p Note #328

Attenuation of Propagating TE Modes
in Rectangular Beam Chambers

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I. Review of Formalism

The understanding of how to attenuate unwanted TE electromagnetic wave modes in rectangular beam chambers is important for improving the response of the pickup stochastic cooling electrodes presently being tested at Argonne National Laboratory. 1

We start the discussion by reviewing the field solutions for a rectangular wave guide filled with a medium having dielectric constant $\varepsilon$ and magnetic permeability $\mu$. 2
The electromagnetic fields

\[
\begin{align*}
\vec{E}(x, y, z, t) & = \{ \vec{E}(x, y) e^{i k_z z - i \omega t} \\
\vec{B}(x, y, z, t) & = \{ \vec{B}(x, y) e^{i k_z z - i \omega t}
\end{align*}
\]  

\tag{1}

\]

are solutions to the wave equation

\[
\left[ \nabla_t^2 + \left( \mu \varepsilon \frac{\omega^2}{c^2} - k_z^2 \right) \right] (\vec{E} / \vec{B}) = 0, \quad (2)
\]

where the transverse Laplacian

\[
\nabla_t^2 = \nabla^2 - \frac{1}{2 z^2}.
\]

\tag{3}
However, to solve Eq. (2) it is convenient to first solve for the two-dimensional scalar eigenfunction \( \psi_{mn} \) from

\[
\left( \nabla_t^2 + \gamma_{mn}^2 \right) \psi_{mn} = 0, \tag{4}
\]

where \( \psi_{mn} \) is \( B_z (E_z) \) for TE (TM) modes and the guide wave number \( k_g \) is related to the eigenvalue \( \gamma_{mn} \) via

\[
k_g^2 = \mu \varepsilon \frac{\omega^2}{c^2} - \gamma_{mn}^2 \]

\[
= \mu_r \varepsilon_r k_0^2 - \gamma_{mn}^2 . \tag{5}
\]

The subscript \( r \) denotes values relative to vacuum. Thus, we can define the cutoff wave number
\[ k_{J_c} = \gamma_{mn} \quad (5a) \]

and the cutoff frequency via

\[ \gamma_{mn}^2 = \frac{\mu \varepsilon}{c^2} \omega_s^2 \quad (5b) \]

so that

\[ k_g = \frac{1}{c} \sqrt{-\varepsilon \sqrt{\omega^2 - \omega_s^2}} \quad (5c) \]

We must solve eq. (4) for \( \psi_{mn} \)

subject to the boundary conditions

\[ \psi (\text{at wall}) = 0 \quad (TM \text{modes}) \quad (6a) \]

\[ \frac{2\psi}{2n} (\text{at wall}) = 0 \quad (TE \text{modes}) \quad (6b) \]

where \( \frac{2}{2n} \) is the derivative normal to the wall. Finally, once we have \( \psi_{mn} \), transverse \( \vec{E}_t \) and \( \vec{B}_t \)
are readily obtained. For the TE modes relevant to our discussion,

$$\mathbf{B}_t = \frac{i k_g}{\gamma^2} \left( \mathbf{\nabla} \times \mathbf{E}_t \right) e^{i k_g z - i \omega t} \quad (7a)$$

$$\mathbf{E}_t = -\frac{\omega}{c k_g} \hat{e}_z \times \mathbf{B}_t \quad . \quad (7b)$$

The solutions to the field equations for the $\text{TE}_{mn}$ modes are derived from the eigenfunctions and eigenvalues

$$\psi_{mn}(x,y) = B_0 \cos \left( \frac{m \pi x}{a} \right) \cos \left( \frac{n \pi y}{b} \right) \quad (8a)$$

$$\gamma_{mn}^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \quad . \quad (8b)$$

Specializing to the Argonne system, the beam pipe has dimensions
a = 22 cm and b = 3 cm. Thus, the cutoff frequencies $f_{c,mn} = \frac{\omega_{c,mn}}{2\pi}$ are given by the following table:

