SRC 10020

Fermilab

$\bar{p}$ Note #449

Ac

Antiproton Yield Calculations

G. Dugan
1/20/86

Antiproton Yield Calculations

G. Dugan

Jan. 20, 1986

In the past, pbar yield calculations have been done using Monte Carlo hadronic cascade programs[1] (based on measured cross sections) which have the advantage of including effects such as secondary production and multiple Coulomb scattering. However, because of their complexity, it is sometimes difficult to understand the dependences of the yield on various parameters of the system, and the development of an intuitive feel for the calculations is thereby inhibited. It is therefore useful to supplement these Monte Carlo calculations with less complex but stil relatively realistic estimates of the yield, based on analytical calculations. As will be shown below, the only significant feature of the Monte Carlo calculations omitted in the analytical development is the secondary production in the target.

The analytical development follows closely the technique discussed in ref 2. Basically, a transport equation for the pbar distribution function in phase space is written down. in addition to pbar production, it includes proton and pbar absorption in the target, but neglects multiple Coulomb scattering and energy loss. Since the pbar production is derived from the primary proton beam only, secondary production is neglected. Multiple Coulomb scattering of both the protons and pbars in the target is also neglected. However, this effect can be shown to be negligible for our case (see Appendix A-2). All other significant features of the pbar production and collection system are included in the calculation.

Based on the parameterization of ref 1, the pbar production cross section's angular dependence is plotted in fig. 1. (See Appendix A-1 for details). This angular dependence is fit very well by a gaussian, $e^{-\theta^2/\theta_0^2}$, with $\theta_0 =$ 75.4 mrad (for 120 GeV incident protons on tungsten, and 8.9 GeV/c pbars). This angular dependence is used throughout the calculations. The proton beam spatial distribution is also modelled as a Gaussian of rms size $\sigma_x = \sigma_y = \sigma_n/\sqrt{2}$.

To obtain the yield of pbars into a given system (e.g., beamline, Debuncher, etc.) the pbar distribution function at the target (the solution to the transport equation) must be integrated over the acceptance of the system, projected back to the target. The determination of the acceptance of a complex system like a beam line is best done using a numerical (ray-tracing) approach; however, for the first and most important element in the line (the lithium lens), the acceptance can be idealized as an ellipse in phase space, and the sensitivity of the yield to the parameters of the lens (and target) can be studied

The details of the integration of the pbar distribution function over the acceptance ellipse are expounded in Appendix A-3. The analytical development leads to a number of integrals which must be evaluated numerically on the computer. As explained in the appendix, under some simplifying (but unrealistic) assumptions, the integrals can be evaluated analytically. For example, the integral over all transverse phase space produces a simple equation for the yield Y, which is defined as

$$Y=\frac{N_{\bar{p}}}{N_p\Delta p} \qquad (1)$$

Here $N_{\bar{p}}$ = number of antiprotons in the momentum interval Δp, and N_p = the number of protons. The dependence of the pbar production cross section on momentum is taken to be flat. The total yield is (see Appendix A-3, p. 4):

$$Y_{tot}(z)=\frac{1}{\lambda}\left(\frac{1}{\sigma_a}\frac{d\sigma}{dp}\right)ze^{-z/\lambda} \qquad (2)$$

Here z=target length, $\frac{d\sigma}{dp}$ =pbar production cross section for 8.9 GeV/c pbars by 120 GeV protons on tungsten, λ = proton absorption length in tungsten and σ_a = proton absorption cross section in tungsten. The pbar absorption length in tungsten ($\bar{\lambda}$) is taken to be equal to λ in this situation (although not in the computer calculations); numerically, λ =9.36 cm and $\bar{\lambda}$ =[illegible] Y_{tot} maximizes at λ

$$Y_{max}=Y_{tot}(\lambda)=\frac{1}{e}\left(\frac{1}{\sigma_a}\frac{d\sigma}{dp}\right) \qquad (3)$$

Numerically, $1/\sigma_a\, d\sigma/dp$ =4.5x10**-3/GeV/c, so Y_{max} =1660 ppm/GeV/c. As will be seen below, the design value of Y is [illegible] ppm/GeV/c, only about 3% of the total

An analytical result for the yield into a finite acceptance area $\pi\varepsilon$ ($=\pi\varepsilon_x=\pi\varepsilon_y$) can be obtained as detailed in Appendix A-3, p. 10-11, in the limit $\sigma_n\rightarrow 0$ and $\varepsilon\gamma<<\theta_0^2$. With γ = $1/\beta_t$ for the upright acceptance ellipse at the center of the target, the result is

$$Y(z)=\frac{e^{-z/\lambda}}{\pi}\left(\frac{1}{\sigma_a}\frac{d\sigma}{dp}\right)\frac{1}{\lambda\theta_0^2}8\varepsilon\tan^{-1}\left(\frac{z\gamma}{2}\right) \qquad (4)$$

Since $\tan^{-1}\frac{z\gamma}{2}$ grows more slowly than z, it is clear that Y(z) optimizes at $z < \lambda$, showing that targets shorter than 9 cm are best for small emittance collection. However, in our case, neither of the conditions $\sigma_n \to 0$ and $\varepsilon\gamma \ll \theta_0^2$ is satisfied, as mentioned above, so this result is only of academic interest. The full expression presented in Appendix A-3 must be integrated numerically over the lens acceptance ellipse.

The acceptance ellipse of the lens is determined by the lens optics. The development of the relation between the acceptance ellipse parameters at the target and the lens parameters (length, radius, gradient) is presented in Appendix A-4. A plot of the variation of the lattice functions β and α from the target to the lens image plane is given in fig. 2.

In the approximation of an elliptical acceptance ellipse, the yield of pbars is entirely determined by the aspect ratio of the upright acceptance ellipse at the target center (β_t) and the area of the ellipse ($\pi\varepsilon$, the emittance of the transmitted pbar beam). The basic relation between β_ℓ (β at the image point, which is approximately equal to β at the lens exit(see fig 2)) and β_t (β at the target) is

$$\beta_\ell = \frac{1}{\beta_t} \frac{1}{k^2 \sin^2\phi} \qquad (5)$$

where k^2 (in m**-2) = $0.3G/p$, G=lens gradient (in T/m), p=pbar momentum (in GeV/c), ϕ = $k\ell$, and l=length of the lens. This relation between β_ℓ and lens gradient G is plotted in fig. 3 for various β_t up to 1 cm (which covers the optimum for the pbar yield as discussed below). The relation between the maximum β_ℓ allowed and the physical lens radius R_o is

$$\beta_\ell(max) = \pi/4 \; R_o^2/\varepsilon \qquad (6)$$

where $\pi\varepsilon$ =pbar beam emittance (see Appendix A-4). Thus, for a given R_o , β_ℓ is limited above by this: these limits are shown (for R_o =10 mm) on fig. 3. From this graph can be obtained the minimum gradient required for a particular emittance, for a given value of β_t . The actual value of β at the image point, once chosen, must of course be able to be matched by AP-2 to the Debuncher. This could be a problem for small values of β_ℓ .

The required values of β_t are shown in fig. 4, which illustrates the variation of the $\overline{p}$ yield vs. β_t for different emittances. This result comes from the integration over the pbar distribution function discussed above, which is presented in gory detail in Appendix A-3. This calculation includes absorption of pbars in the lithium, using $\lambda_{Li} = 136$ cm. Note that there is a broad range of acceptable β_t, over which the yield variation is rather flat: this is especially true at small values of ε. The falloff at large β_t is due to the fact that the ellipse is too squat: this corresponds to too little lens focusing action. As β_t decreases, the acceptance ellipse gets tall and skinny: the falloff at small β_t (large lens gradients) comes from loosing particles at large x, and is a result of the finite size of the proton beam. For a point beam, there could be no falloff at small β_t: this can be seen from eq. 4, where the yield increases monotonically as γ increases.

All the information on the yield dependence on the lens parameters is, in principle, contained in figs. 3 and 4. However, it is useful to display the yield as a function of the lens and targeting parameters more explicitly, to allow the development of some insight. Also, the dependence of the yield on target length, proton spot size and lens length is useful to illustrate. In figs. 5 to 9, the yield is plotted as a function of:

(a). pbar beam emittance, for various lens gradients (fig. 5). The circles with dots on this graph are the points from ref. 1, calculated with the same parameters. The fact that these points lie above the curve for $\varepsilon > 20$ presumably reflects the presence of secondaries, although the secondaries are estimated at 20% in ref. 1 and the differences shown in fig. 5 are considerably less than this.

(b). proton beam size, for various emittances and 2 lens gradients (fig. 6).

(c). lens length, for various emittances and 2 lens gradients (fig. 7).

(d). target length, for various emittances and 2 lens gradients (fig. 8).

(e). lens gradient, for various emittances (fig. 9).

In each case, the value of β_t used for the yield calculation is that which corresponds to the maximum β_ℓ allowable for that gradient, taking $R_0 = 1$ cm. (i.e., the dotted lines in fig. 3)

In fig. 10, we show the variation in yield with lens radius, for various emittances: for each radius, the gradient is scaled like $1/R_0$ as the radius is changed (which corresponds to scaling the current like R_0) and the value of β_t used again corresponds to the maximum β_ℓ allowed for the scaled gradient and the corresponding radius.

Examination of the figures reveals the following facts:

(a). From fig. 5, at $\varepsilon = 20$ there is very little difference between 800 and 1000 T/m. This is seen even more clearly in fig. 9. Lens operation at 800 T/m is considerably easier than at 1000 T/m; thus, it would be very useful to be able to empirically optimize the lens gradient over the range 800 to 1000 T/m. This requires the ability to change the focal length over the range 14.5 to 19.5 cm. This ability is not present in the existing target assembly and should be added. As seen from fig. 9, for emittances larger than 20 pi, higher gradients (i.e., near the design of 1000 T/m) are clearly preferred. Also, for smaller proton spot sizes, larger gradients are required, as discussed below.

(b). From fig. 6, the substantial benefit afforded in principle by smaller spot sizes is clearly seen. Smaller proton spot sizes produce more gain for larger gradients: this is because the smaller β_t associated with larger gradients is required to make optimum use of the higher pbar density achieved by narrowing the proton spot size. Of course, since we are near the limit set by mechanical destruction of the target for the design $\sigma_n = 0.054$ cm, substantial reduction in σ_n would require implementaion of a beam sweeping system. Moreover, there are limits on the effective proton beam size due to multiple scattering in the lens and lens aberrations (see Addenum C) which must also be overcome to push to very small spot sizes (i.e., σ_n under about 0.02 cm).

(c). From figs 7 and 8, it is clear that $\ell_{lens} = 15$ cm and $\ell_{targ} = 5$ cm are near the optima in each case, for either 800 or 1000 T/m gradients. Larger emittances than 20 pi would require longer lenses (for fixed gradient) and longer targets.

(d). Figure 10 shows the interesting feature that, for a current which varies as the radius of the lens and $\varepsilon \lesssim 20$, the yield is almost independent of the lens radius R_0.

This is due to the following: as the radius is decreased, β_ℓ at the lens must decrease (see eq. 6) which means that, to keep β_t (and hence the yield) fixed, the gradient must increase (see fig. 3). But with current varying like R_o, gradient varies like $1/R_o$; this increase in gradient is close to what is required to keep β_t constant. The variation of Y with β_t is slow anyway (see fig. 4), so the result is a very slow change in Y with R_o.

This observation leads one to consider the possibility of smaller radius collection lenses, which have the advantage that they can pulse more rapidly (see ref. 3). As explained in ref. 3, the temperature rise per pulse in the lens

$$\Delta T_p \propto T I_o^2 / R_o^4$$

where T is the pulse duration and I_o is the peak lens current. To acheive the same current penetration ratio, (δ / R_o) should be kept constant, where $\delta \propto \sqrt{T}$ is the skin depth. For constant $(\delta / R_o)^2 \propto T / R_o^2$, T must decrease as R_o^2, so

$$\Delta T_p \propto I_o^2 / R_o^2$$

Thus, to keep ΔT_p constant, I_o must vary like R_o, as in fig. 10.

The cyclic thermal stress in the lens (in the cooling jacket) is (see ref. 3)

$$\Delta\sigma \propto \frac{\Delta T_p R_o}{t\left(1 + 2\gamma R_o / E t\right)}$$

where t is the thickness of the cooling jacket, γ = the compressibility of lithium and E= the elastic modulus of the cooling jacket. If ΔT_p is constant, $\Delta\sigma$ can be kept constant if (R_o /t) is maintained constant as R_o decreases. This implies a thinner cooling jacket. For constant $\Delta\sigma$, the minimum lens cycle time τ_{cyc} is proportional to the thermal time constant for heat transfer across the jacket, τ_R:

$$\tau_{cyc} \propto \tau_R \propto t R_o$$

For constant (R_o /t), $t \propto R_o$, so $\tau_R \propto R_o^2$, for example, to reduce τ_{cyc} from 2 sec (Tev I design) to 1 sec requires reducing R_o by $\sqrt{2} \Rightarrow R_o =$ .707 cm. As seen in fig. 10, the same yield would be produced if I_o scaled like R_o to 350 kA. (Actually, 280 kA would be just as good, as fig. 10 shows).

Other practical considerations resulting from smaller lens radius must also be considered, of course. For example, as R_o decreases and β_ℓ decreases as R_o^2 , AP-2 must still be able to match the lens to the Debuncher. Also, as G increases the focal length decreases, which could present practical problems. For example, for R_o=.7 cm and I_o =350 kA, f=8.4 cm and β_ℓ =1.92 m. Nevertheless, the feasibility of smaller radius lenses is clearly useful to investigate.

References

1. C. Hojvat, A. van Ginneken, "Calculations fo Antiproton Yields for the Fermilab Antiproton Source", NIM 206, 67(1983).

2. T. Vsevolozskaya, "The Optimization and Efficiency of Antiproton Production within a Fixed Acceptance", NIM 190, 479(1981).

3. G. Dugan, "Limits to the Pulse Repetition Rate of a Lithium Lens", in Proceedings of the 1984 Summer Study on Design and Utilization of the SSC, p. 321 (1984).

4. T. Vsevolozskaya et. al., "Optical properties of cylindrical lenses", Sov. Phys. Tech. Phys., vol. 20, num. 12, 1556(1977).

5. A. Lennox, "The Design parameters for a lithium lens as an Antiproton collector", IEEE Trans. on Nucl. Sci., NS-30, 3663 (1983).

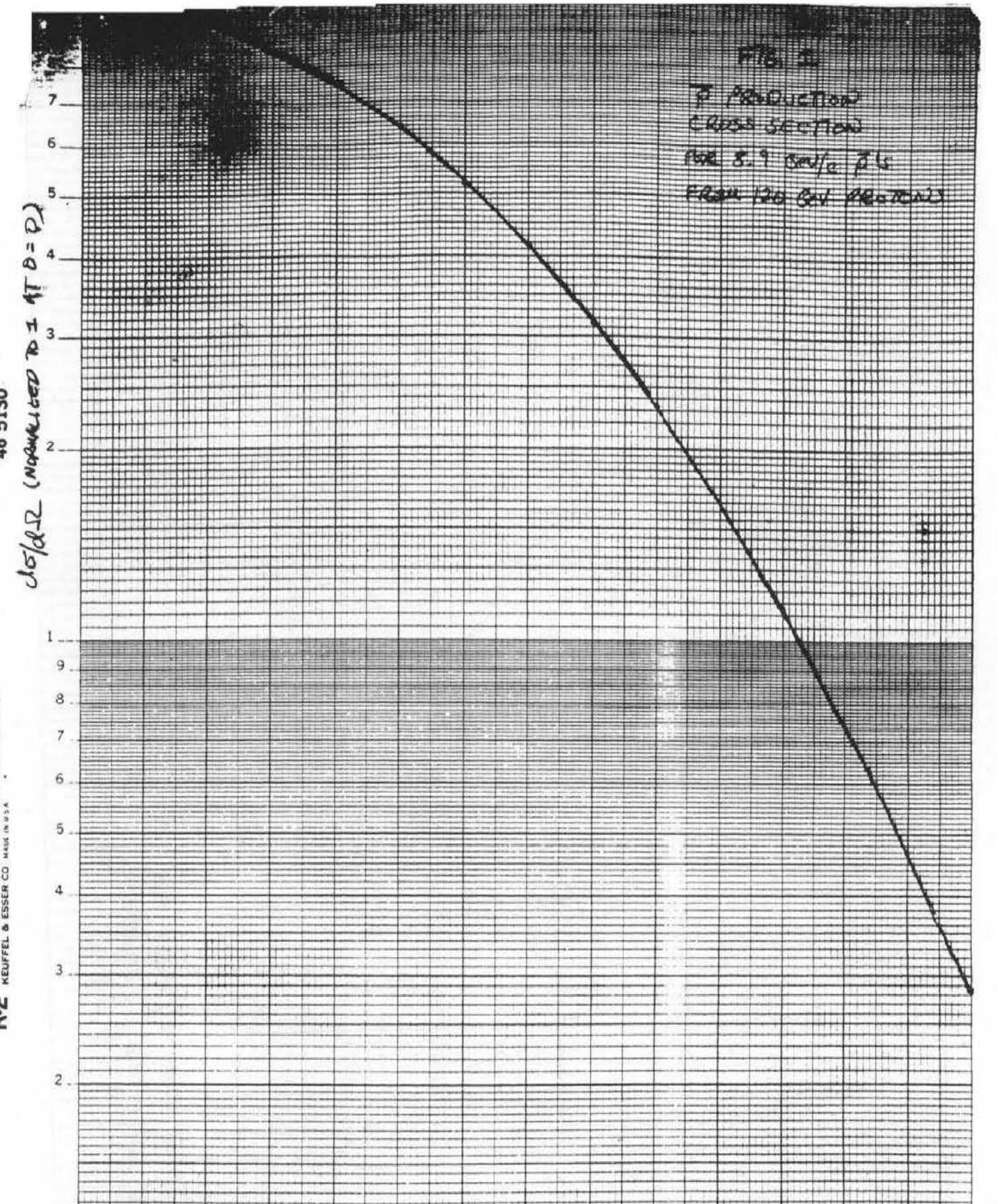


FIG. 2

$\bar{p}$ PRODUCTION CROSS SECTION FOR 8.9 GeV/c $\bar{p}$'s FROM 120 GeV PROTONS

