \) and \( b \) are the radii of the beam and the pipe respectively; \( I_0 \) and \( I_1 \) are modified Bessel functions of order 0 and 1 respectively. Note that, as \( q \rightarrow 0, g_{\text{long}} \rightarrow 1 \).

The pickup plate and ground form a transmission line. The line equations for the supplementary scalar potential \( V_1 \) and longitudinal po-
ential \( A_1 \) (set to zero for ground) produced by the plate are

\[
\frac{1}{c} \frac{\partial V_1}{\partial t} + \frac{\partial A_1}{\partial z} = 0, \tag{7}
\]

\[
\frac{\partial V_1}{\partial z} + \frac{1}{c} \frac{\partial A_1}{\partial t} = 0. \tag{8}
\]

Equation (7) expresses the charge conservation law and is homogeneous because the linear velocity and the phase velocity of a longitudinal disturbance are exactly equal. Equation (8) is homogeneous because the plate is assumed to be a perfect conductor.

The most general solution of the transmission equation is

\[
\begin{aligned}
\{ V_1(z,t) \} &= (ae^{j\omega(z_s-z)} \pm be^{-j\omega(z_s-z)})e^{j\omega t} \\
\{ A_1(z,t) \} &
\end{aligned}
\tag{9}
\]

subject to the boundary conditions that the currents at both ends of the plate extending from \( z = z_0 \) to \( z_s + \ell \) must be zero:

\[
\frac{A_1(z_s,t)}{Z_0} = -\frac{V_1(z_s,t)}{Z_r} - 2\phi_0 b J_1(z_s,t), \tag{10}
\]

\[
\frac{A_1(z_s + \ell,t)}{Z_0} = \frac{V_1(z_s + \ell,t)}{Z_r} - 2\phi_0 b J_1(z_s + \ell,t). \tag{11}
\]

If we match \( Z_0 = Z_T \), we get

\[
a = -\phi_0 b Z_0 J_s, \tag{12}
\]

\[
b = \phi_0 b Z_0 J_s e^{-j\omega z_s - jk\ell}, \tag{13}
\]

with

\[
J_s = \frac{I_{10}}{2\pi b} g_{\text{long}} e^{-jkz_s}. \tag{14}
\]

2. Longitudinal Impedance

Elsewhere (not necessarily on the plate), the supplementary potentials, since obeying Eqs. (7) and (8), can be written as
\[
\begin{align*}
\{V_1(\rho, \phi, z, t)\} &= \sum_{p, h} \phi_p \{V_h\} \frac{i_P(q \rho)}{I_p(q b)} \cos p \phi e^{j(\omega t - \frac{h}{R} z)} \\
\{A_1(\rho, \phi, z, t)\} &= \sum_{p, h} \phi_p \{A_h\} \frac{i_P(q \rho)}{I_p(q b)} \cos p \phi e^{j(\omega t - \frac{h}{R} z)}.
\end{align*}
\]

As all modes \(p \neq 0\) and \(h \neq n\) are orthogonal to the fundamental mode of the disturbance, the forces produced by these modes are ineffective over a complete turn in the machine. We therefore retain only \(h = n\) and \(p = 0\). Defining
\[
\phi_0 = \frac{2 \phi_0 b}{2\pi} \int_{-\phi_0}^{\phi_0} d\phi = \frac{2 b \phi_0^2}{\pi},
\]
we get
\[
\begin{align*}
\{V_n\} e^{j \omega t} &= \frac{1}{2\pi R} \frac{1}{2b \phi_0} \int_{z_s}^{z_s + z} dz \ e^{j n z} \ \{V_1\} \\
\{A_n\} e^{j \omega t} &= \frac{k}{4\pi R} \frac{Z_0 J_s e^{j k z_s}}{j \theta (1 - \beta_w^2)} \ \{A_1\} e^{j \omega t}
\end{align*}
\]

with
\[
\begin{align*}
C_1 &= -\sin 2\phi \sin 2\theta - j \sin 2\phi \cos 2\phi, \\
C_2 &= 1 - \cos 2\phi \cos 2\theta + j \cos 2\phi \sin 2\phi,
\end{align*}
\]

\(2\theta = k z, \quad 2\phi = \frac{\omega}{c} z.\) (17)

We note that Eq. (15) is independent of \(z_s\), the position of the plate along the beam pipe. Thus for \(M\) identical plates, \(B\) just multiply Eq. (15) by \(M\).