**TABLE of $f_{c,mn}$ (in GHz)**

<table>
<thead>
<tr>
<th>$n \rightarrow$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m \rightarrow$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.00</td>
<td>5.00</td>
<td>10.00</td>
<td>15.00</td>
</tr>
<tr>
<td>1</td>
<td>0.68</td>
<td>5.05</td>
<td>10.02</td>
<td>15.02</td>
</tr>
<tr>
<td>2</td>
<td>1.36</td>
<td>5.18</td>
<td>10.09</td>
<td>15.06</td>
</tr>
<tr>
<td>3</td>
<td>2.05</td>
<td>5.40</td>
<td>10.21</td>
<td>15.14</td>
</tr>
<tr>
<td>4</td>
<td>2.73</td>
<td>5.70</td>
<td>10.37</td>
<td>15.25</td>
</tr>
<tr>
<td>5</td>
<td>3.41</td>
<td>6.05</td>
<td>10.57</td>
<td>15.38</td>
</tr>
<tr>
<td>6</td>
<td>4.09</td>
<td>6.46</td>
<td>10.80</td>
<td>15.55</td>
</tr>
<tr>
<td>7</td>
<td>4.77</td>
<td>6.91</td>
<td>11.08</td>
<td>15.74</td>
</tr>
<tr>
<td>8</td>
<td>5.45</td>
<td>7.40</td>
<td>11.39</td>
<td>15.96</td>
</tr>
</tbody>
</table>

$a = 22 \text{ cm}, b = 3 \text{ cm}$
$f_{c} = f_{11} = 5.05 \text{ GHz}$

It is most important for the antenna package system.

The TM modes are derived from

\[ E_{x} = \frac{B_{0}}{B_{0} \cos \left( \frac{m \pi}{a} \right)} e^{i k_{y} y - i \omega t} \]

\[ B_{z} = B_{0} \cos \left( \frac{m \pi}{a} \right) e^{i k_{y} y - i \omega t} \]

\[ \psi_{m1}(x, y, z) = E_{0} \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \ln \left( \frac{a}{x} \right) \]

\[ T_{E_{0}} \text{ mode having} \]
II. Attenuation of TE Modes

We now load the rectangular wave guide with resistive strips of thickness t as shown in Fig. 2.

\[ x_2 = x_1 + t \]

\[ F = G, \ 2 \]

In Region III, \( E_y \) must be symmetric or antisymmetric about \( x = \frac{a}{2} \). This
is demanded by symmetry and the observation that as \( \mu, \epsilon \to \mu_0 \epsilon_0 \), the resistive strips disappear and we have to retrieve the hollow wave guide case.

To further study this point, let us first consider the \( TE_{m,0} \) modes for the hollow wave guide and change variables to

\[
x = \tilde{x} + \frac{a}{2},
\]

so that

\[
E_y \propto \sin \frac{m\pi x}{a} = \sin \frac{m\pi}{a} \left( \tilde{x} + \frac{a}{2} \right)
\]

\[
= \cos \frac{m\pi}{2} \sin m\pi \tilde{x} + \sin m\pi \cos \frac{m\pi}{2} \tilde{x}
\]

\[
= \cos \frac{m\pi}{2} \sin m\pi \left( x - \frac{a}{2} \right)
\]

\[
+ \sin \frac{m\pi}{2} \cos \frac{m\pi}{a} \left( x - \frac{a}{2} \right). \tag{12}
\]
Thus,

\[ m \text{ odd} \rightarrow \text{symmetric modes} \]
\[ m \text{ even} \rightarrow \text{antisymmetric modes}. \]

After loading the guide with resistive strips, \( Y_{mn} \) is perturbed from the value \( \frac{m\pi}{a} \) and must be solved for via the new boundary conditions. A pictorial representation of what happens is shown in Fig. 3.
FIG. 3

Empty Guide

$m = 1$

$E_r$

Loaded Guide

$m = 2$

$m = 3$

$\alpha/2$
Let us calculate the attenuation lengths for any $TE_{m,0}$ mode. The wave numbers are

$$k_{g,i}^2 = k_{g,iii}^2 = \frac{\mu_0 \varepsilon_0 \omega^2}{c^2} - \gamma^2 = k_{g,i}^2 - \gamma^2 \quad (13a)$$

$$k_{g,ii}^2 = \frac{\mu \varepsilon_0 \omega^2}{c^2} - \gamma^2 = k_{g,ii}^2 - \gamma^2 \quad (13b)$$

We define

$$k_g \equiv k_{g,i} = k_{g,ii} = k_{g,iii} \quad (14)$$

$\mu_0$ and $\varepsilon_0$ are real while in general $\mu$ and $\varepsilon$ are complex, causing $k_g$ to have an imaginary part which causes the attenuation. We also have
Next, we write down the field solutions in the three regions:

**REGION I**

\[ E_y = e^{ik_z} \sin \gamma_1 x \quad (17a) \]
\[ \sqrt{\frac{\mu_0}{\varepsilon_0}} H_z = \frac{i \gamma_1}{k_{01}} e^{ik_z} \cos \gamma_1 x \quad (17b) \]