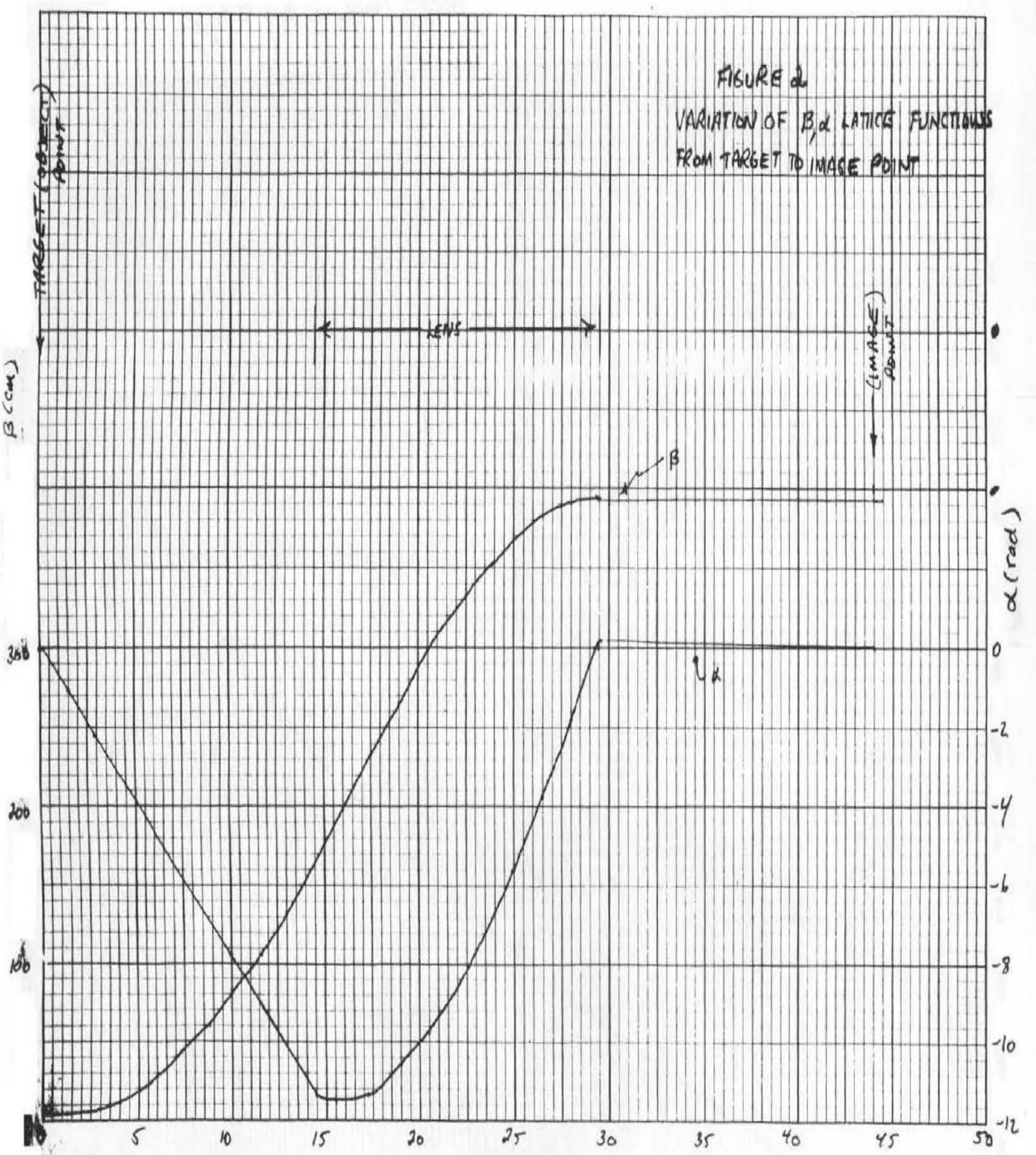


FIGURE 2

VARIATION OF β, α LATICE FUNCTIONS FROM TARGET TO IMAGE POINT

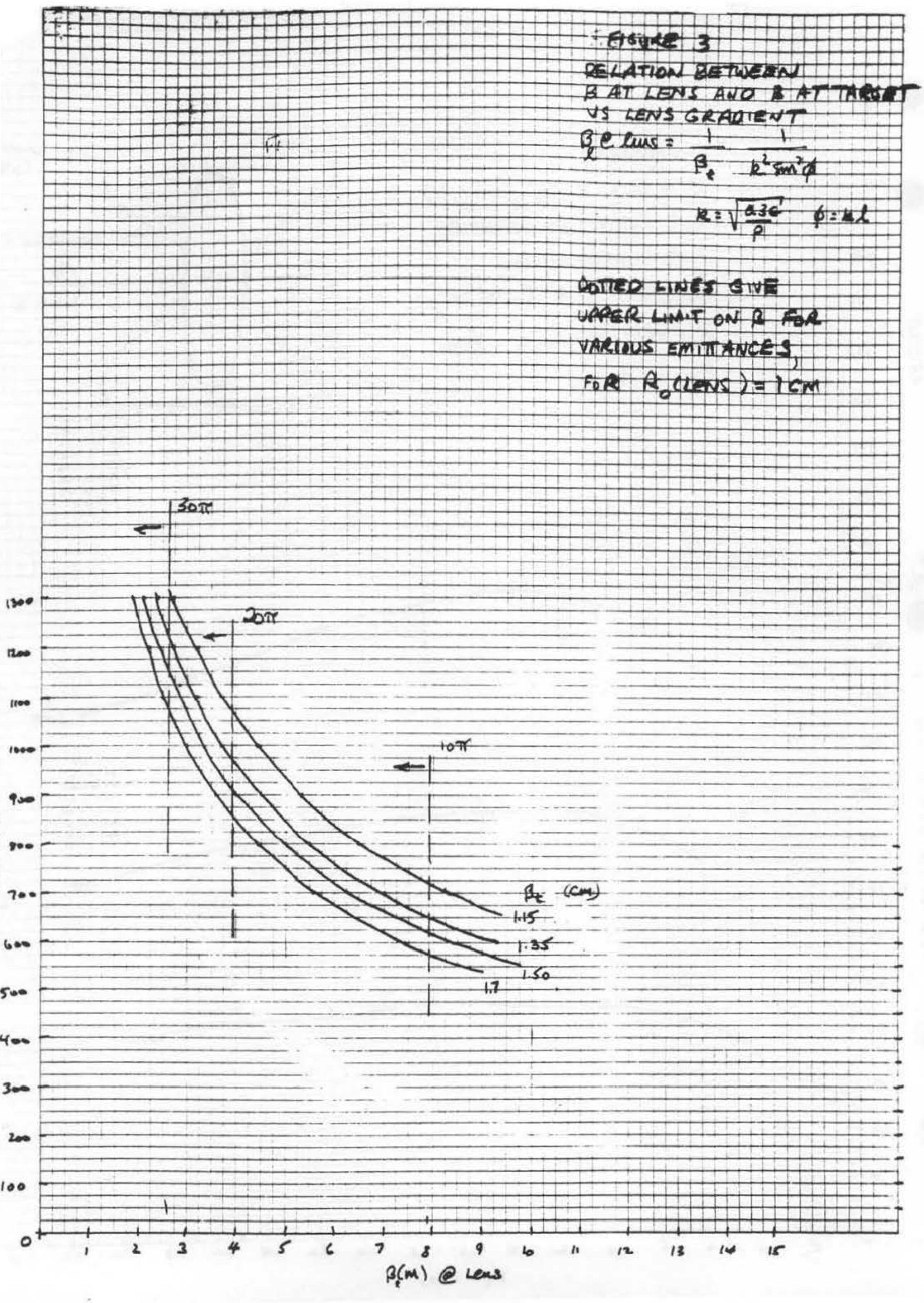


FIGURE 3

RELATION BETWEEN
β AT LENS AND β AT TARGET
VS LENS GRADIENT

$$\beta_\ell \text{ @ lens} = \frac{1}{\beta_t}\,\frac{1}{k^2\sin^2\phi}$$

$$k = \sqrt{\frac{0.3G}{p}} \qquad \phi = k\ell$$

DOTTED LINES GIVE
UPPER LIMIT ON β FOR
VARIOUS EMITTANCES,
FOR R_0 (LENS) = 1 CM

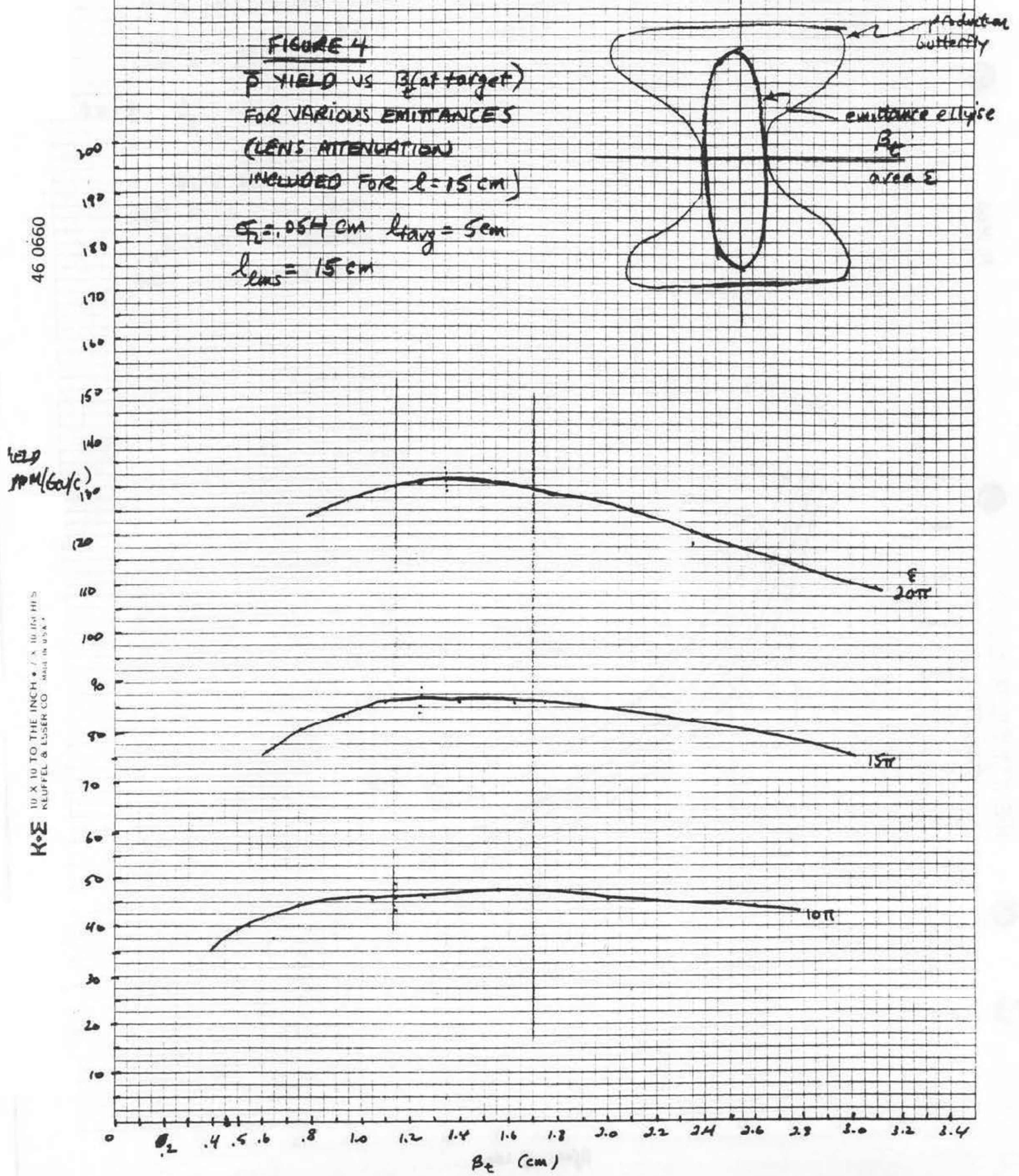


FIGURE 4

$\bar{p}$ YIELD vs β_t(at target) FOR VARIOUS EMITTANCES (LENS ATTENUATION INCLUDED FOR $l = 15$ cm)

$\sigma_{T_2} = .054$ cm $l_{t\,avg} = 5$ cm

$l_{lens} = 15$ cm

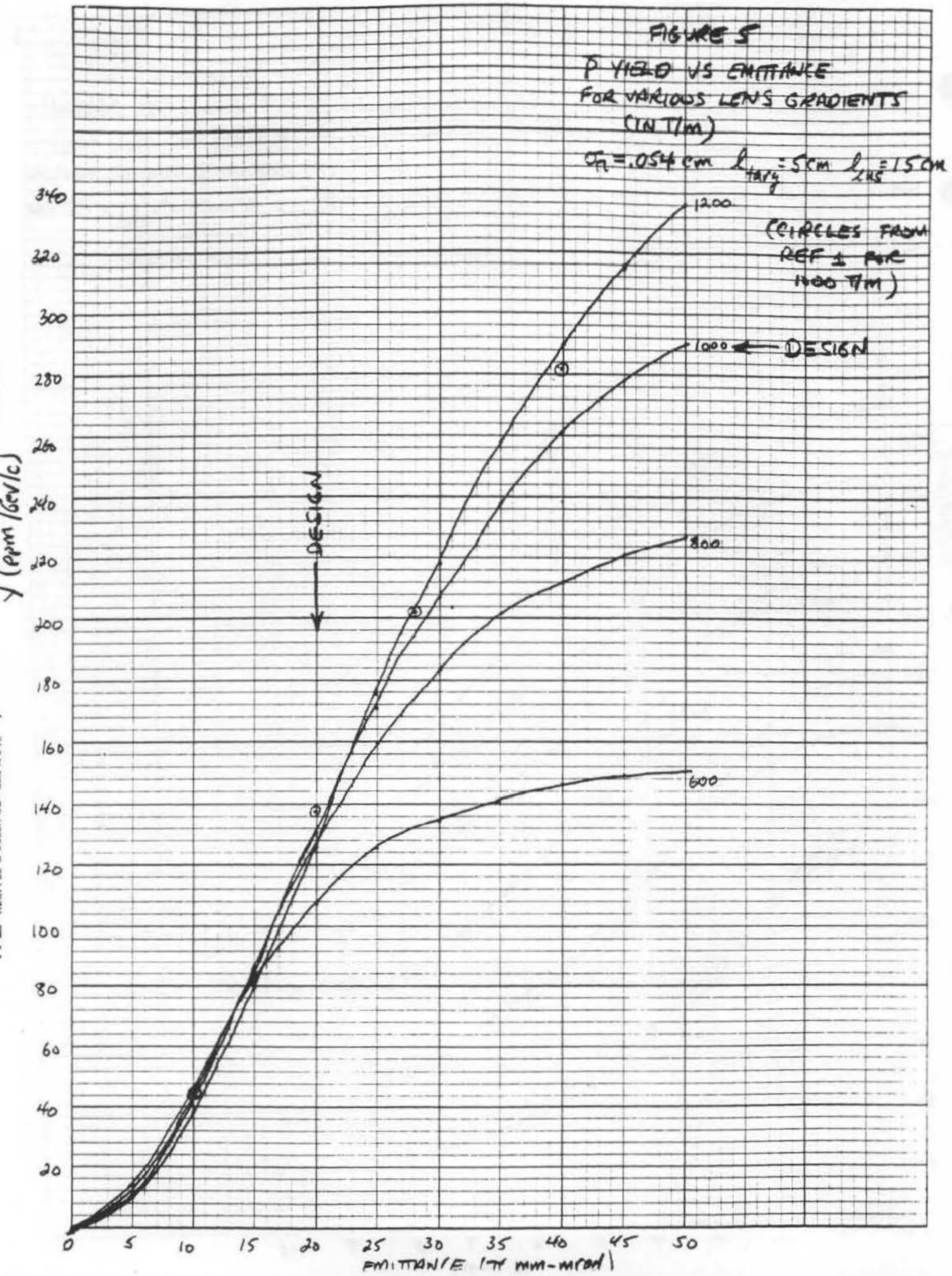


FIGURE 5

$\bar{p}$ YIELD VS EMITTANCE FOR VARIOUS LENS GRADIENTS (IN T/m)

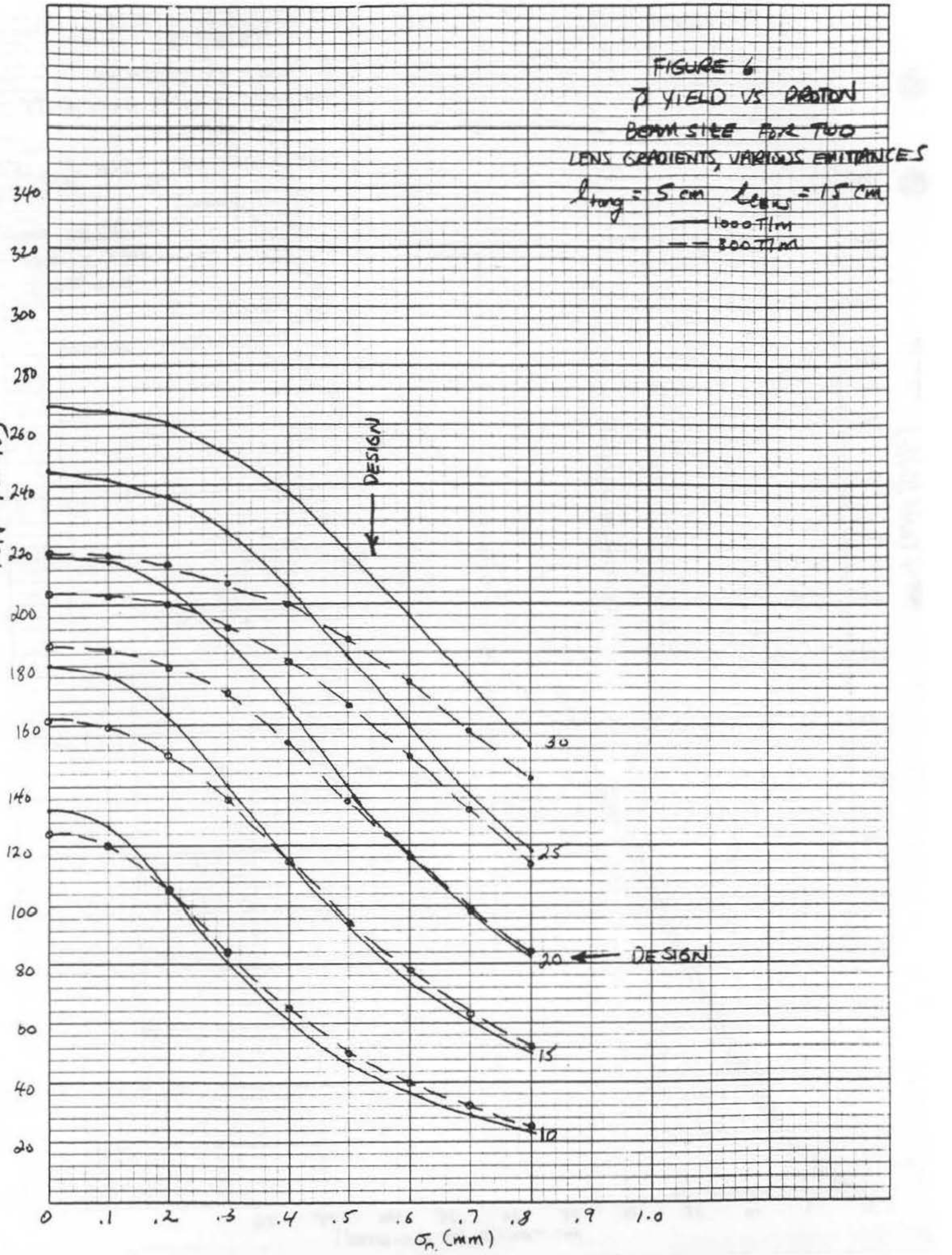
$\sigma_r = .054$ cm $\ell_{targ} = 5$cm $\ell_{LNS} = 15$ cm

(CIRCLES FROM REF 1 FOR 1000 T/m)

FIGURE 6

$\bar{p}$ YIELD VS PROTON BEAM SIZE FOR TWO LENS GRADIENTS, VARIOUS EMITTANCES

$\ell_{targ} = 5$ cm $\ell_{lens} = 15$ cm



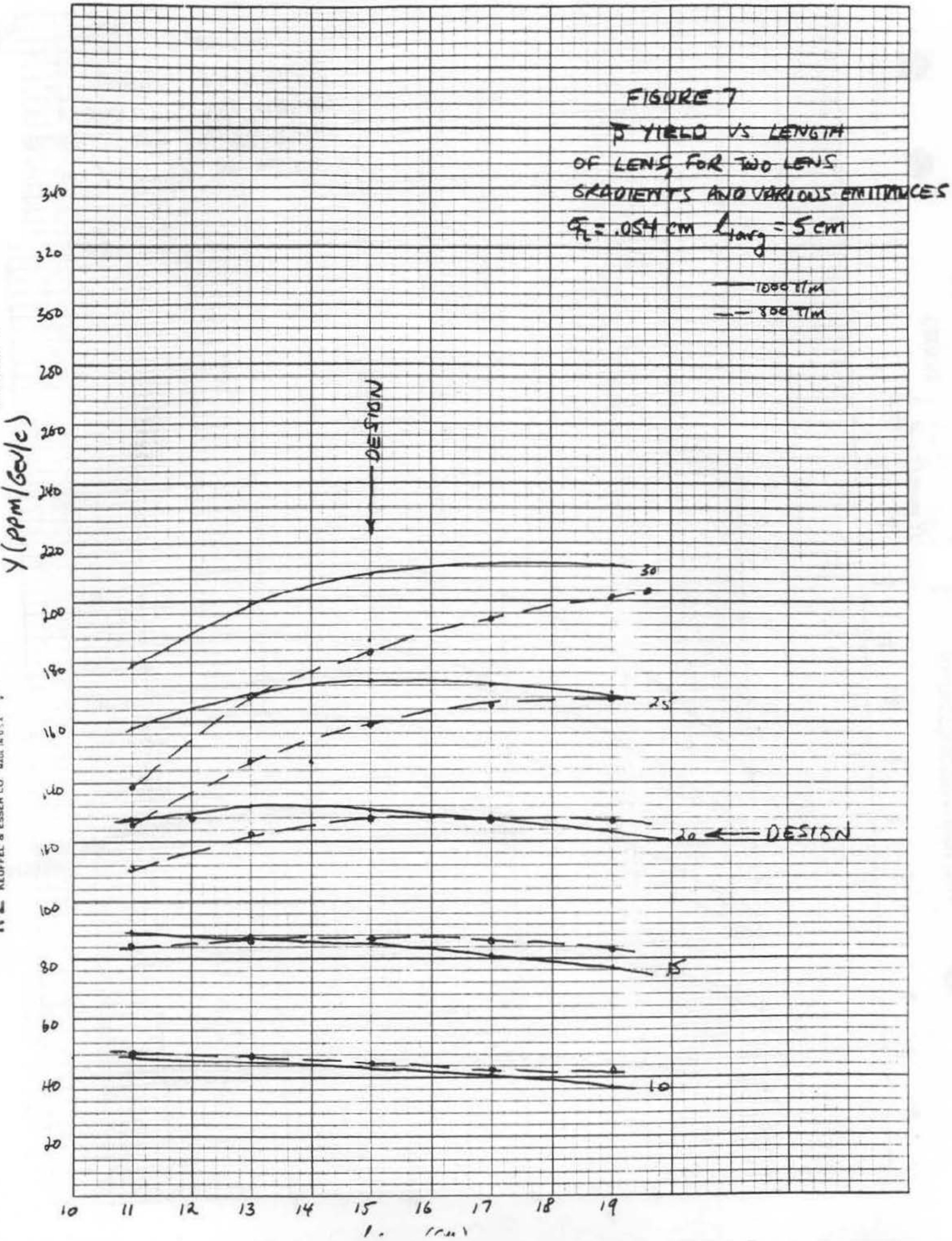


FIGURE 7

$\bar{p}$ YIELD VS LENGTH OF LENS FOR TWO LENS GRADIENTS AND VARIOUS EMITTANCES

σ_t = .054 cm ℓ_{targ} = 5 cm

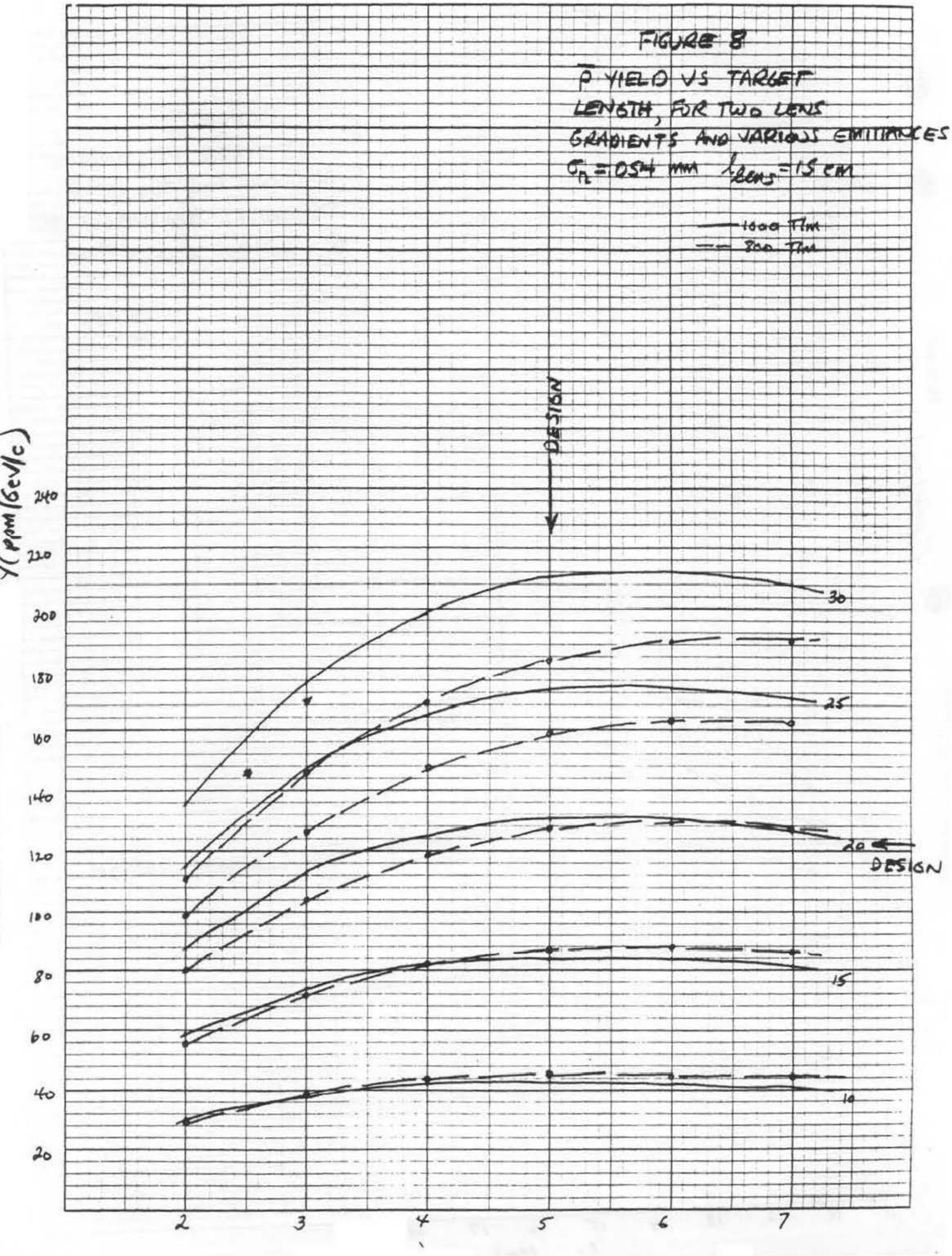


FIGURE 8

$\bar{p}$ YIELD VS TARGET LENGTH, FOR TWO LENS GRADIENTS AND VARIOUS EMITTANCES

$\sigma_R = .054$ mm $l_{lens} = 15$ cm

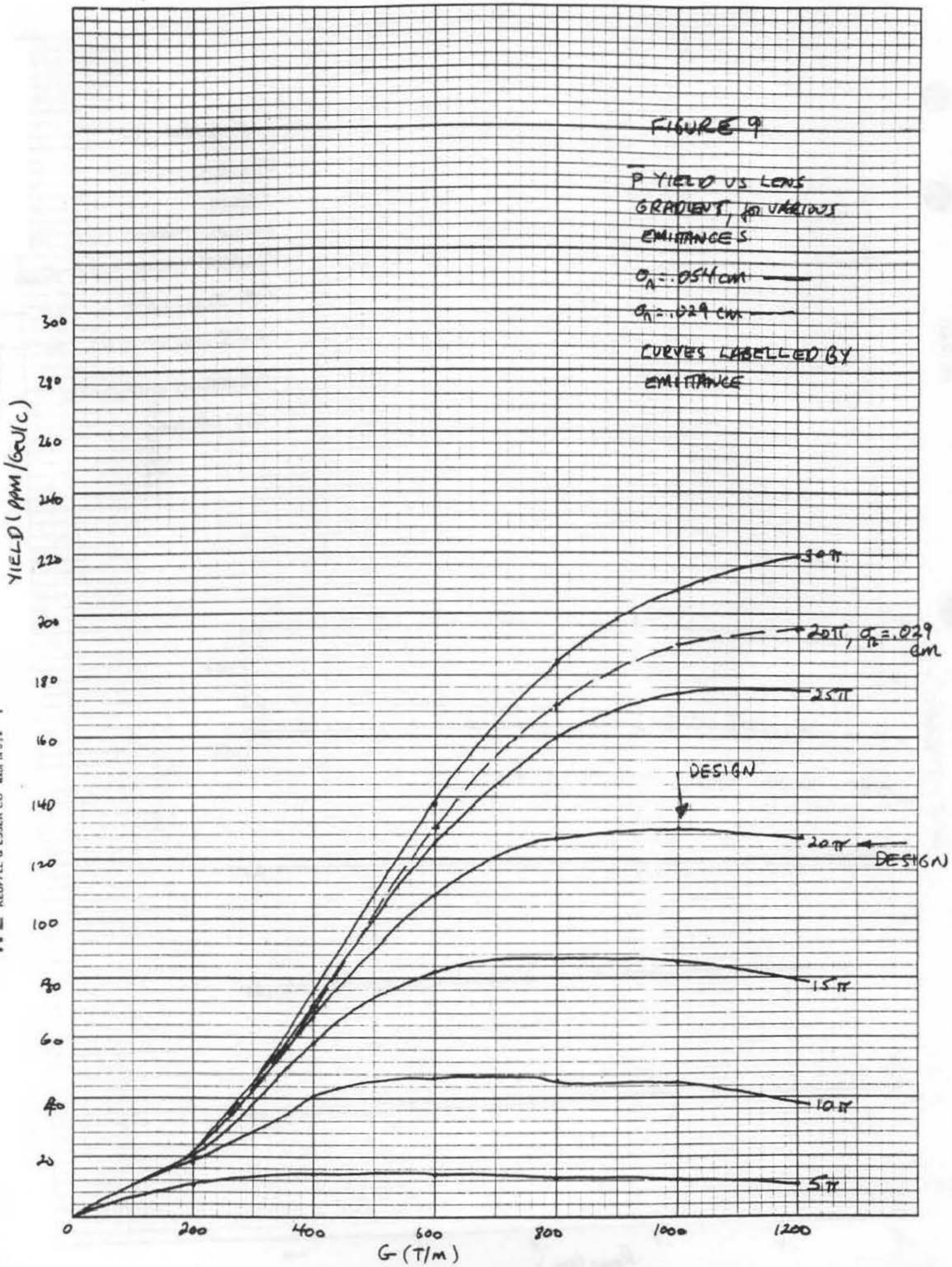


FIGURE 9

$\bar{p}$ YIELD VS LENS GRADIENT, for VARIOUS EMITTANCES.