At the center of the beam, \(\rho = 0\), the supplementary longitudinal electric field due to the \(M\) plates is
\[
E_z(\rho = 0) = \frac{3y_1}{3z} - \frac{1}{c} A_1
\]

\[
= jk \frac{V_n - \beta_w A_n}{I_0(q b)} \phi_0 e^{j(\omega t - k z)}
\]

\[
= \frac{M}{2\pi R} \left( \frac{2 b \phi_0^2}{\pi} \right) \frac{Z_0 J_s}{I_0(q b)} \ C_2. \quad (18)
\]

The potential seen by the beam in one resolution is
\[
U_s = 2\pi R E_z(\rho = 0). \quad (19)
\]

Therefore, the longitudinal impedance due to the plate is
\[ Z_L = -\frac{U_S}{I_1} \]
\[ = M \left( \frac{\phi_0}{\pi} \right)^2 Z_0 \frac{g_{long}}{I_0(qb)} C_2. \]  

(20)

We now let \( \varepsilon_w \to 1 \), then

\[ g_{long}/I_0(qb) \to 1, \]

\[ \theta \to \phi, \]

\[ C_2 \to \sin^2 2\phi + j \sin 2\phi \cos 2\phi. \]

Thus

\[ Z_L = M \left( \frac{\phi_0}{\pi} \right)^2 Z_0 \left( \sin^2 2\phi + j \sin 2\phi \cos 2\phi \right), \]

(21)

agreeing exactly with Shafer's result\(^2\). When the wavelength of the disturbance is long compared with \( R/n \), we get

\[ \frac{Z_L}{\sqrt{\lambda}} = jM \left( \frac{\phi_0}{\pi} \right)^2 Z_0 \frac{g}{R} \]

(22)

which is inductive.

3. Voltage along a pickup plate

Substituting (12) and (13) into Eq. (9), we get the voltage along a plate

\[ V_1(z,t) = -\phi_0 bZ_0 J_s \left[ e^{i\omega(z_0-z)} - e^{-i\omega(z_0-z+2\ell)} \right] e^{i\omega t} \]

(23)

assuming \( \varepsilon_w = 1 \). Obviously \( V_1(z_0+\ell,t) = 0 \), i.e., the downstream end of the plate is floating, a result predicted by Shafer\(^2\). Thus the downstream termination can be removed without affecting the whole system. As a result, our result can be compared with that obtained by Ruggiero\(^1\) for pickup plates with only one termination situated at a distance \( \frac{\ell}{2} (1+\delta) \) from the upstream end (\( \delta \) ranges from -1 to 1). His value of longitudinal force per unit charge at the center of the beam is

\[ F_{long} = E_z(\rho=0) = jB \frac{M_2}{2\pi R} \left( \frac{\phi_0^2}{2\pi^2} \right) \left( \frac{g_{long}}{I_0(qb)} \right) (p_{long}) \frac{\lambda_1}{\ell C}. \]

(24)
where \( C = (cZ_0)^{-1} \) is the capacitance per unit length of the plate with respect to ground and

\[
p_{\text{long}} = \frac{\beta_w}{2} \frac{2j\sin(2\phi - \cos 2\phi) - \sin 2\phi}{2 \cos 2\delta + \cos 2\phi + 2j\sin 2\phi},
\]

with \( r = Z_r/Z_0 \). When the impedances match \( (\delta = 1) \), the termination is at the upstream end of the plate \( (\delta = 1) \), and \( \beta_w = 1 \), we get

\[
p_{\text{long}} = -\frac{1}{4} (\sin 2\phi \cos 2\phi - j \sin^2 2\phi).
\]

Using Eqs. (2) and (3), (19) and (20), we arrive at

\[
Z_L = M \left( \frac{\phi_0}{\pi} \right)^2 Z_0 \left( \sin^2 2\phi + j \sin 2\phi \cos 2\phi \right).
\]

which agrees with Eq. (21).

From Eq. (23), the voltage at the upstream end of a plate is

\[
V_1(z_s, t) = \frac{\phi_0 b}{2\pi} I_0 Z_0 \left( \sin^2 2\phi + j \sin 2\phi \cos 2\phi \right) e^{j(\omega t - nz_s/R)}.
\]

Thus, the average power consumed for \( M \) plates is

\[
\langle P \rangle = \frac{1}{2} \frac{|V_1(z_s, t)|^2}{\text{Re}Z_T} = \frac{1}{2} M \left( \frac{\phi_0}{\pi} \right)^2 Z_0 |I_0|^2 \sin^2 2\phi
\]

which equals, as it should, \( \frac{1}{2} \text{Re}Z_L |I_0|^2 \), the average power lost by the beam.

4. Arbitrary \( Z_T \)

Matching boundary conditions (10) and (11), line equations (7) and (8) lead to a supplementary longitudinal impedance (due to \( M \) plates) of

\[
Z_L = M \left( \frac{\phi_0}{\pi} \right)^2 Z_0 \frac{\alpha_{\text{long}}}{I_0(qb)} C^2
\]
with

\[ C_2^- = \frac{r^2 (\cos 2\phi - \cos 2\theta) + j r \sin 2\phi}{r \cos 2\phi + \frac{1}{2} j (1 + r^2) \sin 2\phi} \]  

(31)

As \( \beta_w \to 1 \), we get

\[ C_2^- = \frac{j \sin 2\phi}{\cos 2\phi + \frac{1}{2} j (1 + r^2) \sin 2\phi} \]  

(32)

\[ \left| \frac{Z_L(r)}{Z_L'(r=1)} \right| = \left[ 1 + \left( \frac{1-r^2}{2r} \right)^2 \sin^2 2\phi \right]^{-1/2} \]  

(33)

Therefore \( Z_L \) can be decreased and stability improved by not matching \( Z_T \) and \( Z_0 \).

REFERENCES