**REGION II**

\[ E_y = e^{ik_z} \left( A \cos \gamma_2 x + B \sin \gamma_2 x \right) \quad (18a) \]
\[ \sqrt{\frac{\mu_0}{\varepsilon}} H_z = \frac{i \gamma_2}{k_{02}} e^{ik_z} \left( -A \sin \gamma_2 x + B \cos \gamma_2 x \right) \quad (18b) \]

\[ t = \text{thickness of strip} \ll 1 \quad (15) \]
\[ k_{0,II} = k_{0,II} \sqrt{\mu_\varepsilon} \quad (16) \]
REGION III

\[ E_y = Ce^{ikz} \left[ \cos \frac{m\pi}{2} \sin \gamma \left( x - \frac{a}{2} \right) + \sin \frac{m\pi}{2} \cos \gamma \left( x - \frac{a}{2} \right) \right] \]  

\[ \sqrt{\frac{\mu_0}{\varepsilon_0}} H_z = i C e^{ikz} \frac{y^m}{k_0} \left[ \cos \frac{m\pi}{2} \cos \gamma \left( x - \frac{a}{2} \right) - \sin \frac{m\pi}{2} \sin \gamma \left( x - \frac{a}{2} \right) \right] \]  

Note that we have deleted the \( m, o \) subscript from the \( y \)'s. Also, we have applied our symmetry arguments [cf. eq. (12)] to obtain the form of the fields in Region III.

Next, we impose the boundary conditions that \( E_y \) and \( H_z \) are
continuous at \( x = x_1 \) and \( x = x_2 = x_1 + t \):

\[
\sin y_1 x_1 = A \cos y_2 x_1 + B \sin y_2 x_1 \quad (20a)
\]

\[
\frac{\gamma x_2}{y_2} - A \sin y_2 x_1 + B \cos y_2 x_1 \quad (20b)
\]

\[
C \left[ \cos m\frac{\pi}{2} \sin y_2 x_1 \left( x_2 - \frac{a}{2} \right) + \sin m\frac{\pi}{2} \cos y_2 x_1 \left( x_2 - \frac{a}{2} \right) \right]

= A \cos y_2 x_2 + B \sin y_2 x_2 \quad (20c)
\]

\[
\frac{\gamma x_2}{y_2} - C \left[ \cos m\frac{\pi}{2} \cos y_2 \left( x_2 - \frac{a}{2} \right) - \sin m\frac{\pi}{2} \sin y_2 \left( x_2 - \frac{a}{2} \right) \right]

= -A \sin y_2 x_2 + B \cos y_2 x_2 \quad (20d)
\]

Thus, we obtain 4 equations in the
3 unknowns \( A, B, C \). Upon eliminating
\( A, B, C \), we obtain 2 equations for the
\( y \)'s depending upon whether \( m \) is even or
odd.
\[ m \text{ even} \]
\[
\tan \left[ \frac{\gamma_{21}}{2} (x_2 - x_1) \right] + \left( \frac{\gamma_{21}}{\gamma_{22}} \right) \frac{\left\{ \tan \left( \frac{\gamma_{21}}{2} x_1 \right) - \tan \left[ \frac{\gamma_{21}}{2} (x_2 - \frac{a}{2}) \right] \right\}}{1 + \left( \frac{\gamma_{21}}{\gamma_{22}} \right)^2 \tan \left( \frac{\gamma_{21}}{2} x_1 \right) \tan \left[ \frac{\gamma_{21}}{2} (x_2 - \frac{a}{2}) \right]} \]
\[ = 0 \]  
\[ (21a) \]

\[ m \text{ odd} \]
\[
\tan \left[ \frac{\gamma_{21}}{2} (x_2 - x_1) \right] + \left( \frac{\gamma_{21}}{\gamma_{22}} \right) \frac{\left\{ 1 + \tan \left( \frac{\gamma_{21}}{2} x_1 \right) \tan \left[ \frac{\gamma_{21}}{2} (x_2 - \frac{a}{2}) \right] \right\}}{\tan \left[ \frac{\gamma_{21}}{2} (x_2 - \frac{a}{2}) \right] - \left( \frac{\gamma_{21}}{\gamma_{22}} \right)^2 \tan \left( \frac{\gamma_{21}}{2} x_1 \right) \tan \left( \frac{\gamma_{21}}{2} x_1 \right) \tan \left[ \frac{\gamma_{21}}{2} (x_2 - \frac{a}{2}) \right]} \]
\[ = 0 \]
\[ (21b) \]

Typically, for attenuators,
\[
\epsilon_r \sim -i \delta \lambda_{0,2} \sigma \sim \text{large}, \quad (22)
\]

where \( \sigma \) is the conductivity and
\[
k_{0,2} = \frac{2\pi}{\lambda_{0,2}}. \quad \text{This implies}
\]
\[ k_{0,\Pi} = k_0 \sqrt{\frac{\varepsilon}{\mu}} \sim \chi_{1} \sim \text{large and complex}. \tag{23} \]

Let us define

\[ k_{c,1} = \chi_{1} , \tag{24} \]

[cf. eq. (5a)].