$\sigma_n = .054$ cm ——

$\sigma_n = .029$ cm – – –

CURVES LABELLED BY EMITTANCE

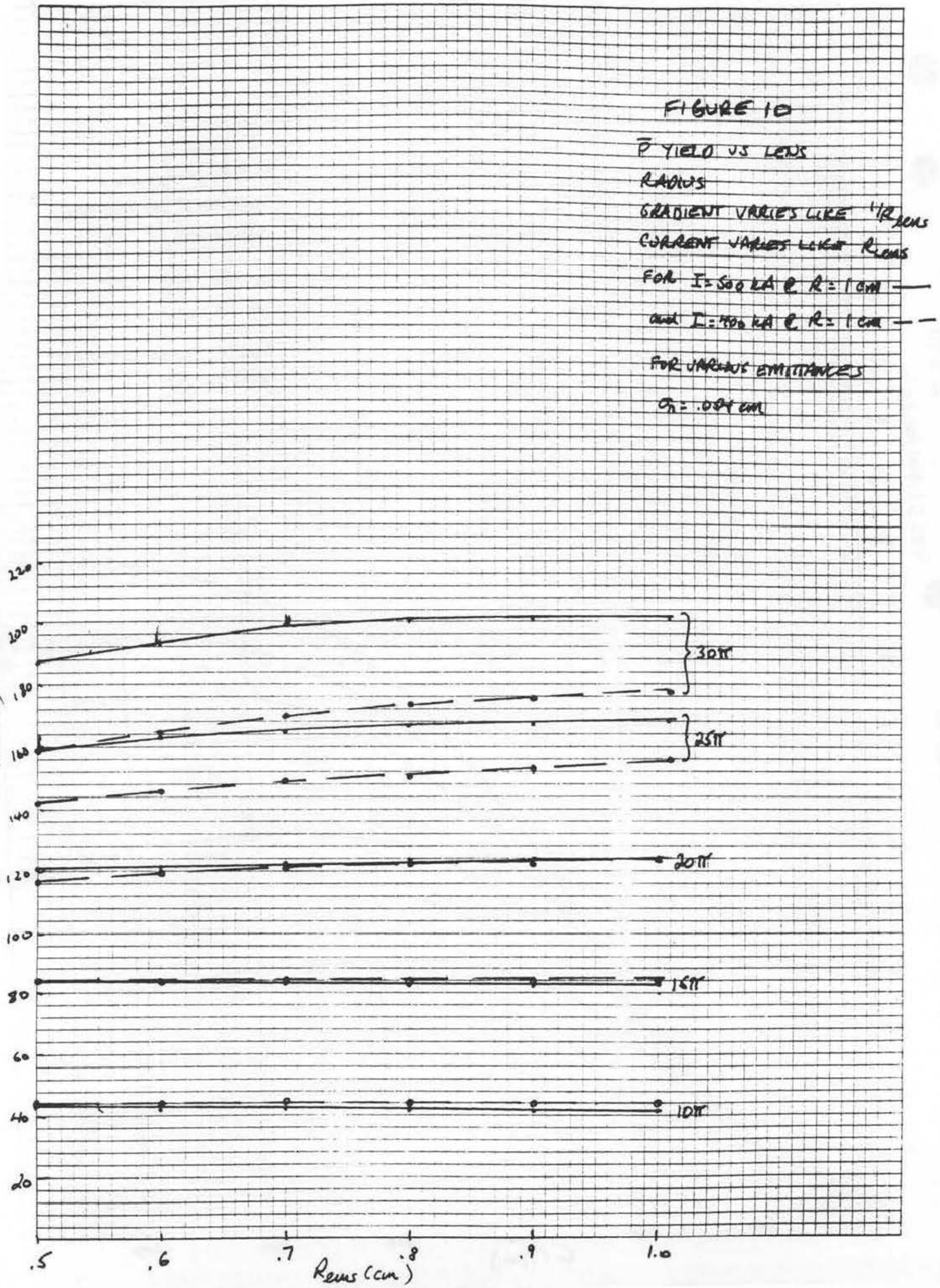


FIGURE 10

$\bar{P}$ YIELD VS LENS RADIUS

GRADIENT VARIES LIKE $1/R_{LENS}$

CURRENT VARIES LIKE R_{LENS}

FOR I = 500 kA @ R = 1 cm ——

and I = 400 kA @ R = 1 cm — —

FOR VARIOUS EMITTANCES

σ_r = .084 cm

Addenum A: the pbar distribution fucntion

An effort to understand AP-2 by Monte Carlo ray tracing of pbars from the target obviously needs an expression for the density distribution in phase space at the target center. This expression is derived in Appendix A-5, basically by guessing at a Green's function, using it to derive an answer, and then verifying that it satifies the transport equation.

Fig. A-1 is a projection of the differential yield at the target center

$$\frac{dY}{dx\,dy\,d\theta_x\,d\theta_y} = \frac{\bar{P}(x,y,\theta_x,\theta_y)}{N_p \Delta p}$$

on the x-axis (for $y=\theta_y=0$, $z=5$ cm) for various θ_x. The broadening of the distribution as θ_x increases is the depth-of-field effect (due to the finite target length). Figs. A-2 and A-3 illustrate a 3-dimensional view of the surface $\bar{P}(x,\theta_x, y=0, \theta_y=0, z=5\,\mathrm{cm})$ from two angles; figure A-4 is a projection of the 3-dimensional surface onto the (x, θ_x) plane. The shape gives rise to the term "buttefly" to describe the density distribution in phase space from a finite-length target.

The distribution fucntion at any point (i) along the beam line can then be computed fromt the expression given in Appendix A-5, once the optical transformation M which connects $\vec{X_t}$ (at the target) with $\vec{X_i}$ is known. For example, at the target image (close to the lens exit)

$$x_i = \theta_{xt} / k \sin\mu \qquad y_i = \theta_{yt} / k \sin\mu$$

and

$$\theta_{x_i} = -x_t k \sin\mu \qquad \theta_{y_i} = -y_t k \sin\mu$$

So, at the target image the pbar density distribution is

$$\bar{P}(-\theta_x / k\sin\mu,\ x k\sin\mu,\ -\theta_y / k\sin\mu,\ y k\sin\mu,\ z)$$

where x, y, θ_x, and θ_y are the phase space variables at the target image. The Jacobian is unity since the determinant of M is one. Figure A-5 illustrates the projection of the differential yield onto the x-axis, for various θ_x values (and $y=\theta_y=0$). The arrows indicate the lens aperture limits (averaged over y). Figs. A-6 and A-7 illustrate 3-dimensional views of the surface for $y=\theta_y=0$, from two angles. The sharp edges correspond to the lens aperture limits.

A simpler but less complete description of the density distribution function can be specified by computing the change in yield into a fixed emittance vs the emittance (ε_x or ε_y). Although the result is a differential in only one quantity (transverse emittance in one plane), the result obviously depends on the aspect ratio of the acceptance ellipse at the target center (i. e., β_t), and so has less information in it than the full differential yield expression. Nevertheless, since, as seen from fig. 4, the yield vs β_t is rather flat near the optimum, this dependence is probably not very strong. Since the emittance is invariant, the distribution of yield vs emittance in the Debuncher is presumably the same as at the target, except for the effects of aperture limitations.

Fig. A-9 shows $dY/d\varepsilon_x$ vs ε_x , where the integration over the (y, θ_y) variables is to ε_y =30, and the other parameters are the TeV I design values. Also shown is the exponential $dY/d\varepsilon_x = 10e^{-\varepsilon/\varepsilon_0}$, with ε_0 =23, for comparison.

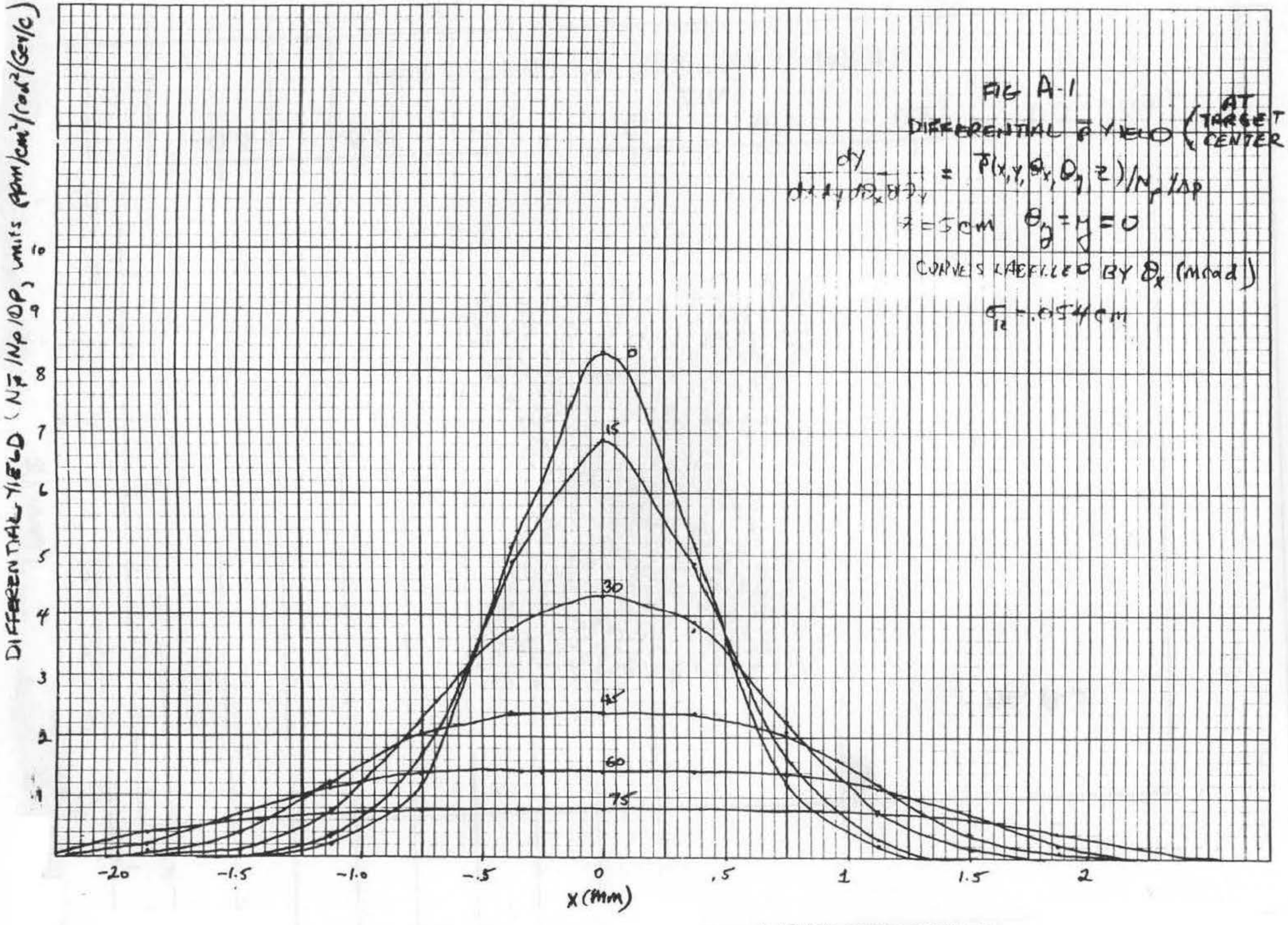


FIG A-1

DIFFERENTIAL $\bar{p}$ YIELD (AT TARGET CENTER)

$$\frac{dY}{dx\,dy\,d\theta_x\,d\theta_y} = \bar{p}(x, y, \theta_x, \theta_y, z)/N_p/\Delta p$$

$z = 5$ cm $\theta_y = y = 0$

CURVES LABELLED BY θ_x (mrad)

$\sigma_{12} = .054$ cm

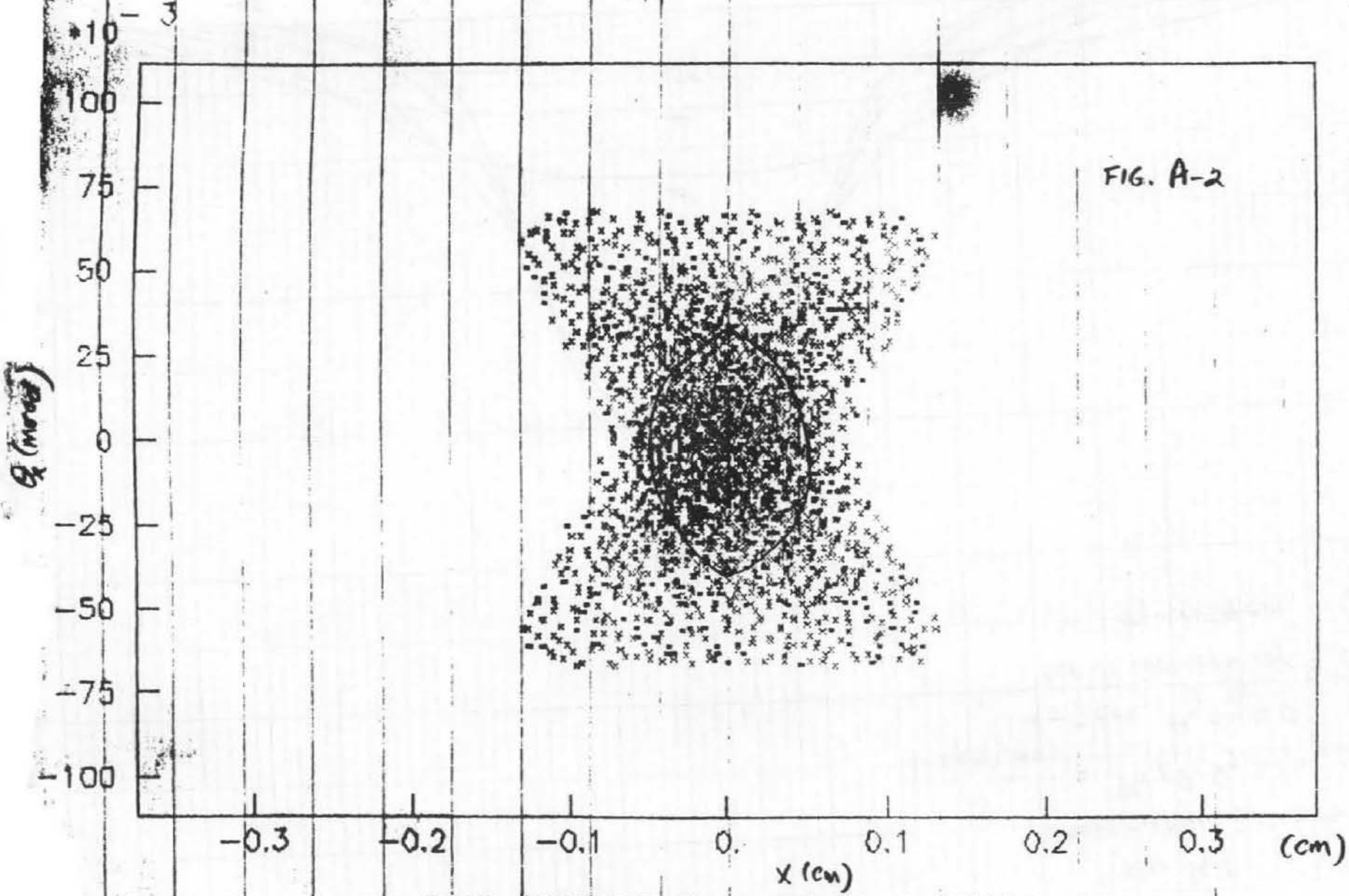
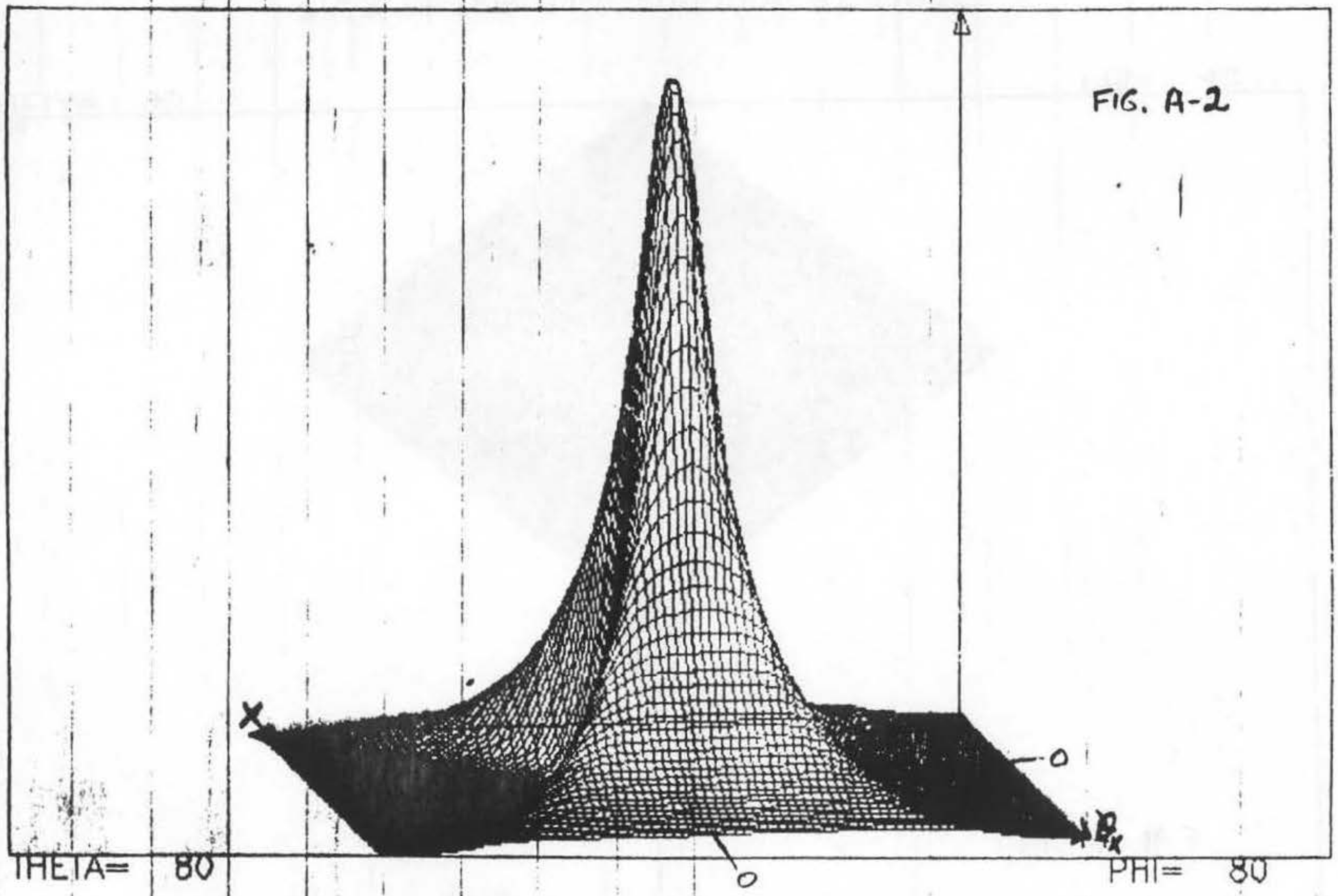


FIG. A-2

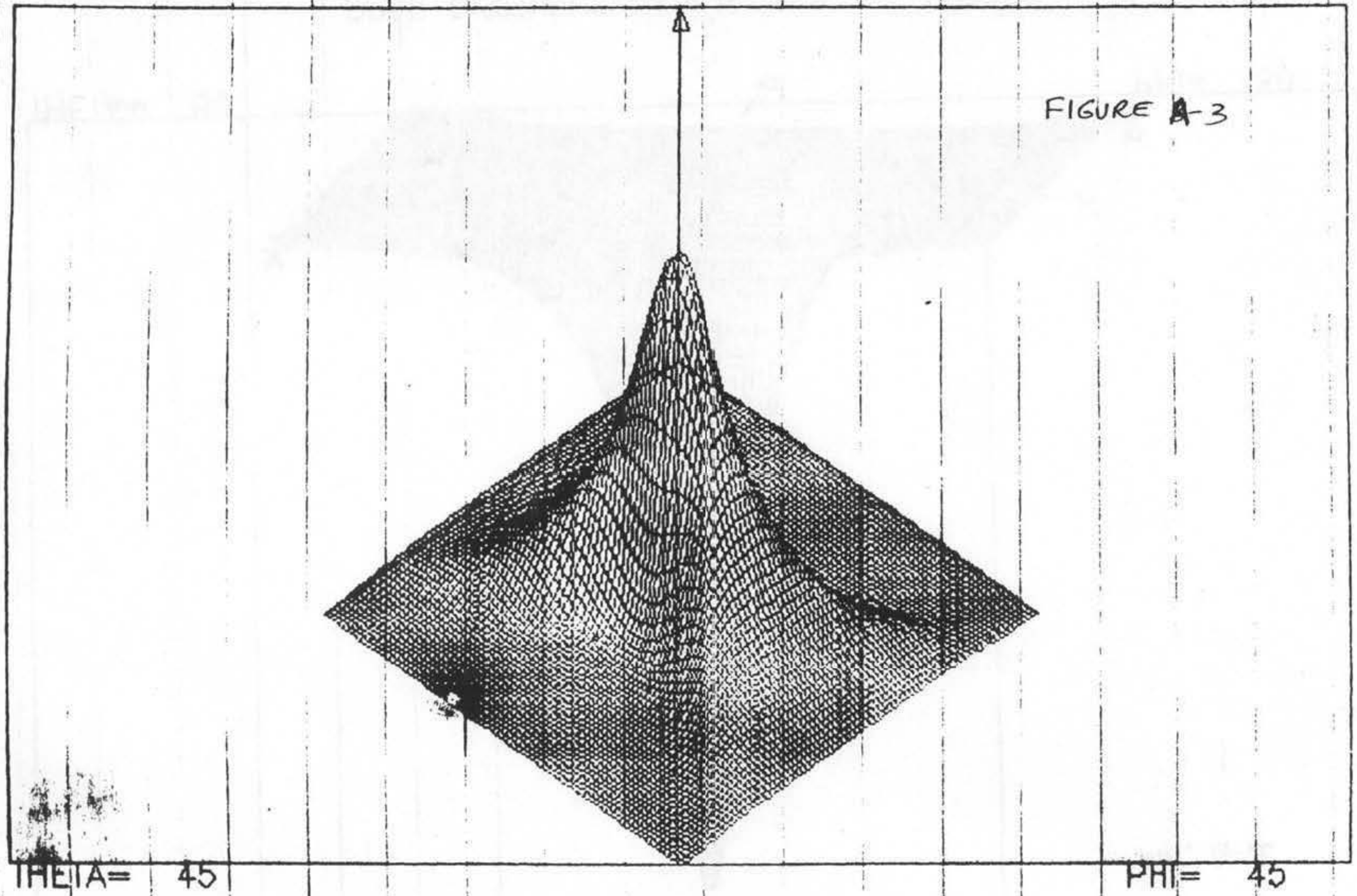
PBAR DISTRIBUTION FUNCTION AT TARGET

FIG. A-2



PBAR DISTRIBUTION FUNCTION AT TARGET

FIGURE A-3



PBAR DISTRIBUTION FUNCTION AT TARGET

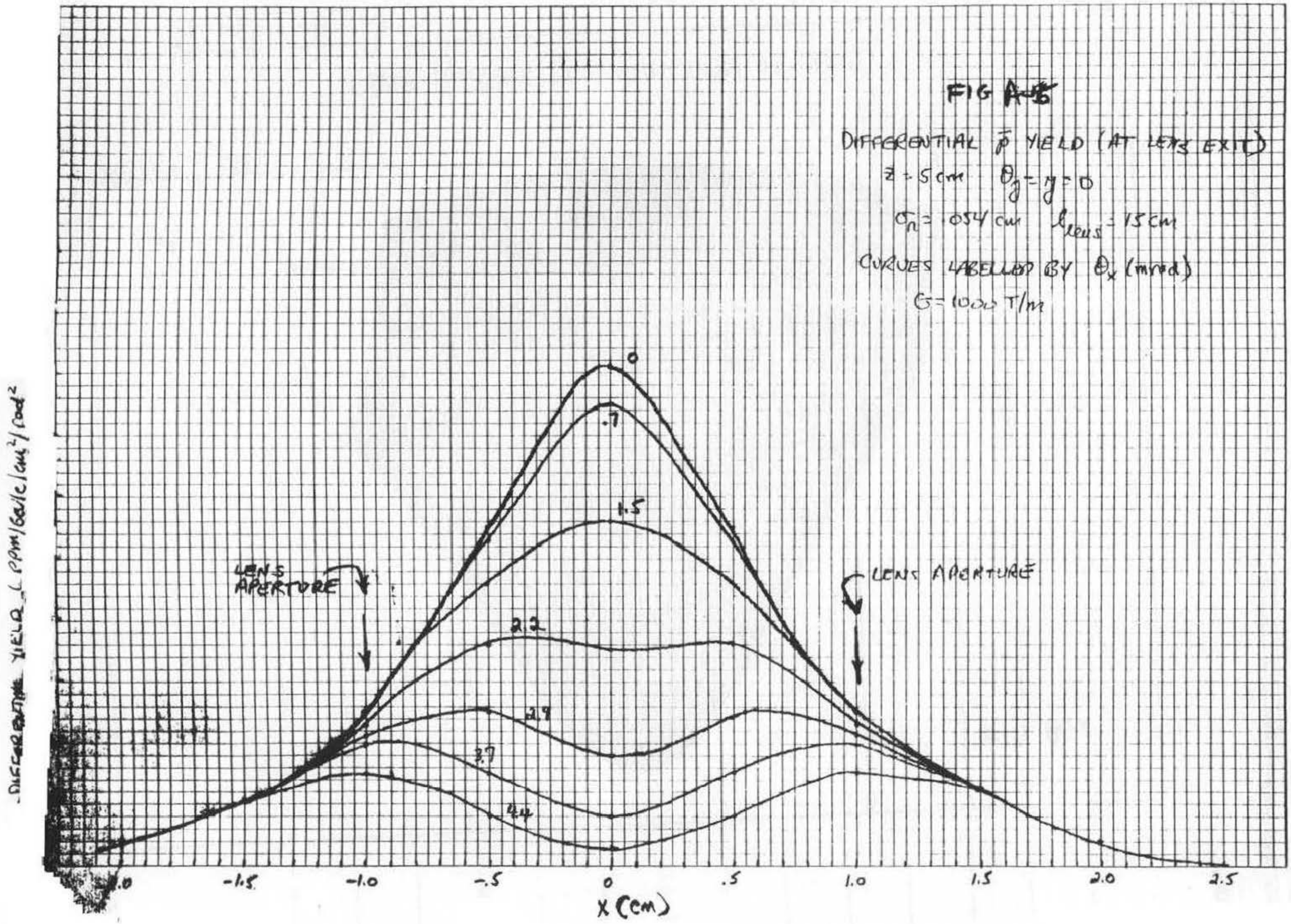


FIG A-5

DIFFERENTIAL $\bar{p}$ YIELD (AT LENS EXIT)

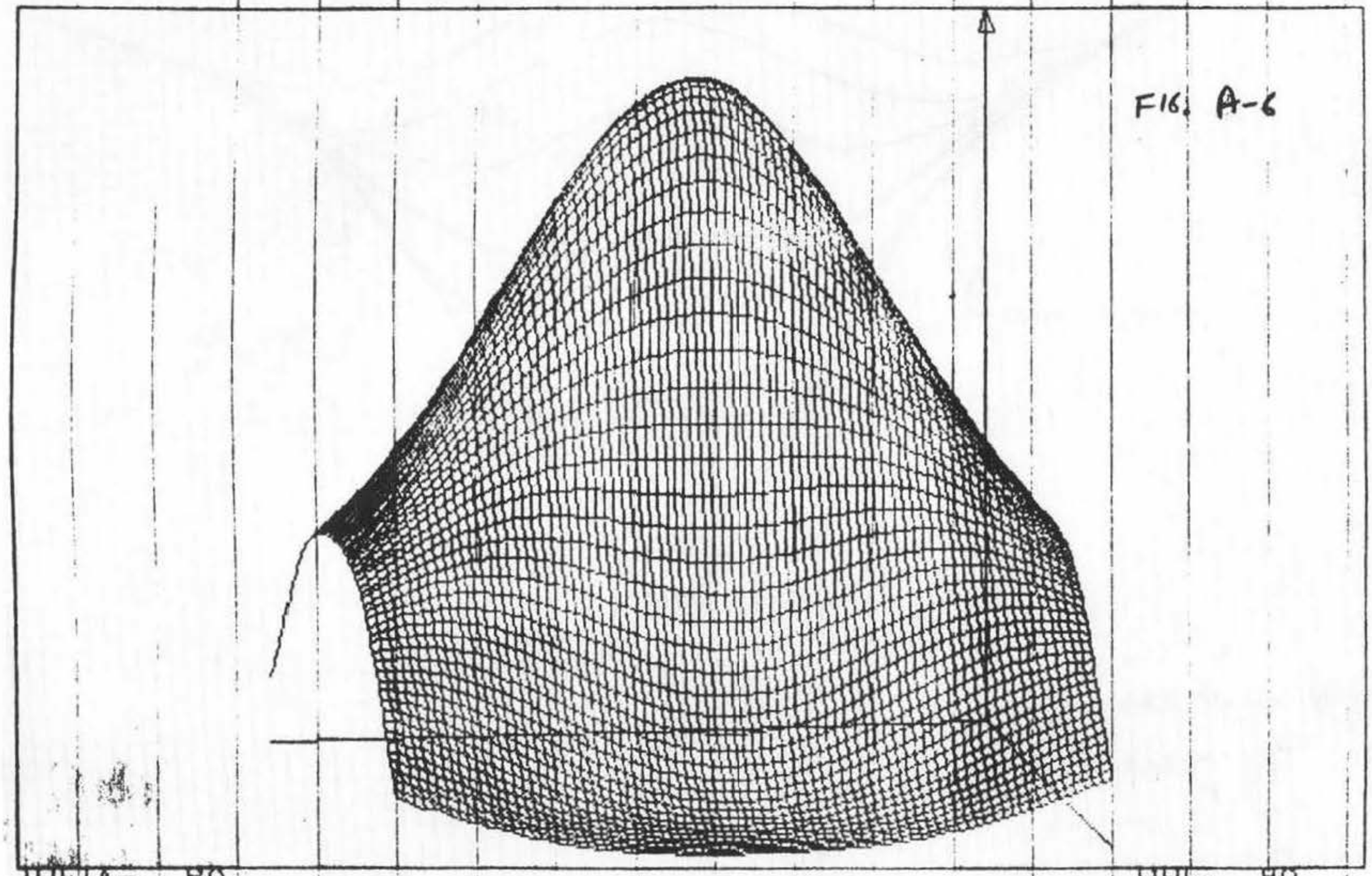
$z = 5$ cm $\theta_y = y = 0$

$\sigma_n = .054$ cm $l_{lens} = 15$ cm

CURVES LABELLED BY θ_x (mrad)

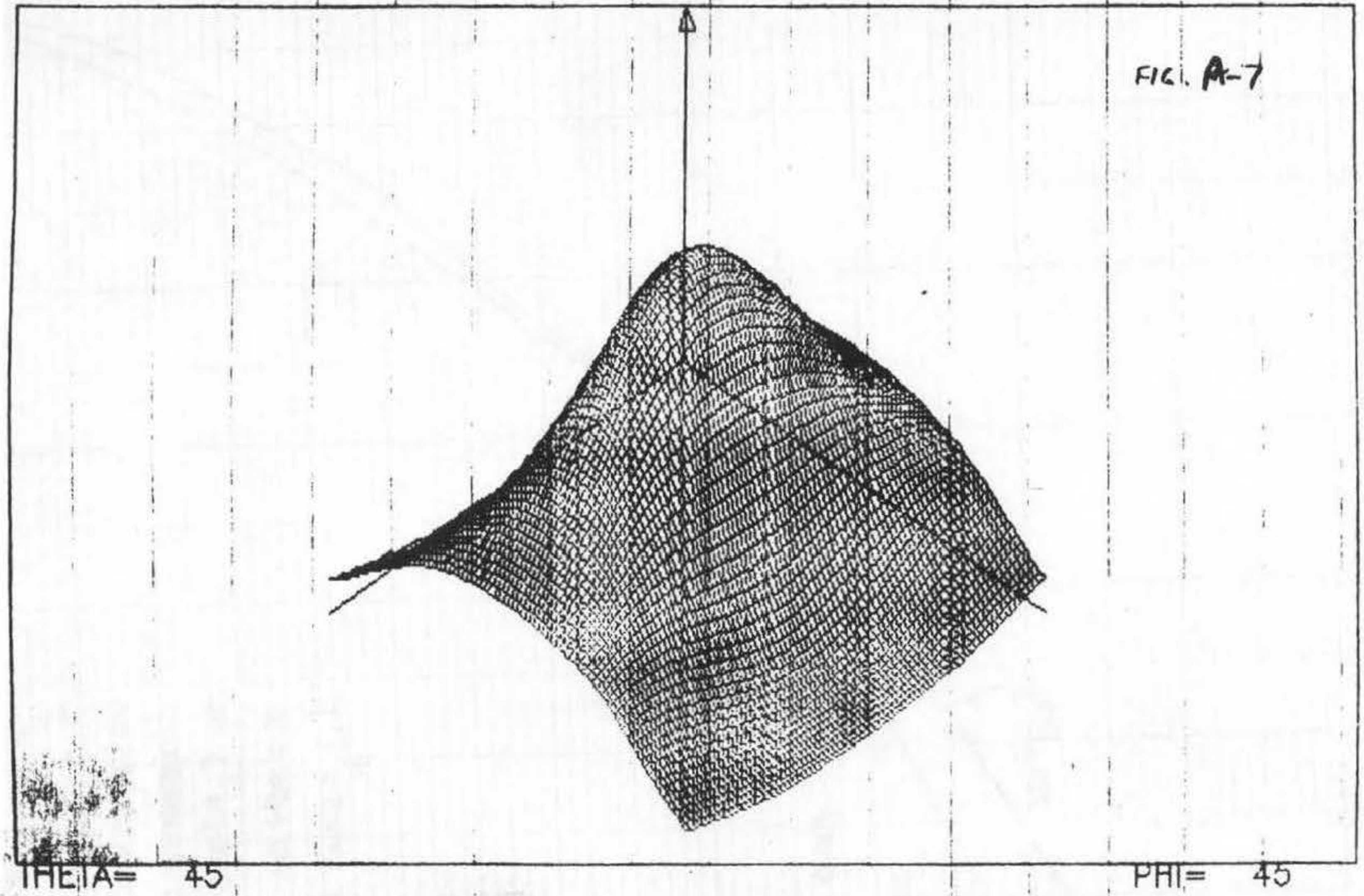
$G = 1000$ T/m

F16. A-6



PBAR DISTRIBUTION FUNCTION AT LENS

FIG. A-7



PBAR DISTRIBUTION FUNCTION AT LENS

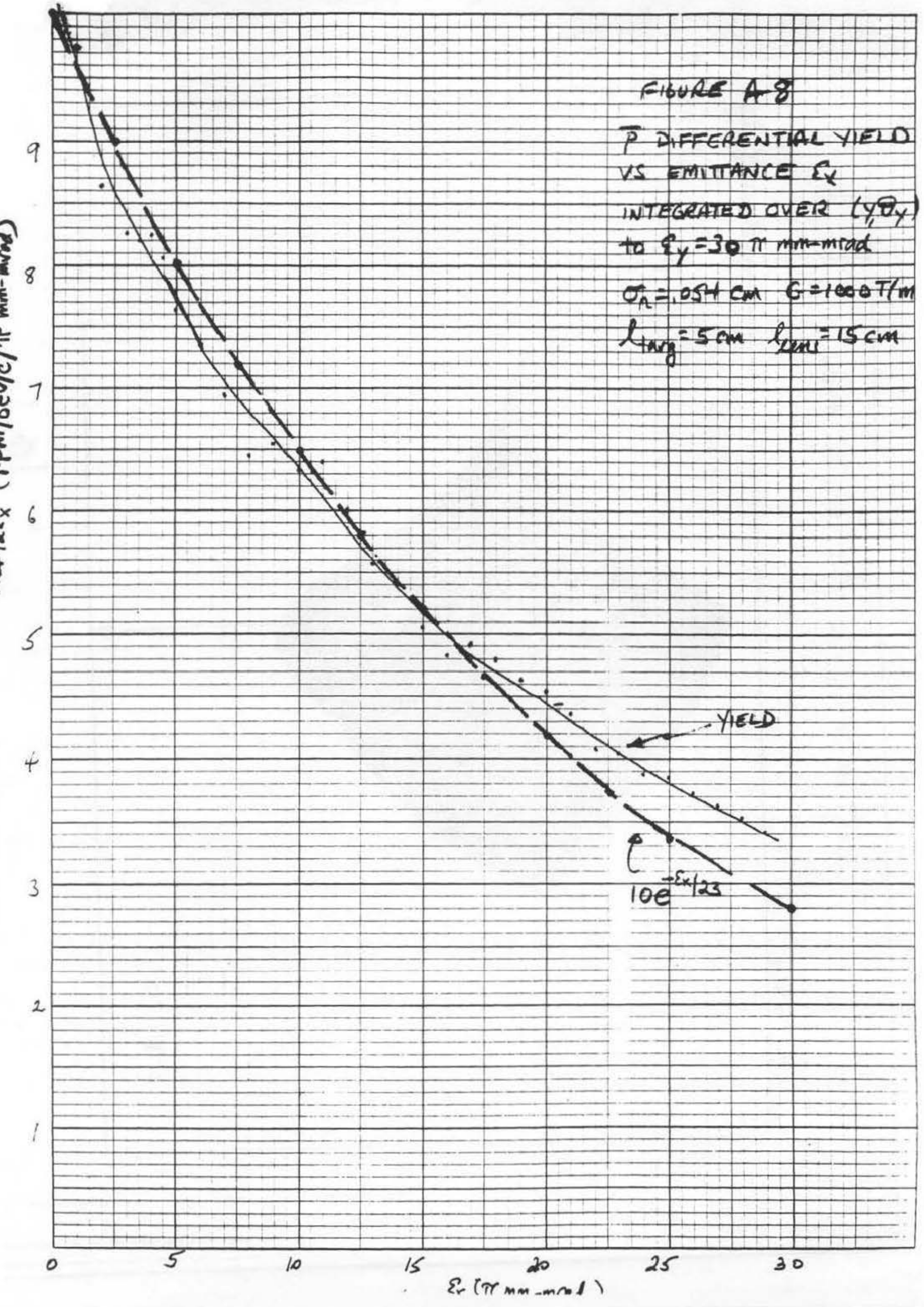


FIGURE A-8

$\bar{p}$ DIFFERENTIAL YIELD VS EMITTANCE ε_x INTEGRATED OVER (y, θ_y) to $\varepsilon_y = 30\,\pi$ mm-mrad

$\sigma_R = .054$ cm $G = 1000$ T/m

$l_{targ} = 5$ cm $l_{lens} = 15$ cm

Addendum B: optics of longer lenses

Plasma lenses are an alternative to lithium lenses as cylindrically symmetric focusing devices. The advantage of the plasma lens is its negligible density, which allows the length to be increased beyond that of the lithium lens. The length of the lithium lens is limited by absorption in the lithium (see fig. 7 above). To evaluate the parameters of a plasma lens which would have the same optical characteristics as the Tev I design lithium lens, we display the quantities I(lens current), l(lens length), and f(focal length) vs. ϕ =phase advance across the lens, in fig B-1. The calculations for these curves are presented on the next two pages: we have required a constant emittance ellipse at the target (ε =20, β_t =1.29 cm) and a plasma radius of 1 cm. As can be seen from the figure, the required current decreases substantially as ϕ and l increase. ϕ =90° corresponds to the minimum current; here, f=0, which is somewhat impractical (although combined target-plasma lens assemblies could be considered). A more practical solution is ϕ =70°, which requires f=7.7 cm, l=25.8 cm and I=330 kA. This peak lens current is substantially less than the roughly 600 kA needed for the lithium lens; the 13% absorption in the lithium is also eliminated. These are the principal advantages of the plasma lens

Equations for [illegible] f, I as functions of the phase advance ϕ

Given: R ([illegible]), β_t (target β), ε (emittance)

[illegible] Lens β $\quad \beta_\ell = \pi/4\, R^2/\varepsilon$

$k = \sqrt{\frac{0.3G}{p}}$ $\quad$ G in T/m $\rightarrow$ k in m^{-1}
p in GeV/c

$G = B_0/R$ $\quad B_0(2\pi R) = \mu_0 I$, $B_0 = \frac{\mu_0 I}{2\pi R}$

$G = \frac{\mu_0 I}{2\pi R^2}$ $\quad$ so $k^2 = \frac{0.3\mu_0 I}{2\pi R^2 p}$ $\qquad G = \frac{\overset{2}{\cancel{4\pi}}\times 10^{-7}}{\cancel{2\pi} R^2} I = \frac{200\, I\,(kA)}{R^2(mm)}$ T

Define $\quad I_m = \left(\varepsilon/\beta_t\right)\frac{2\pi p}{0.3\mu_0}$ $\qquad \mu_0 = 4\pi\times 10^{-7}$

then $k^2 = \frac{I}{R^2}\;\frac{0.3\mu_0}{2\pi p}$

$$= \frac{I}{R^2}\left(\frac{\varepsilon}{\beta_t}\right)\frac{1}{I_m} = \left(\frac{\varepsilon}{R^2\beta_t}\right)\frac{I}{I_m}$$

But $\quad \beta_\ell = \frac{1}{\beta_t k^2 \sin^2\phi} = \frac{\pi}{4}\frac{R^2}{\varepsilon}$

so $\quad \frac{\varepsilon}{\beta_t R^2} = \frac{\pi}{4} k^2 \sin^2\phi$

$$I/I_m = k^2/\varepsilon/R^2\beta_t = \frac{k^2}{\pi/4\, k^2 \sin^2\phi} = \frac{4}{\pi}\frac{1}{\sin^2\phi}$$

$$\boxed{I = I_m\left(\frac{4}{\pi}\right)\frac{1}{\sin^2\phi}} \qquad G = \frac{\mu_0 I}{2\pi R^2} = \frac{\mu_0}{2\pi R^2} I_m \frac{4}{\pi}\frac{1}{\sin^2\phi}$$

$f = \frac{1}{k\tan\phi}$ • $\quad k^2 = \frac{1}{\beta_\ell \beta_t \sin^2\phi}$

$$f = \frac{\sqrt{\beta_\ell\beta_t}\,\sin\phi}{\tan\phi} = \boxed{\sqrt{\beta_\ell\beta_t}\,\cos\phi = f}$$

$$\ell = k\ell/k = \boxed{\phi\,\sqrt{\beta_\ell\beta_t}\,\sin\phi = \ell}$$

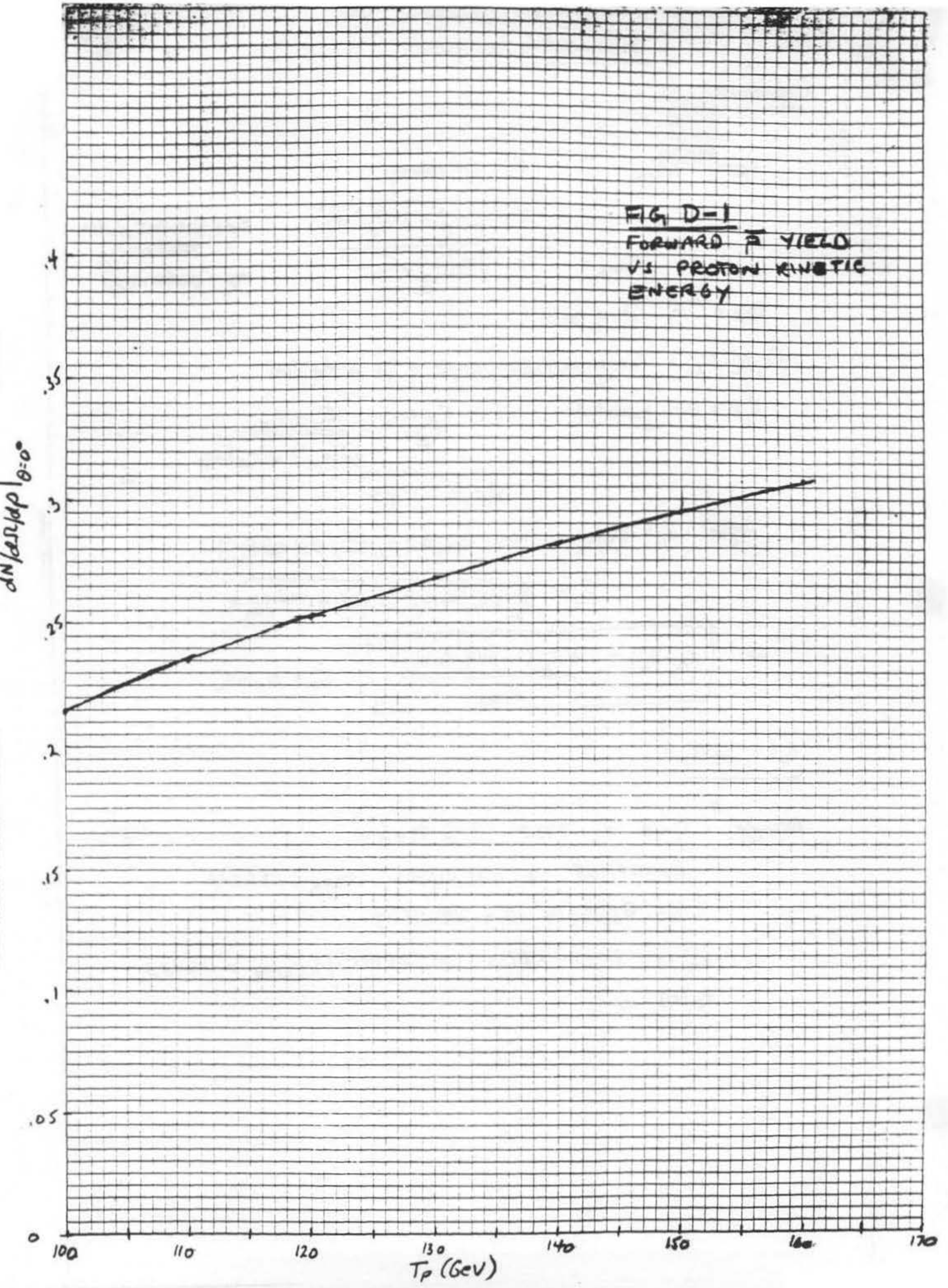


FIG D-1
FORWARD $\bar{p}$ YIELD
VS PROTON KINETIC
ENERGY

Appendix #1

$\bar{p}$ production cross section

Definitions

$x_R =$ [illegible] $/E_{max}$ ⌢ $E' =$ CM energy

$$E' = \sqrt{2m_p^2 + 2(E_p)m_p}$$

E_p = incident proton total energy

$$E_{max} = E'/2 = \sqrt{\frac{m_p^2 + m_p E_p}{2}}$$

m_p = proton mass

$$E_{cm} = \gamma(E_L - \beta c p_L \cos\theta_L)$$

E_L, P_L, θ_L = energy momentum, angle of $\bar{p}$ in the lab

γ, β = CM quantities $\gamma = \frac{E_p + m_p}{E'} = \frac{E_p + m_p}{\sqrt{2m_p^2 + 2(E_p)m_p}}$

$$\beta = \sqrt{1 - 1/\gamma^2}$$

Define $\beta_L = cP_L/E_L$ then $E_{cm} = \gamma E_L(1 - \beta\beta_L \cos\theta_L)$

$$\beta_L = \frac{1}{E_L}\sqrt{E_L^2 - m_p^2} = \sqrt{1 - m_p^2/E_L^2}$$

so $\boxed{x_R(\theta_L) = \frac{\gamma E_L(1 - \beta\beta_L \cos\theta_L)}{E_{max}}}$ $E_L = m_p + T_L$

$\boxed{P_t = P_L \sin\theta_L}$

Example * $T_L = 8$ $E_L = 8.938$ $\beta_L = .994478$

$E_p = 120.938$ $E' = 15.12083$ $E_{max} = 7.5604$

$\gamma = 8.06014$ $\beta = .992274$

$x_R(0) = 8.06014 \times 8.938\ (1 - .986795)/7.5604 = .12583$

$P_t(0) = 0$

Case [redacted]

[redacted] $(0.065)(1-x_\rho)^8 \exp(-3\rho_t^2)(1.69)\exp(1.38\rho_t^2)\exp(-1.79x_\rho)$

$= .10985(1-x_\rho)^8 \exp(-1.62\rho_t^2)\exp(-1.79x_\rho)$

[redacted] $= \beta_L^2/E_L(.10985)(1-x_\rho)^8\exp(-1.79x_\rho)$ ~~exp~~ $\exp(-1.62\rho_L^2)$

$= \beta_L^2 E_L(.10985)(1-x_\rho)^8\exp(-1.79x_\rho)\exp(-1.62\rho_L^2)$

At $\theta_L = 0$

$$\left.\frac{dN}{dp\,d\Omega}\right|_{0} = (.9944478)^2(8.938)(.10985)(.87417)^8\exp(-.225236)$$

$= .26435$ (5% high vs. .252)

~~Approximations $\beta\beta_L = 1-\delta$~~

~~$\cos\theta_L = 1-\theta_L^2/2$~~

~~$\Rightarrow x_\rho = \frac{\gamma E_L}{E_{max}}\left(1-(1-\delta)(1-\theta_L^2/2)\right)$~~

~~$= \frac{\gamma E_L}{E_{max}}\left(1-1+\delta+\theta_L^2/2-\delta\theta_L^2/2\right)$~~

~~$\approx \frac{\gamma E_L}{2E_{max}}(2\delta+\theta_L^2)$ $\delta = .013205$~~

~~$\theta_\delta^2 = 2\delta$ $\theta_\delta = \sqrt{2\delta} = .1625$~~

~~$x_\rho = \frac{\gamma E_L}{2E_{max}}(\theta_\delta^2+\theta_L^2) = \frac{\gamma E_L}{2E_{max}}\theta_\delta^2(1+\theta_L^2/\theta_\delta^2)$~~

~~$1-x_\rho = 1-\frac{\gamma E_L}{2E_{max}}\theta_\delta^2(1+\theta_L^2/\theta_\delta^2) = 1-\frac{\gamma E_L\theta_\delta^2}{2E_{max}}-\frac{\gamma E_L\theta_L^2}{2E_{max}}$~~

~~$1-x_\rho = \left(1-\frac{\gamma E_L\theta_\delta^2}{2E_{max}}\right)\left(1-\underbrace{\frac{\gamma E_L\theta_L^2}{2E_{max}\left(1-\frac{\gamma E_L\theta_\delta^2}{2E_{max}}\right)}}_{k\theta_L^2}\right) =$~~

~~$(1-x_\rho)^n \cong \left(1-\frac{\gamma E_L\theta_\delta^2}{2E_{max}}\right)^n(1-nk\theta_L^2)$~~

[redacted]

Mult[redacted] scattering of (protons) in the target

[redacted] $\frac{}{p\beta}\sqrt{L/L_R}\left(1+\frac{1}{9}\log_{10}L/L_R\right)$

$$\langle y\rangle^2_{plane} = \left(1/\sqrt{3}\,L\theta_0\right)^2 = L^2\theta_0^2/3$$

$$\sigma_p^2 = 2\langle y\rangle^2_{plane} = 2/3\,L^2\theta_0^2$$

so σ_n^2 is modified to

$$\sigma_n^2 \to \left(\frac{1}{\sigma_n^2}+\frac{1}{\sigma_p^2(z)}\right)^{-1}$$

where $\sigma_p^2(z) = 2/3\,z^2\theta_0^2$

$$\theta_0^2 = \left(\frac{4.1}{p\beta}\right)^2 z/L_R\left(1+1/9\log_{10}z/L_R\right)$$

using $p\beta = 120\times10^3$ $L_R = .35$ cm (tungsten)

$$\theta_0^2 = 3.945\times10^{-8}z\left(1+1/9\log_{10}z/.35\right)$$

$$\sigma_p^2(z) = 2/3\,z^2\theta_0^2$$

Example $z = 5$ cm $\theta_0^2 = 2.225\times10^{-7}$ $\theta_0 = .472$ mrad

$$\boxed{\sigma_p(z) = .0019\text{ cm}}$$

<u>Negligble</u> (until $\sigma_n \sim .02$ mm)

For au[redacted]

p[redacted]$\times10^{-}$ MeV $L_R = .35$ cm

$$\theta_0^2 = \left(\frac{14.1}{[redacted]}\right)^2 \frac{z}{.35}\left(1+\frac{1}{9}\log_{10} z/.35\right)$$

$$= 7.171\times10^{-6} z\left(1+ 1/9 \log_{10} z/.35\right)$$

$z = 5$ cm $\theta_0^2 = 4.045\times10^{-5}$ $\underline{\theta_0 = 6.36 \text{ mrad}}$

add quadrature with 7.46 → negligible

~~Lithium $L_R = 155$ cm~~

~~$\theta_0^2 = 1.62\times10^{-3} z\left(1+ 1/9 \log_{10} z/155\right)$~~

~~$z = 15$ $\theta_0^2 = 2.16\times10^{-7}$~~

~~$\underline{\theta_0 = .46 \text{ mrad}}$~~

~~Beryllium [redacted] $L = 35.3$ [redacted] cm~~
~~1.5 cm~~

~~$\theta_0^2 = 9.04\times10^{-8}$~~

~~$\theta_0 = .30$ mrad~~

~~Total $\theta_0 = .55$ mrad : only ~1% effect~~

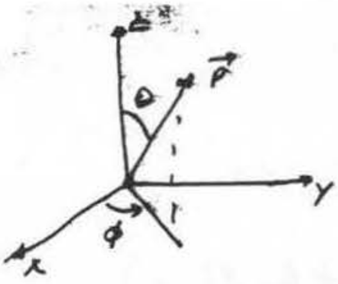
~~$\underline{7.4 \text{ mrad}}$~~

~~This is equivalent to an rms position spread~~

~~9 $\frac{.4\times10^{-3}}{.k \sin\mu} = \frac{.4\times10^{-3}}{.058\times\sin .87}$ cm $= \boxed{.009 \text{ cm}}$~~

~~$= .09$ mm~~

Defin[illegible]



$\tan\theta_x : \tan\theta\cos\phi$
$\tan\theta_y : \tan\theta\sin\phi$
$\tan^2\theta = \tan^2\theta_x + \tan^2\theta_y$
$x^2 + y^2 = r^2$

$dA = dx\,dy$
$d\Omega = \sin\theta\,d\theta\,d\phi = d\theta_x\,d\theta_y$
dp = momentum increment
dN = increment # of protons
$d\bar{N}$ = " " " pbars

Distribution functions $\bar{P}(x, y, z, \theta_x, \theta_y, p) = \bar{P}(\vec{x}, \vec{\theta}, p, z) = \dfrac{d\bar{N}}{dA\,d\Omega\,dp}$ ($\bar{p}$'s [illegible]/GeV/c)

$P(x, y, z) = P(\vec{x}, z) = \dfrac{dN}{dA}$ (p's/cm^2)

Let n_0 = # of target nuclei/vol (#/cm^3)
σ_a = proton absorption cross section (cm^2)
$\bar{\sigma}_a$ = pbar " " " (cm^2)
$\dfrac{d^2\sigma}{d\Omega\,dp}$ = pbar production " " (cm^2/GeV/c)

Evaluation of the Distribution functions

Neglect multiple scattering, energy loss
Assume no angular spread in proton beam

Protons: $dP/dz = -\sigma_a n_0 P$ (1)

Pbars: $\dfrac{d\bar{P}}{dz} = \partial\bar{P}/\partial z + \theta_x\,\partial\bar{P}/\partial x + \theta_y\dfrac{\partial\bar{P}}{\partial y} = -\bar{\sigma}_a n_0\bar{P} + n_0 P(x, y, z)\dfrac{d^2\sigma}{d\Omega\,dp}$

(in small angle approx)

Integrate (1) dz: $\int dP/P = -\int\sigma_a n_0\,dz \Rightarrow P(z) = P(0)\,e^{-z/\lambda}$

$P(x, y, z) = P_0(x, y)\,e^{-z/\lambda}$

~~$N_0 = 2\int dx\,dy\,P(x, y, z) = 2e^{-z/\lambda}\int dx\,dy\,P_0(x, y)$~~

$\Rightarrow \int dx\,dy\,P(x, y, z) = N_0\,e^{-z/\lambda}$ where $N_0 = \int P_0(x, y)\,dx\,dy$

= initial # of protons

integ [redacted] $d\theta_y$

$\int$ [redacted] $\frac{d}{dz}\bar{P} = \frac{d}{dz}\int dx\,dy\,d\theta_x\,d\theta_y \frac{d\bar{N}}{dx\,dy\,d\theta_x\,d\theta_y\,dp} = \frac{d}{dz}\left(\frac{d\bar{N}}{dp}\right)$

$\int dx\,dy\,d\theta_x\,d\theta_y\,\theta_x \frac{\partial \bar{P}}{\partial x} =$ ~~$\frac{d\bar{N}}{dp\,d\theta_x\,dx}$~~ $\int dy\,d\theta_y\,d\theta_x\,\theta_x\,\bar{P}(x\to\infty, \theta_x\, y, \theta_y) \to 0$ if $\bar{P}\to 0$ as $x\to\infty$

~~if $\frac{d\bar{N}}{dp\,d\theta_x\,dx}$ is even in θ_x then the integral vanishes; same for $\theta_y \frac{\partial\bar{P}}{\partial y}$~~

same for $\int dy \frac{\partial\bar{P}}{\partial y}$

$\int dx\,dy\,d\theta_x\,d\theta_y\,n_0\,\rho(x,y,z)\frac{d^2\sigma}{d\Omega\,dp} = \frac{d\sigma}{dp}\,n_0 N_0 e^{-z/\lambda}$

$\int dx\,dy\,d\theta_x\,d\theta_y\,\bar{P}(x,y,z) = \frac{d\bar{N}}{dp}$

$\Rightarrow \quad d/dz\,(d\bar{N}/dp) = -\bar{\sigma}_a n_0\, d\bar{N}/dp + n_0 N_0 e^{-z/\lambda}\, d\sigma/dp$

This is a differential equation of the form

$$dy/dx + y/\lambda = f(x)$$

$$d/dx\,(e^{x/\lambda} y) = e^{x/\lambda} y/\lambda + e^{x/\lambda} dy/dx = e^{x/\lambda}(y/\lambda + dy/dx) = e^{x/\lambda} f(x)$$

$$e^{x/\lambda} y \Big|_0^x = \int_0^x e^{x/\lambda} f(x)\,dx$$

$$e^{x/\lambda} y - y^{(0)} = \int_0^x e^{x/\lambda} f(x)\,dx$$

$$y = \frac{1}{e^{x/\lambda}} \int_0^x e^{x/\lambda} f(x)\,dx \quad \text{if } y(0) = 0$$

$$y \to d\bar{N}/dp \quad x \to z \quad \lambda \to 1/\bar{\sigma}_a n_0 = \bar{\lambda} \quad f(x) \to \frac{d\sigma}{dp} n_0 N_0 e^{-z/\lambda}$$

$$\Rightarrow d\bar{N}/dp = \frac{1}{e^{z/\bar{\lambda}}} \int_0^z e^{z/\bar{\lambda}} \frac{d\sigma}{dp} n_0 N_0 e^{-z/\lambda}\,dz$$

$$= \frac{d\sigma}{dp} n_0 N_0 \frac{\int_0^z e^{z(1/\bar{\lambda} - 1/\lambda)}\,dz}{e^{z/\bar{\lambda}}}$$

$$= \frac{d\sigma}{dp} n_0 N_0 \frac{e^{z/\bar{\lambda}} e^{-z/\lambda} - 1}{e^{z/\bar{\lambda}}} \frac{1}{1/\bar{\lambda} - 1/\lambda}$$

$$1/\bar{\lambda} - 1/\lambda = \frac{\lambda - \bar{\lambda}}{\lambda \bar{\lambda}} = \frac{1/\sigma_a n_0 - 1/\bar{\sigma}_a n_0}{1/\sigma_a n_0 \bar{\sigma}_a n_0}$$

$$= \frac{(\bar{\sigma}_a - \sigma_a) n_0}{\cancel{\sigma_a \bar{\sigma}_a n_0^2}} \cdot \cancel{\sigma_a \bar{\sigma}_a n_0^2} = (\bar{\sigma}_a - \sigma_a) n_0$$

$$\frac{d\bar{N}}{dp} = \frac{d\sigma}{dp} N_0 \frac{e^{-z/\lambda} - e^{-z/\bar{\lambda}}}{\bar{\sigma}_a - \sigma_a}$$

$1/\lambda - 1/\bar{\lambda} = (\sigma_a - \bar{\sigma}_a) n_0$; let $\sigma_a - \bar{\sigma}_a = \delta\sigma << \sigma_a$

$$1/\lambda = 1/\bar{\lambda} + n_0 \delta\sigma$$

$$e^{-z/\lambda} = e^{-z/\bar{\lambda} - n_0 \delta\sigma z} = e^{-z/\bar{\lambda}} e^{-n_0 \delta\sigma z}$$

[redacted] $\left(\frac{e^{-z/\lambda}(e^{-n_0 \delta\sigma z} - 1)}{\delta\sigma}\right) = N_0 \frac{d\sigma}{d\rho} \frac{e^{-z/\lambda}}{\delta\sigma}\left(1 - e^{-n_0\delta\sigma z}\right)$

[redacted] $\approx 1 - n_0 \delta\sigma z$

$\Rightarrow \frac{dN}{d\rho} \cdot N_0 \frac{d\sigma}{d\rho} e^{-z/\lambda} \frac{n_0 \delta\sigma z}{\delta\sigma} = N_0 \frac{d\sigma}{d\rho} z e^{-z/\lambda} n_0$

$dN/d\rho$ peaks at $d/dz\,(z e^{-z/\lambda}) = 0$

$$e^{-z/\lambda} - 1/\lambda\, z e^{-z/\lambda} = 0$$

$$1 - z/\lambda = 0, \boxed{z = \lambda}$$

$$(dN/d\rho)_{max} = N_0 n_0\, d\sigma/d\rho\, \lambda/e = N_0\, d\sigma/d\rho \frac{1}{e\sigma_a}$$

Y = Yield $= \Delta N/N_0$ $\quad Y_{max} = \frac{1}{e\,\sigma_a}\left(\frac{d\sigma}{d\rho}\right)\Delta\rho$

Example $\quad \frac{d^2\sigma}{d\rho\, d\Omega} = \frac{1}{\pi\theta_0^2} e^{-\theta^2/\theta_0^2} \frac{d\sigma}{d\rho}$

$$\int d\Omega \frac{d^2\sigma}{d\rho\, d\Omega} = 2\pi \int_0^\infty \theta\, d\theta\, e^{-\theta^2/\theta_0^2} \frac{1}{\pi\theta_0^2}\, d\sigma/d\rho$$

$$= \frac{\pi}{\pi\theta_0^2} \int_0^\infty du\, e^{-u/\theta_0^2}\, d\sigma/d\rho = \frac{d\sigma}{d\rho} \quad (u = \theta^2)$$

Take $\quad \left.\frac{d^2\sigma}{d\rho\, d\Omega} \frac{1}{\sigma_{abs}}\right|_{\theta=0} = \frac{1}{\sigma_{abs}} \frac{1}{\pi\theta_0^2} \frac{d\sigma}{d\rho} = .252$ /sr/GeV/c

with $\theta_0 = .0578 \Rightarrow \frac{1}{\sigma_a} d\sigma/d\rho = .252 \times (.0578)^2 \pi$

$$= 2.64 \times 10^{-3} \text{ /GeV/c}$$

$$Y_{max} = \frac{2.64 \times 10^{-3}}{2.718} \Delta\rho = 9.73 \times 10^{-4} \Delta\rho$$

For $\theta_0 = .0754$
$\frac{1}{\sigma_a}\frac{d\sigma}{d\rho} = 4.5 \times 10^{-3}$
$Y_{max} = 1.656 \times 10^{-3}\, \Delta\rho$

For $\rho = 8.9$ GeV/c, $\delta = \Delta\rho/\rho$

$Y_{max} = 8.66 \times 10^{-3}\, \delta \quad$ e.g. $\delta = .03 \Rightarrow Y_{max} = 2.5 \times 10^{-4}$

Integrate [redacted] accepted phase space area ε:

$$d/d[redacted] dx = \int_{a(z)}^{b(z)} d/dz\, f(x,z)\, dx + f(b,z)\frac{db}{dz} - f(a,z)\frac{da}{dz}$$

So $$\int^{\varepsilon(z)} dx\,dy\,d\theta_x\,d\theta_y \frac{d\bar{P}}{dz} = d/dz \int_0^{\varepsilon(z)} dx\,dy\,d\theta_x\,d\theta_y\, \bar{P} - \bar{P}(\varepsilon(z), z)\frac{d\varepsilon(z)}{dz}$$

Ignore 2nd term for the moment. Define $\bar{N}_\varepsilon(z) = \int_0^{\varepsilon(z)} dx\,dy\,d\theta_x\,d\theta_y\, \bar{P}\, \Delta\rho$
(See 5a)

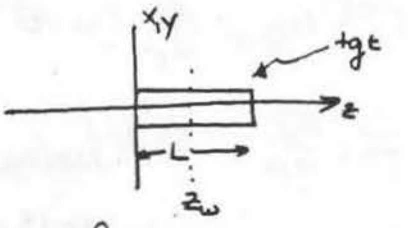
then $$\frac{d\bar{N}_\varepsilon}{dz} = n_0 e^{-z/\lambda} \Delta\rho \int_0^{\varepsilon(z)} dx\,dy\,d\Omega\, P(x,y)\frac{d^2\sigma}{d\Omega\, d\rho} - \bar{\sigma}_a n_0 \bar{N}_\varepsilon$$

use $y = e^{-x/\lambda}\int_0^x e^{x/\lambda} f(x)\,dx$ again, (solution of $dy/dx + y/\lambda = f(x)$)

to get $$\bar{N}_\varepsilon(z) = e^{-z/\bar{\lambda}} \int_0^z n_0 e^{-z'/\lambda} \Delta\rho\, e^{z'/\bar{\lambda}} \int_0^{\varepsilon(z)} dx\,dy\,d\Omega\, P(x,y)\frac{d^2\sigma}{d\Omega\, d\rho}\, dz$$

Phase space integration

In a drift space γ is constant;

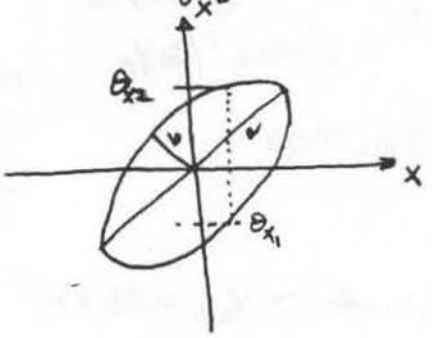


z_w is the waist, where $\alpha = 0$

$$\alpha(z) = \alpha(0) - \gamma z$$

$$\alpha(z_w) = 0 = \alpha(0) - \gamma z_w \Rightarrow \alpha(0) = \gamma z_w$$

$$\alpha(z) = \gamma(z_w - z) = -\gamma(z - z_w)$$



Limits of θ_x integration are $\theta_{x2}, \theta_{x1}\,(x)$

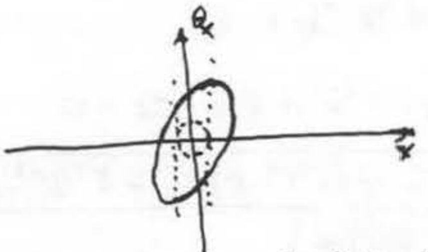
Ellipse area $\varepsilon = \pi ab$

[redacted] $d\theta_x \int_{\theta_1(y,z)}^{\theta_2(y,z)} d\theta_y \frac{d\bar{P}}{dz} =$

$$d/dz \int dx\,dy\,d\theta_x\,d\theta_y\,\bar{P} - \Big[\int dx\,dy \int d\theta_y\, \bar{P}(x,y,\theta_y,\theta_x=\theta_2(x,z))$$

$$- \int dx\,dy \int d\theta_y\, \bar{P}(x,y,\theta_y,\theta_x=\theta_1(x,z)) + \int dx\,dx \int d\theta_x\, \bar{P}(x,y,\theta_x,\theta_y=\theta_2(x,z))$$

$$- \int dx\,dy \int d\theta_x\, \bar{P}(x,y,\theta_x,\theta_y=\theta_1(y,z)) \Big]$$



Each of the integrals is essentially the line integral of the distribution function over either the top or bottom of the acceptance ellipse. Because of symmetry the line integral over the top half cancels that over the bottom half => all extra terms are zero invariant emittance $\quad \varepsilon = \gamma x^2 + 2\alpha x\theta_x + \beta\theta_x^2$

$$\beta = \frac{1+\alpha^2}{\gamma} = \frac{1+\gamma^2(z-z_w)^2}{\gamma}$$

$$\Rightarrow \gamma x^2 + 2x\theta_x(-\gamma)(z-z_w) + \theta^2/\gamma\,(1+\gamma^2(z-z_w))^2 = \varepsilon$$

$$\gamma^2 x^2 - 2x\theta_x\gamma^2(z-z_w) + \theta^2(1+\gamma^2(z-z_w)^2) = \varepsilon\gamma$$

Let $z' = 2(z-z_w)$

$$\gamma^2 x^2 - x\theta_x\gamma^2 z' + \theta_x^2(1+\gamma^2(z')^2/4) = \varepsilon\gamma$$

$$\theta_x^2(1+\gamma^2(z')^2/4) - \theta_x x\gamma^2 z' + \gamma^2 x^2 - \varepsilon\gamma = 0$$

$$\theta_x = \frac{x\gamma^2 z' \pm \sqrt{x^2\gamma^4(z')^2 - 4(\gamma^2 x^2 - \varepsilon\gamma)(1+\gamma^2(z')^2/4)}}{2(1+\gamma^2(z')^2/4)}$$

$$\sqrt{} = \sqrt{\cancel{x^2\gamma^4(z')^2} - 4\gamma^2x^2 - \cancel{4\gamma^2(z')^2/4\,x^2} + 4\varepsilon\gamma + \varepsilon\gamma^3(z')^2}$$

$$= \sqrt{\varepsilon\gamma(4+\gamma^2(z')^2) - 4\gamma^2x^2}$$

$$\theta_x = \frac{x\gamma^2 z'/2 \pm \sqrt{\varepsilon\gamma(1+\gamma^2(z')^2/4) - \gamma^2x^2}}{1+\gamma^2(z')^2/4}$$

limits on z': $\quad z' = 2(z-z_w) \qquad z=0 \Rightarrow z' = -2z_w$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad z=L \Rightarrow z' = 2(L-z_w)$

$dz' = 2dz \qquad z = z_w + z'/2$

$$N_\varepsilon(\ell) = e^{-\ell/\bar\lambda}\int_{-2z_w}^{2(\ell-z_w)} n_0 e^{-(z_w+z/2)/\lambda}\,\Delta p\, e^{+(z_w+z/2)/\bar\lambda}\,\frac{dz}{2}\int_{-\infty}^{\infty} dx \int_{\theta_{x1}}^{\theta_{x2}(x,z)} d\theta_x$$

$$\times \int_{-\infty}^{\infty} dy \int_{\theta_{y1}}^{\theta_{y2}(y,z)} P(x,y)\,\frac{d^2\sigma}{d\Omega\, dp}$$

$N_c(\ell) =$ [illegible] $e^{-z_w/\lambda} e^{+z_w/\bar\lambda} \dfrac{n_0 \Delta\rho}{2} \displaystyle\int_{-2z_w}^{2(\ell-z_w)} e^{+z/2(1/\bar\lambda - 1/\lambda)} dz \int_{-\infty}^{\infty} dx \int_{\theta_{x_1}}^{\theta_{x_2}(x,z)} d\theta_x$

$$\times \int_{-\infty}^{\infty} dy \int_{\theta_{y_1}}^{\theta_{y_2}(y,z)} \rho(x,y) \frac{d^2\sigma}{d\Omega d\rho} d\theta_y$$

$$\frac{d^2\sigma}{d\Omega d\rho} = \frac{1}{\pi\theta_0^2} e^{-\theta^2/\theta_0^2} d\sigma/d\rho \quad \& \quad \left.\frac{d^2N}{d\rho d\Omega}\right|_{\theta=0} \sigma_a = \left.\frac{d^2\sigma}{d\rho d\Omega}\right|_{\theta=0} = \frac{d\sigma/d\rho}{\pi\theta_0^2}$$

$$n_0\, d\sigma/d\rho = n_0 \pi\theta_0^2 \left.\frac{d^2\sigma}{d\rho d\Omega}\right|_{\theta=0}$$

$$= \pi\theta_0^2 \frac{1}{\lambda\sigma_a} \left.\frac{d^2\sigma}{d\rho d\Omega}\right|_{\theta=0} = \frac{\pi\theta_0^2}{\lambda} \left.\frac{d^2N}{d\rho d\Omega}\right|_{\theta=0}$$

$$= \frac{\pi\theta_0^2}{\lambda} X, \quad X = \left.\frac{d^2N}{d\rho d\Omega}\right|_{\theta=0}$$

$$n_0 \frac{d^2\sigma}{d\rho d\Omega} = n_0\, d\sigma/d\rho \frac{1}{\pi\theta_0^2} e^{-\theta^2/\theta_0^2} = \frac{\pi\theta_0^2}{\lambda} X \frac{1}{\pi\theta_0^2} e^{-\theta^2/\theta_0^2} = (X/\lambda) e^{-\theta^2/\theta_0^2}$$

Integration over angles: $e^{-\theta^2/\theta_0^2} = e^{-(\theta_x^2+\theta_y^2)/\theta_0^2}$

$$\int_{-\infty}^{\infty} dx \int_{\theta_1}^{\theta_2} d\theta \frac{d^2\sigma}{d\Omega d\rho} \rho(x,y) \quad : \quad \rho(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-x^2/\sigma_x^2 2} e^{-y^2/\sigma_y^2 2}$$

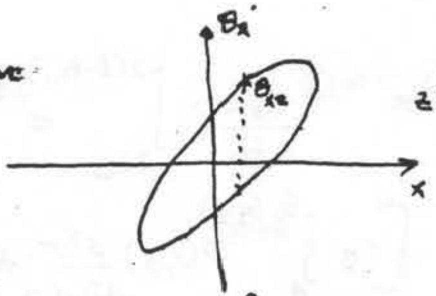
$$\int_{-\infty}^{\infty} dx\, \rho(x) \int_{\theta_1}^{\theta_2} d\theta_x e^{-\theta_x^2/\theta_0^2} = 2\int_{-\infty}^{\infty} dx\, \rho(x) \int_0^{\theta_2} d\theta_x e^{-\theta_x^2/\theta_0^2}$$ if $\rho(x)$ is symmetric about $x=0$

$$\frac{2}{\sqrt{\pi}} \int_0^{\theta_2} d\theta_x e^{-\theta_x^2/\theta_0^2} = 2/\sqrt{\pi} \int_0^{\theta_2/\theta_0} du\, e^{-u^2} \theta_0 \qquad u = \theta_x/\theta_0$$

$$= \mathrm{erf}\left(\theta_2/\theta_0\right) \theta_0$$

$$\text{so} \quad \int_0^{\theta_2} e^{-\theta_x^2/\theta_0^2} d\theta_x = \frac{\sqrt{\pi}}{2} \mathrm{erf}\left(\theta_2/\theta_0\right) \theta_0$$

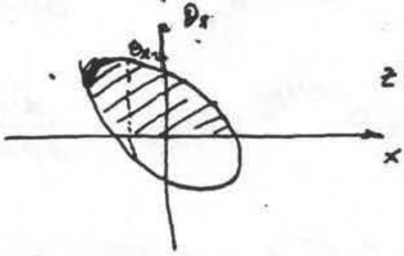
[redacted] Limits [redacted] integration



$z>0$: $0 \rightarrow \theta_{x2}$ x 2

θ_{x2} has (+) sign

$x \geq 0, z \geq 0$



$z<0$ θ_{x2} has ~~negative~~ (+) sign

$x<0$ and $z<0$

gives large θ_{x2}

For $z<0$ [redacted] the limits on θ_x are the same as for $z>0$ (i.e, function is symmetric in z) when integrated over x

$N_E(l)$ [redacted] $e^{-z_w/\bar\lambda} e^{+z_w/\bar\lambda} \; \frac{n_0 \Delta p}{2} \int_{-2z_w}^{2(l-z_w)} e^{+z/2(1/\bar\lambda - 1/\lambda)} dz$

$$\times \left\{ \int_{-\infty}^{\infty} dx \; \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{\theta_2(x,z)}{\theta_0}\right) \theta_0 \, 2 \int_{-\infty}^{\infty} dy \, \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{\theta_2(y,z)}{\theta_0}\right) \theta_0 \, 2 \, P(x,y) \right\} \frac{d\sigma}{dp} \frac{1}{\pi \theta_0^2}$$

$$\Rightarrow N_\varepsilon(l) = e^{-l/\bar\lambda} e^{-z_w/\lambda} e^{+z_w/\bar\lambda} \; \frac{n_0 \Delta p}{2} \int_{-2z_w}^{2(l-z_w)} e^{z/2(1/\bar\lambda - 1/\lambda)} dz$$

$$\times \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; P(x,y) \operatorname{erf}\left(\frac{\theta_2(x,z)}{\theta_0}\right) \operatorname{erf}\left(\frac{\theta_2(y,z)}{\theta_0}\right) \frac{d\sigma}{dp}$$

For $z_w = l/2$: $\int_{-l}^{l} e^{z/2(1/\bar\lambda - 1/\lambda)} dz \, f(z) \longrightarrow 2 \int_0^l e^{z/2(1/\bar\lambda - 1/\lambda)} dz \, f(z)$

$$\longrightarrow 2 \int_0^l dz \, f(z) \text{ as } \bar\lambda \to \lambda$$

Define $f_5(\theta) = \exp(-\theta^2/\theta_0^2)$

~~$f_4(\phi, r, z) = \int_{\theta_1(r,\phi,z)}^{\theta_2(r,\phi,z)} f_5(\theta) d\theta \int_{\theta_1(y,\phi,z)}$~~

$$f_4(\phi, r, z) = \int_{\theta_1(x,z)}^{\theta_2(x,z)} f_5(\theta) d\theta \int_{\theta_1(y,z)}^{\theta_2(y,z)} f_5(\theta) d\theta$$

$y = r \sin\phi$
$x = r \cos\phi$

$$\theta_2(x,z) = \frac{-\alpha(z)\gamma x + \sqrt{\varepsilon\gamma(1+\alpha^2(z)) - x^2\gamma^2}}{1+\alpha^2(z)}$$

$$\theta_1(x,z) = \frac{-\alpha(z)\gamma x - \sqrt{\varepsilon\gamma(1+\alpha^2(z)) - x^2\gamma^2}}{1+\alpha^2(z)}$$

$$\alpha(z) = \underline{-\gamma z'/2} = -\gamma/2 \cdot 2(z - z_w) = -\gamma(z - z_w)$$

$f_3(r,$ [redacted] $z)\, r\, \rho_0(r)$ $\qquad \rho_0(r) = \frac{N_0}{\pi\sigma_r^2}\exp(-r^2/\sigma_r^2)$

$f_2(r, z)$ [redacted] $f_4(r, \phi_1 z)$

$$N_\Sigma(l) = e^{-l/\bar\lambda} e^{z_0(1/\bar\lambda - 1/\lambda)} \frac{n_0\Delta\rho N_0}{2}\int_{-2z_0}^{2(l-z_0)} e^{z/2(1/\bar\lambda-1/\lambda)}\,dz \int_{-\infty}^{\infty} dx \int_{\theta_1}^{\theta_2} d\theta_x$$

$$\int_{-\infty}^{\infty} dy \int_{\theta_1}^{\theta_2} \rho(x,y)\,\frac{d^2\sigma}{d\Omega\, d\rho}\, d\theta_y$$

$$N_\Sigma(l) = N_0\,\frac{n_0\Delta\rho}{2}\,\frac{d\sigma}{d\rho}\,\frac{1}{\pi\theta_0^2}\, e^{-l/\bar\lambda} e^{z_\omega(1/\bar\lambda-1/\lambda)}\int_{-2z_\omega}^{2(l-z_\omega)} e^{z/2(1/\bar\lambda-1/\lambda)}\,dz$$

$$\int_0^{2\pi} d\phi \int_0^{r_0} r\,dr \int_{\theta_1}^{\theta_2} d\theta_x\, d\theta_y\, \rho_0(r)\, e^{-\theta_x^2/\theta_0^2} e^{-\theta_y^2/\theta_0^2}$$

$$= \frac{n_0\Delta\rho}{2}\,\frac{d\sigma}{d\rho}\,\frac{N_0}{\pi\theta_0^2}\, e^{-l/\bar\lambda} e^{z_\omega(1/\bar\lambda-1/\lambda)}\int_{-2z_\omega}^{2(l-z_\omega)} e^{z/2(1/\bar\lambda-1/\lambda)}\,dz \int_0^{r_0} dr\, f_3(r,z)$$

$$f_1 = \int_0^{r_0} f_3(r)\,dr = f_1(z)$$

$1/\lambda - 1/\bar\lambda = (\bar\sigma_a - \bar\sigma_a)n_0$; let $\bar\sigma_a - \bar\sigma_a = \delta\sigma$

$1/\bar\lambda - 1/\lambda = -\delta\sigma n_0$ $\qquad$ also $\quad n_0\, d\sigma/d\rho = \frac{\pi\theta_0^2}{\lambda} X$

$$N_\Sigma(l) = \frac{\Delta\rho}{2}\,\frac{\cancel{\pi\theta_0^2}}{\lambda} X\, \frac{N_0}{\cancel{\pi\theta_0^2}}\, e^{-l/\bar\lambda} e^{-\delta\sigma n_0 z_\omega}\int_{-2z_\omega}^{2(l-z_\omega)} e^{-\frac{\delta\sigma n_0 z}{2}} f_1(z)\,dz$$

$$\frac{N_\Sigma(l)}{N_0\Delta\rho} = \frac{e^{-l/\bar\lambda} e^{-\delta\sigma n_0 z_\omega}}{2\lambda}(X)\int_{-2z_\omega}^{2(l-z_\omega)} e^{-\delta\sigma n_0 z/2} f_1(z)\,dz$$

$$\bar\lambda = \lambda + \Delta\lambda$$

$$\frac{1}{\bar\lambda} = \frac{1}{\lambda(1+\Delta\lambda/\lambda)} = \frac{1}{\lambda}\left(1-\frac{\Delta\lambda}{\lambda}\right) = \frac{1}{\lambda} - \frac{\Delta\lambda}{\lambda^2}$$

$$\delta\sigma n_0 = \frac{1}{\lambda} - \frac{1}{\bar\lambda} = \frac{1}{\lambda} - \frac{1}{\lambda} + \frac{\Delta\lambda}{\lambda^2}$$

$$\boxed{\delta\sigma n_0 = \frac{1}{\lambda}\frac{\Delta\lambda}{\lambda}} \qquad \frac{1}{n_0\delta\sigma} = \lambda(\lambda/\Delta\lambda)$$

Approximat[illegible] 1) $z_\omega = \ell/2 \quad \delta\sigma = 0$

$$\frac{N(\ell)}{N_0 \Delta\rho} = \frac{e^{-\ell/\lambda}}{2\lambda} \times \int_{-\ell}^{\ell} f_1(z)\,dz$$

Since $f_1(z)$ is <u>even</u> in z $\quad \int_{-\ell}^{\ell} f_1(z)\,dz = 2\int_0^{\ell} f_1(z)\,dz$

$$\frac{N_\varepsilon(\ell)}{N_0 \Delta\rho} = \frac{e^{-\ell/\lambda}}{\lambda} \times \int_0^{\ell} f_1(z)\,dz$$

2) $\sigma_r \to 0$ then $\rho_0(r) = \frac{1}{2\pi r}\delta(r)$

~~$f_3(0,z) = \frac{1}{2\pi} f_2(0)$~~

$$f_1(z) = \int_0^{r_0} dr\, f_3(r,z) = \int_0^{r_0} dr\, f_2(r,z)\frac{r\,\delta(r)}{2\pi r} = \frac{1}{2\pi} f_2(0,z)$$

$$f_2(0,z) = \int_0^{2\pi} d\phi\, f_4(0,\phi,z) = 2\pi f_4(0,0,z)$$

$$f_4(0,\phi,z) = \left[\int_{\theta_1}^{\theta_2} f_5(\theta)\,d\theta\right]^2 \qquad \theta_2 = \frac{\sqrt{\varepsilon\gamma(1+\alpha^2(z)}}{1+\alpha^2(z)}$$

$$\theta_1 = -\theta_2$$

$$f_4(0,\phi,z) = \left[2\int_0^{\theta_2} d\theta \exp\left(-\theta^2/\theta_0^2\right)\right]^2 = \left(\sqrt{\pi}\,\theta_0 \operatorname{erf}(\theta_2/\theta_0)\right)^2$$

$$\frac{N_\varepsilon(\ell)}{N_0 \Delta\rho} = \frac{e^{-\ell/\lambda}}{\lambda} \times \int_0^{\ell} dz\, \pi\theta_0^2 \left[\operatorname{erf}(\theta_2/\theta_0)\right)^2$$

$$\theta_z = \text{[illegible]} \pm \frac{\sqrt{\varepsilon\gamma(1+\gamma^2 z^2/4) - x^2\gamma^2}}{1+\gamma^2 z^2/4}$$

$$\frac{\theta_z(0)}{\theta_0} = \frac{\sqrt{\varepsilon\gamma(1+\gamma^2 z^2/4)}}{\theta_0(1+\gamma^2 z^2/4)} \quad \text{for } \varepsilon\gamma \ll \theta_0^2$$

$$\frac{\theta_z(0)}{\theta_0} = \sqrt{\frac{\varepsilon\gamma}{\theta_0^2}}\left(1+\gamma^2 z^2/4\right)^{-1/2} \qquad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du \rightarrow \frac{2x}{\sqrt{\pi}} \text{ for } x \ll$$

$$\mathrm{erf}(x) \sim x\frac{2}{\sqrt{\pi}} \text{ for } x \ll 1$$

$$\mathrm{erf}\left(\sqrt{\frac{\varepsilon\gamma}{\theta_0^2}}\left(1+\gamma^2 z^2/4\right)^{-1/2}\right)^2 \approx \frac{\varepsilon\gamma}{\theta_0^2}\frac{1}{1+\gamma^2 z^2/4}\frac{4}{\pi}$$

$$\frac{N_\varepsilon(\ell)}{N_0\Delta p} = \frac{e^{-\ell/\lambda}}{\lambda} \times \pi\cancel{\theta_0^2}\frac{\varepsilon\gamma}{\cancel{\theta_0^2}}\int_0^L \frac{1}{1+\gamma^2 z^2/4}dz \times \frac{4}{\pi}$$

$$\int\frac{dx}{X} = \frac{2}{\sqrt{q}}\tan^{-1}\frac{2cx+b}{\sqrt{q}} \qquad q = 4ac-b^2 \qquad \begin{matrix} b=0 \\ a=1 \end{matrix} \quad c = \gamma^2/4$$

$$\int_0^L \frac{1}{1+\gamma^2 z^2/4}dz = \frac{2}{\gamma}\tan^{-1}\left\{\frac{\gamma^2/2\,x}{\gamma}\right\}\Bigg|_0^L = 2/\gamma\tan^{-1}\frac{L\gamma}{2}$$

$$\frac{N_\varepsilon(\ell)}{N_0\Delta p} = \frac{e^{-\ell/\lambda}}{\lambda}\times\cancel{\pi}2\varepsilon\tan^{-1}\frac{\ell\gamma}{2}\times\frac{4}{\cancel{\pi}} = \frac{8e^{-\ell/\lambda}}{\lambda}\times\varepsilon\tan^{-1}\frac{\ell\gamma}{2}$$

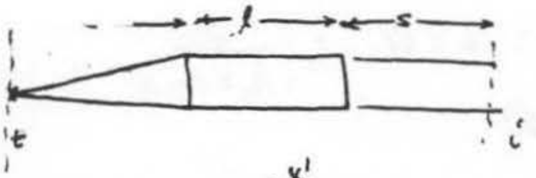
$$= e^{-\ell/\lambda}\frac{n_0\frac{d\sigma}{dp}}{\pi\theta_0^2}\pi 2\varepsilon\tan^{-1}\frac{\ell\gamma}{2}\times\frac{4}{\pi} \qquad \lambda = \frac{1}{n_0\sigma_a}$$

$$= e^{-\ell/\lambda}\cancel{n_0}\frac{d\sigma}{dp}\frac{1}{\theta_0^2}\frac{1}{\lambda\cancel{n_0}\sigma_a}2\varepsilon\tan^{-1}\frac{\ell\gamma}{2}\times\frac{4}{\pi}$$

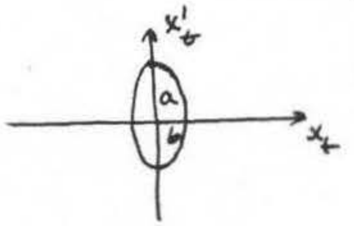
$$= \frac{e^{-\ell/\lambda}}{\pi}\frac{d\sigma}{dp}\frac{1}{\theta_0^2\lambda\sigma_a}8\varepsilon\tan^{-1}\frac{\ell\gamma}{2}$$

Appendix A-[illegible]

Lens focus[illegible]



At z_m



Ellipse area ε

Aspect ratio γ, β

At lens principal plane
$$\begin{pmatrix} x_i \\ x_i' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu/k \\ -k\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ x_t' \end{pmatrix}$$

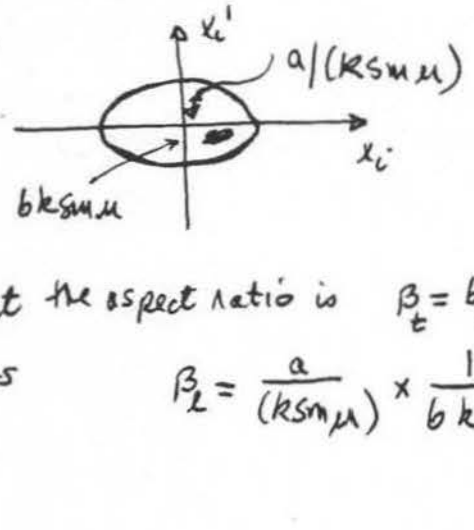
$$x_t = x_i(\cos\mu - ks\sin\mu) - (2s\cos\mu + \sin\mu/k - ks^2\sin\mu)x_i'$$
$$x_t' = x_i \quad k\sin\mu \quad + \quad (\cos\mu - ks\sin\mu)x_i'$$

The focal distance is $f = \dfrac{1}{k\tan\mu}$

and for $s = f$, $\quad x_t = -x_i'/k\sin\mu$
$$x_t' = x_i\, k\sin\mu$$

$\mu = kl$ = lens phase advance

At i,



At the target the aspect ratio is $\beta_t = b/a$ and $\gamma_t = a/b$

After the lens $\quad \beta_l = \dfrac{a}{(k\sin\mu)} \times \dfrac{1}{b\,k\sin\mu} = \dfrac{\gamma_t}{k^2\sin^2\mu}$

We require that $r_0 = \frac{a}{k \sin\mu}$ = radius of the lens

then $a = r_0 k \sin\mu$

and $\varepsilon = ab \Rightarrow b = \varepsilon/a = \frac{\varepsilon}{r_0 k \sin\mu}$

so $\beta_t = b/a$

$$\beta_t = \frac{\varepsilon}{(r_0 k \sin\mu)^2}$$

$$\gamma_t = 1/\beta_t$$

$$\beta_\ell = \frac{\gamma_t}{k^2 \sin^2\mu} = \frac{r_0^2 k^2 \sin^2\mu}{\varepsilon k^2 \sin^2\mu} = r_0^2/\varepsilon$$

$k = \sqrt{\frac{0.3G}{p}}\ m^{-1}$, G in T/m, p in GeV/c $G = B_0/r_0$ [redacted]

$\mu = k\ell = \ell\sqrt{\frac{0.3G}{p}}$

See next page for a modification of the relation between β_ℓ and r_0 due to the cylindrical-symmetry of the lens.

Addendum C: Lens Abberations

The discussions in the previous sections assumed a perfectly linear field in the lens, and neglected multiple Coulomb scattering in the lens and chromatic effects. Of these three principal sources of aberrations, the most significant in the Tev I design is the first one. This arises from the pulsed nature of the lens current, which takes some time to diffuse from the periphery of the lens into the center. This diffusion, and the constraints that it sets on the lens operating parameters, has been described in detail elsewhere (see ref. 4 and 5). As discussed in these references, the deviation from linearity of the lens field is a function of the time t after the beginning of the lens pulse:

$$\Delta^2(\omega t) = \int_0^{r_0} dr\, r\, \frac{(B(r,\omega t) - G(\omega t) r)^2}{(B(r_0,\omega t))^2} \Big/ \int_0^{r_0} r\, dr$$

Here Δ^2 is the mean square relative deviation of the field $B(r,\omega t)$ from linearity at the time t; $T = \pi/\omega$ is the duration of the half-sine wave current pulse. $G(\omega t)$ is the effective gradient, defined by minimizing the function $\Delta^2(\omega t)$; i.e.,

$$d/dG\,(\Delta^2(\omega t)) = 0$$

is the equation for $G(\omega t)$. $\sqrt{\Delta^2}$ vs. $\phi = \omega t$ is shown in fig. C-1 for the Tev I design lens parameters; it minimizes at about 2.1% for $\phi = 1.98$ radians. As shown in Appendix A-6, the angular variation from linear focusing produced by Δ^2 in the lens is roughly equivalent to a position spread at the target of

$$\delta r_t = R_0 \sqrt{\Delta^2}$$

For the Tev I design lens (see Appendix A-6), $\delta r_t \sim .21$ mm. This appendix also shows that the efect of the multiple Coulomb scattering in the lens is equivalent to a position spread of about .16 mm, and chromatic effects for $\Delta p/p = \pm 2.4\%$ are equivalent to a position spread of 0.07 mm at the target.

The total effect of the aberrations (taken in quadrature) is roughly equivalent to a spread in r of about .26 mm at the target. When taken in quadrature with the design $\sigma_r = .54$ mm, this is a 10% increase in beam size, or a 10% decrease in pbar yield according to fig. 6. However, if plans are made to reduce the spot size possible on the target utilizing beam sweeping techniques, it must be realized that there will be a limit to the effective beam size at about .26 mm, unless Δ^2 is reduced by increasing the pulse width of the current pulse applied to the lens (see ref. 4, 5). This results in greater energy depostion in the lens and has implications for the peak current at which the lens can be operated.

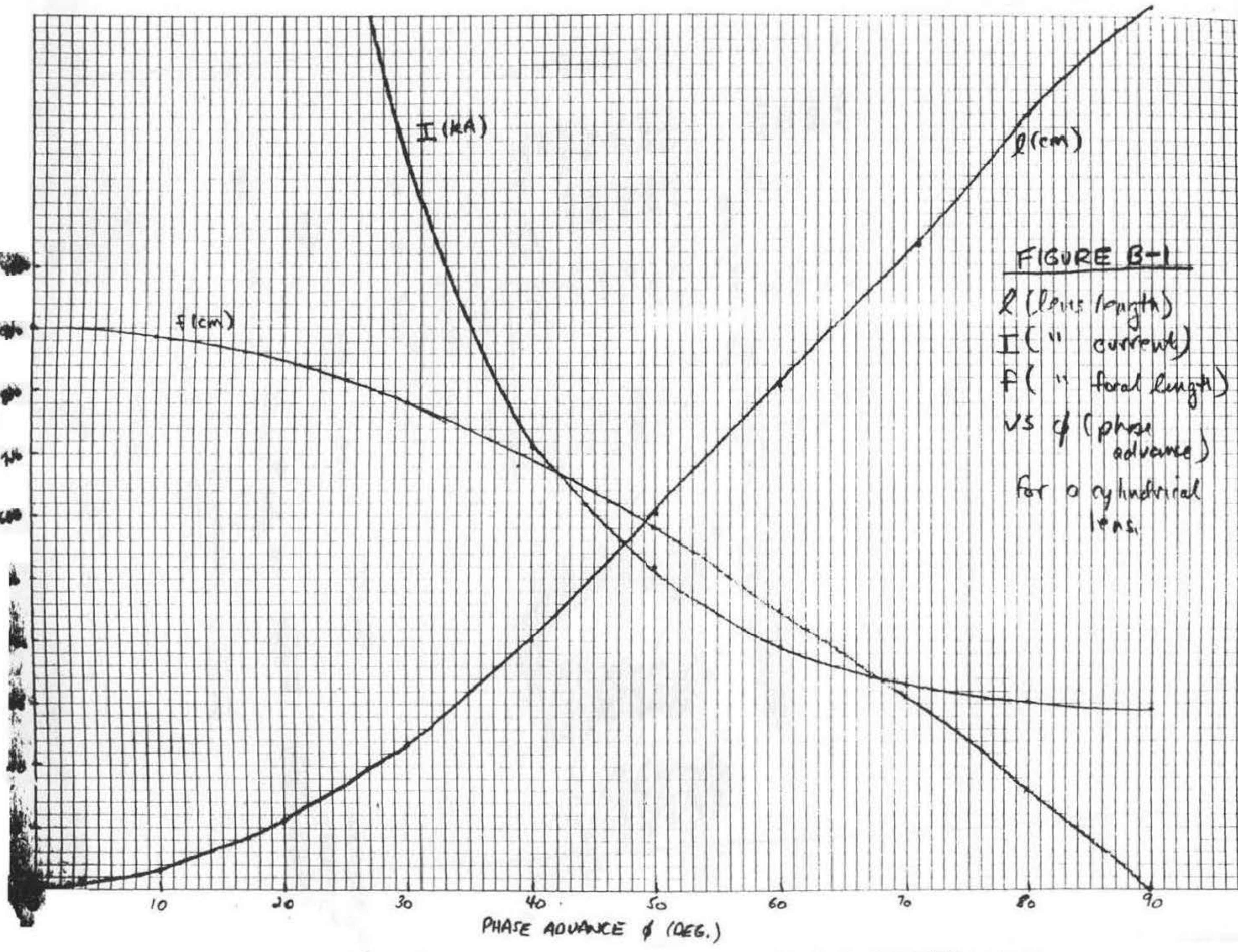


FIGURE B-1

l (lens length)
I (" current)
f (" focal length)
vs ϕ (phase advance)
for a cylindrical lens.

[redacted]

$$\varepsilon/\beta_t = \frac{20\times10^{-?}}{1.3\times10^{-2}} = 1.54\times10^{-3}$$

$$\rho = 8.9$$

$$I_{th} = (1.54\times10^{-3}) \times \frac{2\pi\times8.9}{0.3\times\frac{4\pi\times10^{-7}}{2}} = \underline{230\,kA}$$

R = 10 mm $\beta_\ell = \frac{\pi}{4}\,10^2/20$ [redacted] cm = $\underline{3.93\ m}$ $\beta_\ell\beta_t = 3.93\,m \times 1.29\times10^{-2}\,m = 5.07\times10^{-2}\,m^2$

$$\sqrt{\beta_\ell\beta_t} = .225\ m = \underline{22.5\ cm}$$

so $I = 293/\sin^2\phi$ kA

$f = 22.5\cos\phi$ cm

$\ell = 22.5\,\phi\sin\phi$ cm

			R = 10 mm		~~R = 5 mm~~		
ϕ	I (kA)	G(T/m)	ℓ (cm)	f (cm)	~~ℓ (cm)~~	~~f (cm)~~	~~G (T/m)~~
0	∞		0	22.5	~~0~~	~~11.25~~	
20	2505	5010	2.69	21.14	~~1.34~~	~~10.57~~	~~20040~~
40	709	1418	.10.0	17.24	~~5.0~~	~~8.62~~	~~5672~~
60	390	780	20.17	[redacted] 11.25	~~10.0~~	~~5.63~~	~~3120~~
80	302	604	30.94	3.91	~~15.47~~	~~1.96~~	~~2416~~
90	293	586	35.34	0	~~17.67~~	~~0~~	~~2344~~

~~R = 5 mm β_ℓ = .983 m $\sqrt{\beta_\ell\beta_t}$ = 11.25 cm~~

10	9716	.68	22.1
30	1172	5.89	19.5
50	500	15.0	14.5
70	331	25.8	7.7

(a) Aluminum $\boxed{\Delta r_t \doteq .027\ cm}$

[illegible] (b) Lithium $L_R = 155\ cm$

$$\theta_0^2 = 1.62 \times 10^{-8} z \left(1 + 1/9 \log z/155\right)$$

$z = 15\ cm \quad \theta_0^2 = 2.16 \times 10^{-7}$

Beryllium : $z = 1.5\ cm \quad L = 35.3\ cm$

$$\theta_0^2 = 9.04 \times 10^{-8}$$

$\Rightarrow \quad \theta_0^2(\text{total}) = 3.06 \times 10^{-7}$

$\mu_0 = .87 \quad k = 5.8\ m^{-1}$

$$\Delta x_t^2 = \frac{3.06 \times 10^{-7}/2\,(1 + .566)}{(5.8)^2\,(.76)^2} = .123 \times 10^{-7}\ m^2$$

$$\boxed{\Delta x_t = .11\ mm}$$

(c) $\delta p/p = .024$

$$\delta f/f = \frac{.024}{2}\left(1 - \frac{.87}{.76 \times .64}\right) = 9.46 \times 10^{-3}$$

$\underset{m\text{-}rad}{\varepsilon = 20 \times 10^{-6}} \quad \beta_0 = 1.3 \times 10^{-2}_{\ m} \quad f = 14.5\ cm$

$$\Delta x = \sqrt{\frac{20 \times 10^{-6}}{1.3 \times 10^{-2}}} \times 14.5 \times 9.46 \times 10^{-3}\ cm = 5.38 \times 10^{-3}$$

$$\boxed{\Delta x \doteq .05\ mm}$$

[illegible]

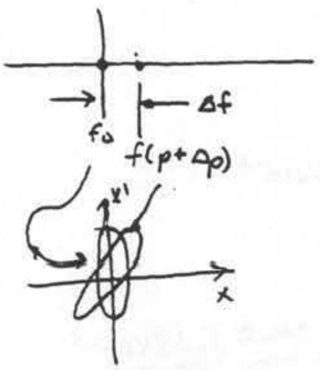
[illegible] $] = C\,ds$ $\qquad \boxed{C\ell = \theta^2_{mcs}}$

[illegible] integrate $\quad \Delta x_t^2 = C\int_0^\ell ds \left(\frac{1}{k\sin k\ell}\right)^2 \cos^2(ks-k\ell)$.

let $\mu_0 = k\ell$

Do the integral $\Rightarrow$ $\boxed{\Delta x_t^2 = \frac{(1/2\,C\ell)\,(1 + 1/2\,\frac{\sin 2\mu_0}{\mu_0})}{k^2\sin^2\mu_0}} \longrightarrow \frac{C\ell}{(k^2\ell)^2}$ in thin-lens approximation ($\mu_0 \ll 1$)

ⓒ Chromatic aberrations: argue that a shift in focal length is equivalent to a tilt of the acceptance ellipse at the target center:



$$x^2 = \beta\varepsilon = \varepsilon(\beta_0 + \Delta\beta) = \varepsilon(\beta_0 + \frac{1}{\beta_0}s^2)$$

where

$$\text{So } \Delta(x^2) = \varepsilon/\beta_0\, s^2 = \varepsilon/\beta_0 (\Delta f)^2$$

$$\Delta x = \sqrt{\varepsilon/\beta_0}\,\Delta f$$

$f = \frac{1}{k\tan k\ell}$ $\qquad df/dk = d/dk\,(k^{-1}(\tan k\ell)^{-1})$

$$= -k^{-2}/\tan k\ell - 1/k\left(\frac{1}{\tan k\ell}\right)^2 \ell\sec^2 k\ell$$

$$= -1/k^2\left(\frac{1}{\tan k\ell} - \frac{k\ell\,\cos^2 k\ell}{\cos^2 k\ell\,\sin^2 k\ell}\right) = -1/k^2\left(\frac{1}{\tan\mu_0} - \frac{\mu_0}{\sin^2\mu_0}\right)$$

$dk = k/2\,dp/p$

$$df/dp = df/dk\;dk/dp = k/2p\;df/dk$$

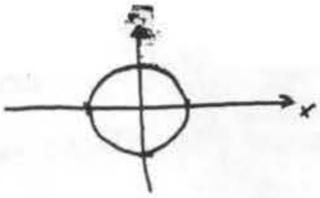
$$df/dp = -k/2p \cdot 1/k^2\left(\frac{1}{\tan\mu_0} - \frac{\mu_0}{\sin^2\mu_0}\right)$$

$$= -\frac{1}{k\tan\mu_0}\left(1 - \frac{\mu_0}{\sin^2\mu_0}\frac{\sin\mu_0}{\cos\mu_0}\right)\frac{1}{2p}$$

$$\boxed{\frac{df}{f} = -\frac{1}{2}\,dp/p\left(1 - \frac{\mu_0}{\sin\mu_0\cos\mu_0}\right)}$$

$$\Delta x = \sqrt{\varepsilon/\beta_0}\,\Delta f$$

[illegible] $-\sqrt{\varepsilon\beta} < \underset{y}{x} < \sqrt{\varepsilon\beta}$ $\quad x^2 + y^2$



$$0 < r^2 < \varepsilon\beta$$

S $\quad \int_0^{\ell} \bar{f}\, ds = \int_0^{\ell} ds \int_0^{\sqrt{\varepsilon\beta(s)}} r\,dr\,(B(r) - G) = 0$

— Neglect the variation of beam size along the lens. Then G is defined as above, and $\overline{\Delta r'} = 0$
Second moment is

$$\overline{(\Delta r')^2} = \left(\frac{0.3}{p}\right)^2 \ell^2\, \overline{\delta^2}$$

$$\frac{\int_0^{r_0} r\,dr \left(\frac{B(r) - Gr}{B(r_0)}\right)^2}{\int_0^{r_0} r\,dr} = \Delta^2$$ This is the rms relative deviation from linearity.

$$\int_0^{r_0} r\,dr = r_0^2/2$$

$$\Delta^2 = 2/r_0^2\, \frac{1}{B_0^2} \int_0^{r_0} r\,dr\,(B(r) - Gr)^2 = \overline{\delta^2}/B_0^2$$

$$\overline{(\Delta r')^2} = \left(\frac{0.3}{p}\right)^2 \ell^2\, B_0^2\, \overline{\Delta^2} \qquad \left(\frac{0.3}{p}\right)^2 = k^4/G^2$$

$$\overline{(\Delta r')^2} = \frac{k^4}{G^2} \ell^2 B_0^2\, \overline{\Delta^2} \qquad B_0^2 = G^2 r_0^2$$

$$= \frac{k^4}{\cancel{G^2}} \ell^2 \cancel{G^2} r_0^2\, \overline{\Delta^2} = k^4 \ell^2 r_0^2\, \overline{\Delta^2}$$

[illegible]

$$(\Delta r')_{RMS} = k^2 \ell r_0 \Delta_{RMS} \qquad (d(\Delta r')_{RMS} = k^2 r_0 \Delta_{RMS}\, ds)$$

$$d(\Delta r_b)_{RMS} = d(\Delta r')_{RMS} \left(-\frac{\cos k(s-\ell)}{k \sin k\ell}\right)$$

$$= k^2 r_0 \Delta_{RMS} \left(-\frac{\cos k(s-\ell)}{k \sin k\ell}\right) ds$$

$$(\Delta r_b)_{RMS} = k^2 r_0 \Delta_{RMS} \int_0^{\ell} \frac{\cos k(s-\ell)\,ds}{k \sin k\ell} \qquad \int_0^{\ell} \cos k(s-\ell)\,ds = \int_{-\ell}^{0} \cos kz\,dz = \frac{1}{k} \sin k\ell$$

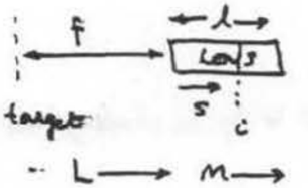
$$(\Delta r_+)_{RMS} = k^2 r_0 \Delta_{RMS}\, \frac{\sin k\ell}{\cancel{k^2}} = \boxed{r_0 \Delta_{RMS} = (\Delta r_b)_{RMS}}$$

At optimum phase for our system $\Delta_{RMS} = 2.1\% \Rightarrow (\Delta r_b)_{RMS} = .021$ cm

APPENDIX A-L

Lens Aberrations

Angular [redacted] referred to the target location, in terms of position spread



$$\vec{x}_i = M(s)\,L\,\vec{x}_t$$

$$L = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad M(s) = \begin{pmatrix} \cos ks & \sin ks/k \\ -k\sin ks & \cos ks \end{pmatrix}$$

$$\vec{x}_t = L^{-1} M^{-1}(s)\,\vec{x}_i \qquad L^{-1}M^{-1} = \begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos ks & -\sin ks/k \\ k\sin ks & \cos ks \end{pmatrix}$$

$$x_t = x_i(\cos ks - kf\sin ks) + x_i'(-1/k\,\sin ks - f\cos ks)$$

$$x_t' = k\sin ks\,x_i + x_i'\cos ks$$

$$dx_t/dx_i' = -1/k\,\sin ks - f\cos ks \qquad f = \frac{\cos kl}{f\sin kl}$$

$$= -\frac{\cos k(s-l)}{k\sin kl}$$

This correlates an angular variation dx_i' at s in the lens with a variation in x at the target dx_t.

Ⓐ Lens non-linearity

$$dr'/ds + k^2 r = 0 \quad \text{Equation of motion in the lens}$$

$$k^2 = \frac{0.3}{p}\left(\frac{B(r)}{r}\right) \qquad B \to T \quad r \to m \quad p \to \text{GeV}/c \quad k^2 \to m^{-2}$$

$$k^2 = k_0^2 + \delta k^2 \qquad B(r) = Gr + \delta$$

$$k^2 = \frac{0.3}{p}\,\frac{Gr+\delta}{r} = \frac{0.3G}{p} + \underbrace{\frac{0.3}{p}\,\frac{\delta}{r}}_{\delta k^2}$$

Deviation from linearity gives rise to

$$\frac{\Delta r'}{\Delta s} = -\delta k^2 r = -\frac{0.3}{p}\delta\,, \quad \delta = B(r) - Gr$$

$$\Delta r' = -\frac{0.3\,\delta l}{p}$$

Average over the beam: $\overline{\delta} = \int r\,dr\,(B(r) - Gr) = 0$ for a uniform beam.

This defines G

[illegible] $\frac{\sigma_r}{\theta}\frac{\sqrt{\pi}}{2}\exp\left(\frac{\omega^2\sigma_r^2}{4\theta^2}-\frac{\omega r}{\theta}\right)\left[\operatorname{erf}\left\{\frac{1}{\sigma_r}(\theta z-r)+\frac{\sigma_r\omega}{2\theta}\right\}-\operatorname{erf}\left\{\frac{\sigma_r\omega}{2\theta}-r/\sigma_r\right\}\right]$

$$\bar{\rho}(r,\theta,z)=e^{-z/\lambda}\left(X/\lambda\right)\Delta\rho\frac{e^{-\theta^2/\theta_0^2}}{\pi\sigma_r^2}I$$

Let $r \to r+L\theta/2$

$\omega r/\theta \to \omega/\theta\,(r+L\theta/2)=\omega r/\theta+\omega L/2$

$\theta z-r=\theta L-(r+L\theta/2)=\theta L/2-r$

$r \to$ [illegible] $L\theta/2$

$$\bar{\rho}(r,\theta,z)=e^{-z/\lambda}\left(X/\lambda\right)\Delta\rho\, e^{-\theta^2/\theta_0^2}\frac{1}{2\sqrt{\pi}\,\theta\sigma_r}\exp(a_1)\left(\operatorname{erf}(a_2)-\operatorname{erf}(a_3)\right)$$

$$a_1=\frac{\omega^2\sigma_r^2}{4\theta^2}-\omega r/\theta-\omega L/2$$

$$a_2=\sigma_r\omega/2\theta+\frac{1}{\sigma_r}\left(\theta L/2-r\right)$$

$$a_3=\sigma_r\omega/2\theta-\frac{1}{\sigma_r}\left(r+\theta L/2\right)$$

Limit $\theta \to 0$ is

$$\rho(r,\theta,z)=e^{-z/\lambda}\left(X/\lambda\right)\frac{\Delta\rho}{\pi\sigma_r^2}e^{-r^2/\sigma_r^2}\left(1-e^{-z\omega}\right)/\omega$$

[redacted]

$$\bar{P}(r,\theta,z) \quad \int_0^z n_0 e^{-z'/\bar\lambda} \Delta\rho\, e^{z'/\lambda} \int r\,dr\,d\phi'\, \theta' d\theta' d\phi''\, P(r) \frac{d^2\sigma}{d\Omega\, d\rho}$$
$$\delta(r' - r + z\theta)\, \delta(\theta - \theta')$$

$$P(r) = \frac{1}{\pi\sigma_r^2} e^{-r^2/\sigma_r^2}$$

$$n_0 \frac{d^2\sigma}{d\Omega\, d\rho} = (X/\lambda)\, e^{-\theta^2/\theta_0^2}$$

$$\bar{P} = e^{-z/\lambda}(X/\lambda)\, \Delta\rho\, \frac{e^{-\theta^2/\theta_0^2}}{\pi\sigma_r^2} \int_0^z e^{-z'/\bar\lambda} e^{z'/\lambda} e^{-(r-\theta z')^2/\sigma_r^2}\, dz'$$

$$\omega = 1/\bar\lambda - 1/\lambda$$

$$\text{arg} = -\omega z - \frac{1}{\sigma_r^2}(r^2 - 2r\theta z + z^2\theta^2)$$
$$= -z^2\theta^2/\sigma_r^2 - z(\omega - 2r\theta/\sigma_r^2) - r^2/\sigma_r^2$$
$$= -\theta^2/\sigma_r^2 \left(z^2 + \sigma_r^2/\theta^2 \left(\omega - \frac{2r\theta}{\sigma_r^2}\right)\right) - r^2/\sigma_r^2$$

Complete the square $-\theta^2/\sigma_r^2 \left(z^2 + \sigma_r^2/\theta^2\, 2(\omega/2 - r\theta/\sigma_r^2) + (\sigma_r^2/\theta^2)^2 \left(\frac{\omega}{2} - r\theta/\sigma_r^2\right)^2\right)$

$$- r^2/\sigma_r^2 + \theta^2/\sigma_r^2 (\sigma_r^2/\theta^2)^2 (\omega/2 - r\theta/\sigma_r^2)^2$$

$$= -\theta^2/\sigma_r^2 \left(z + \sigma_r^2/\theta^2(\omega/2 - r\theta/\sigma_r^2)\right)^2$$
$$- r^2/\sigma_r^2 + \sigma_r^2/\theta^2 \left(\frac{\omega^2}{4} + r^2\theta^2/\sigma_r^4 - \frac{\omega r\theta}{\sigma_r^2}\right)$$

Lost terms are $\cancel{-r^2/\sigma_r^2} + \frac{\omega^2\sigma_r^2}{4\theta^2} + \cancel{r^2/\sigma_r^2} - \frac{\omega r}{\theta} = \frac{\omega^2\sigma_r^2}{4\theta^2} - \frac{\omega r}{\theta}$

so the integral is $\exp\left(\frac{\omega^2\sigma_r^2}{4\theta^2} - \frac{\omega r}{\theta}\right) \int_0^z \exp\left(-\frac{\theta^2}{\sigma_r^2}\left(z + \sigma_r^2/\theta^2\left(\frac{\omega}{2} - r\theta/\sigma_r^2\right)\right)^2\right.$

Let $u = \frac{\theta}{\sigma_r}(z + \sigma_r^2/\theta^2(\omega/2 - r\theta/\sigma_r^2)) = \frac{\theta z}{\sigma_r} + \sigma_r/\theta(\omega/2 - r\theta/\sigma_r^2)$

$$= \theta z/\sigma_r + \sigma_r\omega/2\theta - r/\sigma_r$$

$du = \frac{\theta\, dz}{\sigma_r} \quad dz = \sigma_r/\theta\, dZ$

$$= 1/\sigma_r(\theta z - r) + \sigma_r\omega/2\theta$$

$$\int_{x_1}^{x_2} e^{-u^2} du = \frac{\sqrt{\pi}}{2}(\mathrm{erf}(x_2) - \mathrm{erf}(x_1))$$

for $z = 0$ $u = \sigma_r/\theta(\omega/2 - r\theta/\sigma_r^2) = \frac{\sigma_r\omega}{2\theta} - r/\sigma_r$

$$F(x,y,\theta_x,\theta_y,z) = \underbrace{e^{-z/\lambda}\left(x/\lambda\right)\Delta\rho\,\frac{e^{-\theta_x^2+\theta_y^2/\theta_0^2}}{2\pi\sigma_x\sigma_y}\,\frac{\sigma_n}{\sqrt{\theta_x^2+\theta_y^2}}\,\frac{\sqrt{\pi}}{2}}_{F1} \qquad 2\sigma_x\sigma_y = 2\sigma_x^2 = \sigma_n^2$$

$$\times \exp\{a_1\}\{\operatorname{erf}(a_2)-\operatorname{erf}(a_3)\}$$

$$a_1 = \frac{1}{4}\frac{\sigma_n^2\omega^2}{\theta_x^2+\theta_y^2} - \frac{\omega(\theta_x x+\theta_y y)}{\theta_x^2+\theta_y^2} - \frac{\omega L}{2} - \frac{(\theta_y x-\theta_x y)^2}{\sigma_n^2(\theta_x^2+\theta_y^2)}$$

$$a_2 = \frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}\frac{L}{2} - \frac{\theta_x x+\theta_y y}{\sigma_n\sqrt{\theta_x^2+\theta_y^2}} + \frac{\omega\sigma_n}{2\sqrt{\theta_x^2+\theta_y^2}}$$

$$a_3 = \frac{\omega\sigma_n}{2\sqrt{\theta_x^2+\theta_y^2}} - \frac{L}{2}\frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n} - \frac{(\theta_x x+\theta_y y)}{\sigma_n\sqrt{\theta_x^2+\theta_y^2}}$$

$$\omega = 1/\bar{\lambda} - 1/\lambda$$

$$F1 = \exp(-L/\lambda)\,(x/\lambda)\,\Delta\rho\, e^{-(\theta_x^2+\theta_y^2)/\theta_0^2}\,\frac{\sigma_n}{\sqrt{\pi\sigma_n^2}}\,\frac{\sqrt{\pi}}{2}\,\frac{1}{\sqrt{\theta_x^2+\theta_y^2}}$$

$$= \exp(-L/\lambda)\,(x/\lambda)\,\Delta\rho\, e^{-(\theta_x^2+\theta_y^2)/\theta_0^2}\left(\frac{1}{2\sqrt{\pi}\,\sigma_n\sqrt{\theta_x^2+\theta_y^2}}\right)$$

Limit $\theta_x^2+\theta_y^2 \to 0$ $(\Rightarrow \theta_x, \theta_y \to 0)$

$$\bar{F}(x,y,0,0,z) = e^{-z/\lambda}\left(x/\lambda\right)\frac{\Delta\rho}{2\pi\sigma_x\sigma_y}\,e^{-x^2/\sigma_n^2}e^{-y^2/\sigma_n^2}\underbrace{\int_0^z e^{-z'\omega}\,dz'}_{1/\omega\,(e^{-z\omega}-1)}$$

$$= e^{-z/\lambda}\left(x/\lambda\right)\frac{\Delta\rho}{\pi\sigma_n^2}\,e^{-x^2/\sigma_n^2}e^{-y^2/\sigma_n^2}\,(e^{-z\omega}-1)\left(-\frac{1}{\omega}\right)$$

Limit $\omega \to 0$ in $\frac{1-z\omega-1}{\omega} = -z$

$$\theta_x \frac{d[illegible]}{dx} = [illegible] - \frac{2(\theta_y x - \theta_x y)\theta_y}{\sigma_n^2(\theta_x^2+\theta_y^2)}\Big\}\theta_x$$

$$\theta_y \frac{dI[illegible]}{dy} = I[illegible]\frac{\theta_y}{\theta_x^2+\theta_y^2} + \frac{2(\theta_y x - \theta_x y)\theta_x}{\sigma_n^2(\theta_x^2+\theta_y^2}\Big]\theta_y$$

$$\theta_x\frac{dI_0}{dx}+\theta_y\frac{dI_0}{dy} = I_0\Big\{-\omega + \frac{2}{\sigma_n^2(\theta_x^2+\theta_y^2)}\Big[\cancel{-\theta_y^2\theta_x x}+\cancel{\theta_x^2\theta_y y}+[illegible]$$
$$\cancel{+\theta_y^2\theta_x x} - \cancel{\theta_x^2\theta_y y}\Big\} = -\omega I_0$$

Square the exponential argument:

$$-\left(\frac{\omega^2\sigma_n^2}{4(\theta_x^2+\theta_y^2)} + \frac{\theta_x^2x^2+\theta_y^2y^2+2\theta_x\theta_y xy}{(\theta_x^2+\theta_y^2)\sigma_n^2} - \frac{\cancel{2}\omega\sigma_n(\theta_x x+\theta_y y)}{\cancel{2}(\theta_x^2+\theta_y^2)\sigma_n}\right)$$

Net result is $\exp -\frac{(x^2+y^2)}{\sigma_n^2}$

So $$\frac{d\bar{\rho}}{dz}+\theta_x\frac{d\bar{\rho}}{dx}+\theta_y\frac{d\bar{\rho}}{dy} = e^{-z/\lambda}\frac{X}{\lambda}\Delta\rho\frac{1}{2\pi\sigma_x\sigma_y}e^{-\frac{\theta_x^2+\theta_y^2}{\theta_0^2}}e^{-x^2+y^2/\sigma_n^2}$$

This is the original equation.

For a target of length L, $z = L$. Projection back to mid-target $\Rightarrow$ $x \to x + L\theta_x/2$
$y \to y + L\theta_x/2$

So $$\theta_x x + \theta_y y \to \theta_x(x+L\theta_x/2) + \theta_y(y+L\theta_y/2)$$
$$= \theta_x x + \theta_y y + L/2(\theta_x^2+\theta_y^2)$$

$$\theta_y x - \theta_x y \longrightarrow \theta_y(x+L\theta_x/2) - \theta_x(y+L\theta_y/2)$$
$$= \theta_y x - \theta_x y + L\theta_x\theta_y/2 - L\theta_x\theta_y/2 = \theta_y x - \theta_x y$$

$$\frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}z - \frac{(\theta_x x+\theta_y y)}{\sigma_n\sqrt{\theta_x^2+\theta_y^2}} \longrightarrow \frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}L - \frac{(\theta_x x+\theta_y y)}{\sigma_n\sqrt{\theta_x^2+\theta_y^2}} - \frac{L}{2}\sqrt{\theta_x^2+\theta_y^2}/\sigma_n$$

$$\longrightarrow \frac{L}{2}\sqrt{\theta_x^2+\theta_y^2}/\sigma_n - \frac{\theta_x x+\theta_y y}{\sigma_n\sqrt{\theta_x^2+\theta_y^2}}$$

$$\frac{d\,\text{[illegible]}}{dz}\ \frac{d}{d}\,\text{[illegible]}\,{}^{+\theta_y^2/\theta_0^2} I + P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2}\frac{dI}{dz}$$

$$= -\frac{1}{\lambda}\bar{P} + P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2} I_0 \frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}\exp(a_1)^2/\sqrt{\pi}$$

$$\theta_x\frac{d\bar{P}}{dx} = P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2}\left[\theta_x\frac{dI_0}{dx}(f_1-f_2) - \frac{I_0}{\sqrt{\theta_x^2+\theta_y^2}}\theta_x^2/\sigma_n\,(\exp(a_1)-\exp(a_2))\right]\times 2/\sqrt{\pi}$$

$$\theta_y\frac{d\bar{P}}{dy} = P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2}\left[\theta_y\frac{dI_0}{dy}(f_1-f_2) - \frac{I_0}{\sqrt{\theta_x^2+\theta_y^2}}\theta_y^2/\sigma_n\,(\exp(a_1)-\exp(a_2))\right]\times 2/\sqrt{\pi}$$

So
$$\frac{\partial\bar{P}}{\partial z} + \theta_x\frac{\partial\bar{P}}{\partial x} + \theta_y\frac{\partial\bar{P}}{\partial y} = -\frac{1}{\lambda}\bar{P} + \cancelto{}{P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2} I_0\frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}\exp(a_1)^2/\sqrt{\pi}}$$

$$+ P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2}\left[\theta_x\frac{dI_0}{dx}(f_1-f_2) + \theta_y\,dI_0/dy\,(f_2-f_2)\right]$$

$$+ P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2}\cancelto{}{I_0\Big[-\exp(a_1)}\frac{\theta_x^2+\theta_y^2}{\sqrt{\theta_x^2+\theta_y^2}}\frac{1}{\sigma_n} + \exp(a_2)\frac{\theta_x^2+\theta_y^2}{\sqrt{\theta_x^2+\theta_y^2}}\frac{1}{\sigma_n}\Big]2/\sqrt{\pi}$$

Terms with arrow cancel

From next page, $\theta_x\,dI_0/dx + \theta_y\,dI_0/dy = -\omega I_0$

$\Rightarrow$ middle term is $-P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2}\omega I_0(f_1-f_2) = -\omega P = \left(\frac{1}{\bar\lambda}-\frac{1}{\lambda}\right)P$

Sum with $-\bar{P}/\bar{\lambda} = -P/\lambda$ want $\bar{\lambda} \Rightarrow$ ~~[illegible]~~ interchange $\lambda \Leftrightarrow \bar{\lambda}$

$$\Rightarrow \bar{P}(x,y,\theta_x,\theta_y,z) = e^{-z/\lambda}(x/\lambda)\Delta P\frac{e^{-\theta_x^2+\theta_y^2/\theta_0^2}}{2\pi\sigma_x\sigma_y} I(\theta_x,\theta_y,x,y,z)$$ with $\omega = \frac{1}{\bar\lambda}-\frac{1}{\lambda}$

$\omega = \frac{\lambda-\bar{\lambda}}{\bar{\lambda}\lambda}$

$$\Rightarrow \frac{\partial\bar{P}}{\partial z} + \theta_x\frac{\partial\bar{P}}{\partial x} + \theta_y\frac{d\bar{P}}{dy} = P_0 e^{-\theta_x^2+\theta_y^2/\theta_0^2} I_0\sqrt{\theta_x^2+\theta_y^2}/\sigma_n\exp(a_2)^2/\sqrt{\pi}$$

$$\frac{2}{\sqrt{\pi}}\frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n} I_0\exp(a_2) = \cancel{\frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}}\,\cancel{\frac{\sigma_n}{\sqrt{\theta_x^2+\theta_y^2}}}\,\frac{\sqrt{\pi}}{\cancel{2}}\cdot\cancel{2}/\sqrt{\pi}\,\exp\left(\frac{\sigma_n^2\omega^2}{4(\theta_x^2+\theta_y^2)} - \frac{\omega(\theta_x x+\theta_y y)}{\theta_x^2+\theta_y^2}\right)$$

$$\exp\frac{1}{\sigma_n^2(\theta_x^2+\theta_y^2)} - (\theta_y^2x^2+\theta_x^2y^2-2\theta_x\theta_y xy)\exp-\left\{\frac{\omega\sigma_n}{2\sqrt{\theta_x^2+\theta_y^2}} - \frac{(\theta_x x+\theta_y y)}{\sqrt{\theta_x^2+\theta_y^2}\,\sigma_n}\right\}^2$$

Square

[illegible] $= 2/\sqrt{\pi}\int_0^x e^{-w^2}\,dw$

[illegible]$(x) = 2/\sqrt{\pi}\,(e^{-x^2})$

$x = x($[illegible]$)$ $d/dy\ \mathrm{erf}(x) = 2/\sqrt{\pi}\ e^{-x^2}\,dx/dy$

~~$dI/dz = \exp\{-\frac{1}{2}\frac{\sigma_x^2\omega^2}{\theta_x^2+\theta_y^2}$~~

$dI/dz = I_0\ df_1/dz$

$dI/dx = dI_0/dx\ (f_1 - f_2) + I_0(df_1/dx - df_2/dx)$

$dI/dy = dI_0/dy\ (f_1 - f_2) + I_0(df_1/dy - df_1/dy)$

$$df_1/dz = \frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}\exp\left(\frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}(z) + \frac{\omega\sigma_n}{2\sqrt{\theta_x^2+\theta_y^2}} - \frac{\theta_x}{\sqrt{\theta_x^2+\theta_y^2}}x/\sigma_n - \frac{\theta_y}{\sqrt{\theta_x^2+\theta_y^2}}y/\sigma_n\right)^2 \frac{2}{\sqrt{\pi}}$$

$$= \frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}\exp(a_1)\ 2/\sqrt{\pi}$$

$f_1 = \mathrm{erf}(a_1) \quad f_2 = \mathrm{erf}(a_2)$

$$df_1/dx = \frac{-\theta_x}{\sqrt{\theta_x^2+\theta_y^2}}\frac{1}{\sigma_n}\exp(a_1)\frac{2}{\sqrt{\pi}} \quad df_1/dy = \frac{-\theta_y}{\sqrt{\theta_x^2+\theta_y^2}}\frac{1}{\sigma_n}\exp(a_1)\ 2/\sqrt{\pi}$$

$$df_2/dx = \frac{-\theta_x}{\sqrt{\theta_x^2+\theta_y^2}}\frac{1}{\sigma_n}\exp(a_2)\frac{2}{\sqrt{\pi}} \quad df_2/dy = \frac{-\theta_y}{\sqrt{\theta_x^2+\theta_y^2}}\frac{1}{\sigma_n}\exp(a_2)\ 2/\sqrt{\pi}$$

so $dI/dz = I_0\frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}\exp(a_1)\ 2/\sqrt{\pi}$

$$dI/dx = dI_0/dx\ (f_2 - f_2) - \frac{I_0}{\sqrt{\theta_x^2+\theta_y^2}}\frac{\theta_x}{\sigma_n}(\exp a_1 - \exp a_2)^2/\sqrt{\pi}$$

$$dI/dy = dI_0/dy\ (f_1 - f_2) - \frac{I_0}{\sqrt{\theta_x^2+\theta_y^2}}\frac{\theta_y}{\sigma_n}(\exp a_1 - \exp a_2)^2/\sqrt{\pi}$$

$$\bar{P} = \bar{P}_0(z)\,e^{-\theta_x^2+\theta_y^2/\theta_0^2}I_0(\theta_x, \theta_y, x, y, z)$$

$$\bar{P}_0(z) = e^{-z/\lambda}(x/\lambda)\,\Delta\rho\ /2\pi\sigma_x\sigma_y$$

So the integral

$$I = \exp\left\{\frac{1}{4}\frac{\sigma_n^2\omega^2}{\theta_x^2+\theta_y^2} - \frac{\omega(\theta_x x+\theta_y y)}{\theta_x^2+\theta_y^2}\right\} \exp\left\{\frac{1}{\sigma_n^2}\frac{1}{\theta_x^2+\theta_y^2}\,(\theta_y x-\theta_x y)^2\right\}$$

$$\int_0^{z'} \exp\left[-\frac{(\theta_x^2+\theta_y^2)}{\sigma_n^2}\left(z+\frac{\sigma_n^2\omega}{2(\theta_x^2+\theta_y^2)} - \frac{\theta_x x}{\theta_x^2+\theta_y^2} - \frac{\theta_y y}{\theta_x^2+\theta_y^2}\right)^2\right] dz$$

Let $u = \frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n}\left(z+\frac{\sigma_n^2\omega}{2(\theta_x^2+\theta_y^2)} - \frac{\theta_x x}{\theta_x^2+\theta_y^2} - \frac{\theta_y y}{\theta_x^2+\theta_y^2}\right)$

$du = \frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n} dz \qquad dz = \frac{\sigma_n}{\sqrt{\theta_x^2+\theta_y^2}} du$

$\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2} du = \text{erf}(u) \qquad \int_{x_1}^{x_2} e^{-u^2} du = \frac{\sqrt{\pi}}{2}(\text{erf}(x_2)-\text{erf}(x_1))$

for $z=0 \quad u = \frac{\omega\sigma_n}{2\sqrt{\theta_x^2+\theta_y^2}} - \frac{\theta_x}{\sqrt{\theta_x^2+\theta_y^2}}\frac{x}{\sigma_n} - \frac{\theta_y}{\sqrt{\theta_x^2+\theta_y^2}}\,y/\sigma_n$

$$I = \exp\left[\frac{1}{4}\frac{\sigma_n^2\omega^2}{\theta_x^2+\theta_y^2} - \frac{\omega(\theta_x x+\theta_y y)}{\theta_x^2+\theta_y^2}\right] \exp\left\{\frac{1}{\sigma_n^2}\frac{1}{\theta_x^2+\theta_y^2}\,(\theta_y x-\theta_x y)^2\right\} \frac{\sigma_n}{\sqrt{\theta_x^2+\theta_y^2}}\frac{\sqrt{\pi}}{2}$$

$$\times\left\{\text{erf}\left[\frac{\sqrt{\theta_x^2+\theta_y^2}}{\sigma_n} z + \frac{\omega\sigma_n}{2\sqrt{\theta_x^2+\theta_y^2}} - \frac{\theta_x}{\sqrt{\theta_x^2+\theta_y^2}}\frac{x}{\sigma_n} - \frac{\theta_y}{\sqrt{\theta_x^2+\theta_y^2}}\,y/\sigma_n\right\}\right.$$

$$\left. - \text{erf}\left\{\frac{\omega\sigma_n}{2\sqrt{\theta_x^2+\theta_y^2}} - \frac{\theta_x}{\sqrt{\theta_x^2+\theta_y^2}}\frac{x}{\sigma_n} - \frac{\theta_y}{\sqrt{\theta_x^2+\theta_y^2}}\,y/\sigma_n\right]\right\} = I_0(x, y, \theta_x, \theta_y) \times (f_1(x, y, \theta_x, \theta_y, z) - f_2(x, y, \theta_x, \theta_y)$$

$$\bar{P}(x, y, \theta_x, \theta_y, z) = e^{-z/\lambda}\,(x/\lambda)\,\Delta P\,\frac{e^{-\theta_x^2+\theta_y^2/\theta_0}}{2\pi\sigma_x\sigma_y}\, I(\theta_x, \theta_y, x, y, z)$$

[illegible]

$$-\frac{\theta_x^2+\theta_y^2}{\sigma_n^2}\left\{ z^2 + 2z\left(\frac{\sigma_n^2}{\theta_x^2+\theta_y^2}\right)\left(\frac{\omega}{2} - \frac{\theta_x x+\theta_y y}{\sigma_n^2}\right) + \left(\frac{\sigma_n^2}{\theta_x^2+\theta_y^2}\right)^2 \times \left(\frac{\omega}{2} - \frac{\theta_x x+\theta_y y}{\sigma_n^2}\right)^2 \right\}$$

$$+ \frac{\theta_x^2+\theta_y^2}{\sigma_n^2}\left(\frac{\sigma_n^2}{\theta_x^2+\theta_y^2}\right)^2\left(\frac{\omega}{2} - \frac{(\theta_x x+\theta_y y)}{\sigma_n^2}\right)^2 - \frac{(x^2+y^2)}{\sigma_n^2}$$

$$= -\frac{(\theta_x^2+\theta_y^2)}{\sigma_n^2}\left\{ z + \frac{\sigma_n^2}{(\theta_x^2+\theta_y^2)}\left(\frac{\omega}{2} - \frac{\theta_x x+\theta_y y}{\sigma_n^2}\right)\right\}^2$$

$$+ \frac{\sigma_n^2}{\theta_x^2+\theta_y^2}\left(\frac{\omega^2}{4} - \frac{\omega(\theta_x x+\theta_y y)}{\sigma_n^2} + \frac{(\theta_x x+\theta_y y)^2}{\sigma_n^4}\right) - \frac{(x^2+y^2)}{\sigma_n^2}$$

Last part is

$$\frac{\sigma_n^2\omega^2}{4(\theta_x^2+\theta_y^2)} - \frac{\omega(\theta_x x+\theta_y y)}{\theta_x^2+\theta_y^2} + \frac{(\theta_x^2x^2+\theta_y^2y^2+2\theta_x\theta_y xy)}{\sigma_n^2(\theta_x^2+\theta_y^2)} - \frac{(x^2+y^2)}{\sigma_n^2}$$

$$= \frac{1}{4}\frac{\sigma_n^2\omega^2}{\theta_x^2+\theta_y^2} - \frac{\omega(\theta_x x+\theta_y y)}{\theta_x^2+\theta_y^2} + \underbrace{\frac{1}{\sigma_n^2}\left\{ x^2\left(-1+\frac{\theta_x^2}{\theta_x^2+\theta_y^2}\right) + y^2\left(-1+\frac{\theta_y^2}{\theta_x^2+\theta_y^2}\right) + \frac{2\theta_x\theta_y xy}{\theta_x^2+\theta_y^2}\right\}}_{\frac{1}{\sigma_n^2}\frac{1}{\theta_x^2+\theta_y^2}\left\{ x^2(-\theta_y^2) + y^2(-\theta_x^2) + 2\theta_x\theta_y xy\right\}}$$

~~$(\theta_x x+\theta_y y)^2 = \theta_x^2x^2+\theta_y^2y^2+2\theta_x\theta_y xy$~~

$(\theta_y x-\theta_x y)^2 = \theta_y^2x^2+\theta_x^2y^2-2\theta_x\theta_y xy$

$$= \frac{1}{4}\frac{\sigma_n^2\omega^2}{\theta_x^2+\theta_y^2} - \frac{\omega(\theta_x x+\theta_y y)}{\theta_x^2+\theta_y^2} + \frac{1}{\sigma_n^2}\frac{1}{\theta_x^2+\theta_y^2}\left\{ -(\theta_y x-\theta_x y)^2 \right\}$$

[illegible]
DISTRIBUTION FUNCTION

Solution [illegible] $\bar{P}(x, y, \theta_x, \theta_y, z)$

$$e^{-z/\bar{\lambda}} \int_0^z n_0 e^{-z'/\lambda} \Delta p \, e^{z'/\bar{\lambda}} \int dx'dy' d\Omega' \rho(x, y, z) \frac{d^2\sigma}{d\Omega dp} dz' \delta(\theta_x' - \theta_x) \cdot \delta(x' - x + z\theta_x') \delta(\theta_y' - \theta_y) \delta(y' - y + z\theta_y)$$

The equation is $$\frac{d\bar{P}}{dz} + \theta_x \frac{d\bar{P}}{dx} + \theta_y \frac{d\bar{P}}{dy} = -\bar{P}/\bar{\lambda} + n_0 \rho(x, y, z) \frac{d^2\sigma}{d\Omega dp}$$

$$\rho(x, y, z) = \frac{1}{\sqrt{2\pi}\sigma_x} \frac{1}{\sqrt{2\pi}\sigma_y} e^{-x^2/2\sigma_x^2} e^{-y^2/2\sigma_y^2} e^{-z/\lambda}$$

$$n_0 \frac{d^2\sigma}{d\Omega dp} = \left(X/\lambda\right) e^{-\theta_x^2/\theta_0^2} e^{-\theta_y^2/\theta_0^2}$$

Integrate over the delta-functions:

$$\bar{P}(x, y, \theta_x, \theta_y, z) = \text{[illegible]}$$

$$= e^{-z/\bar{\lambda}} \left(\frac{X}{\lambda}\right) \Delta p \frac{e^{-\frac{\theta_x^2 + \theta_y^2}{2\theta_0^2}}}{2\pi\sigma_x\sigma_y} \int_0^z e^{-z'/\lambda} e^{z'/\bar{\lambda}} \, \text{[illegible]} \, e^{-(x - z\theta_x)^2/\sigma_n^2} e^{-(y - z\theta_y)^2/\sigma_n^2}$$

exponential argument:

$$-z\left(\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\right) - \frac{1}{\sigma_n^2}\left\{x^2 - 2z\theta_x x + z^2\theta_x^2 + y^2 - 2z\theta_y y + z^2\theta_y^2\right\}$$

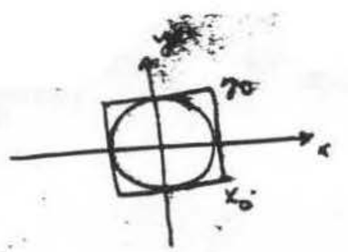
Let $1/\lambda - \frac{1}{\bar{\lambda}} = n_0\delta\sigma =$ [illegible] ω

$$\text{then arg} = \text{[illegible]} - z^2\left\{\frac{\theta_x^2 + \theta_y^2}{\sigma_n^2}\right\} - z\left\{\omega - \frac{2\theta_x x + 2\theta_y y}{\sigma_n^2}\right\} - \frac{(x^2 + y^2)}{\sigma_n^2}$$

$$= -\frac{z^2\theta_x^2}{\sigma_n^2} - z\left\{\omega/2 + \frac{2\theta_x x}{\sigma_n^2}\right\} - x^2/\sigma_n^2$$

$$-\frac{z^2\theta_y^2}{\sigma_n^2} - z\left\{\omega/2 + \frac{2\theta_y y}{\sigma_n^2}\right\} - y^2/\sigma_n^2$$

[redacted]



Assume $N(\varepsilon) \propto$ area (A) $\quad A = 4x_0 y_0$ for a square

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = \pi r_0^2$ for a circle

$\Rightarrow (x_0)_{eff}$ defined by $4(x_0^{eff})^2 = \pi r_0^2$

$$x_0^{eff} = \sqrt{\pi/4}\, r_0$$

$$(x_0^{eff})^2 = \pi/4\, r_0^2$$

$$B_\ell = \frac{\gamma_t}{k^2 \sin^2 \mu} = (x_0^{eff})^2/\varepsilon$$

$$\Rightarrow \beta_t = \frac{1}{\gamma_t} = \frac{\varepsilon}{(x_0^{eff})^2 k^2 \sin^2 \mu} = \left(\frac{4}{\pi}\right) \frac{\varepsilon}{r_0^2 k^2 \sin^2}$$

$$\beta_\ell = \pi/4\, r_0^2/\varepsilon$$