The pickup electrodes should most easily excite the odd \( m \) modes containing the most important one \( TE_{1,0} \). For \( m \) odd, eq. (21b) can be cast into the more useful form:

\[
\frac{1}{2} A \left( \frac{1}{2} k_{c,1} a \right) \cos \left( \frac{1}{2} k_{c,1} a \right) \\
+ i \sin \left[ \left( \frac{1}{2} k_{c,1} a \right) \left( \frac{2 \chi_{1}}{a} \right) \right] \cos \left[ \left( \frac{1}{2} k_{c,1} a \right) \left( \frac{2 \chi_{1}}{a} - 1 \right) \right] = 0, \tag{25} \]
where we have exploited eqs. (15) and (22-24), and a' la Marcuvitz we define

$$A = \frac{R}{377} \left( \frac{2 \pi \eta \mathbf{i}}{\pi a} \right)$$

(26a)

$$R = \frac{1}{\sigma t} = \frac{P}{t} \text{ (in m/square).}$$

(26b)

We must solve eq. (25) for complex $k_c \mathbf{i}$. But first write

$$\delta = \frac{2 x_1}{a}$$

(27a)

$$\frac{1}{2} k_c \mathbf{i} a = \xi_1 + i \xi_2$$

(27b)

Hence, eq. (25) becomes 2 independent equations.
\[
\frac{1}{2} A \left( \xi \cosh \xi - \frac{1}{2} \sinh \xi \sinh \xi \right) \\
+ \sinh(\xi) \cosh(\xi) \sin \left( (d-1)\xi \right) \sinh \left( (d-1)\xi \right) \\
- \cosh(\xi) \sinh(\xi) \cosh \left( (d-1)\xi \right) \cosh \left( (d-1)\xi \right) = 0 \\
(28a)
\]

\[
\frac{1}{2} A \left( \xi \cosh \xi - \frac{1}{2} \sinh \xi \sinh \xi \right) \\
+ \sinh(\xi) \cosh(\xi) \cosh \left( (d-1)\xi \right) \cosh \left( (d-1)\xi \right) \\
+ \cosh(\xi) \sinh(\xi) \sin \left( (d-1)\xi \right) \sinh \left( (d-1)\xi \right) = 0 \\
(28b)
\]

We must solve these 2 equations simultaneously for \( \xi \) and \( \xi \), keeping in mind that for

\[
m = 1, \quad \frac{\pi}{2} < \xi < \pi \\
(29a)
\]

\[
m = 3, \quad \frac{3\pi}{2} < \xi < 2\pi \\
(29b)
\]

(etc.)
(Remember that for the empty wave

guide, for m = 1 we have \( k_{c,x} = \frac{\pi}{a} \)

and hence \( \frac{1}{2} k_{c,x} a = \xi = \frac{\pi}{2} \).

An excellent routine for solving

the above system of simultaneous

nonlinear equations is NEWTON,

found in the CERNLIB computer

program library.

Once we know complex

\( k_{c,x} \), we find complex

\[
k_g = k + i \alpha \tag{30}
\]

via e.g. (13a). Thus the fields will

propagate according to
\[ E(x, y, z, t) = E(x, y) e^{-\alpha k z - i\omega t}, \]  
\[ B(x, y, z, t) = B(x, y) e^{-\alpha k z - i\omega t}, \]

and the attenuation length will be given by \( 1/\alpha \).

Defining

\[ x = \frac{2 \xi_1 \xi_2 \left( \frac{\lambda_0 \omega}{\pi a} \right)^2}{1 - (\xi_1 - \xi_2)^2 \left( \frac{\lambda_0 \omega}{\pi a} \right)^2}, \]  

the final result is

\[ \alpha = \frac{2\pi \sqrt{1 - (\xi_1 - \xi_2)^2 \left( \frac{\lambda_0 \omega}{\pi a} \right)^2}}{\sinh \left( \sinh^{-1} \frac{x}{2} \right)}, \]  
\[ \lambda_0, \omega \]
III. Numerical Results

for the Argonne system with $a = 22 \text{cm}$ and $b = 3 \text{cm}$, we have searched for the values of $x_1$ (see Fig. 2) and $R = \frac{5}{4}$ giving the shortest attenuation length $\frac{1}{2} \phi$ of the unwanted TE$_{1,0}$ mode, which according to the Table of $f_c, m_n$ is the most important odd $m$ mode in the 1-2 6Hz range.

The numerical results are summarized in Figs 4-9. We moved the resistive strips toward the center of the beam pipe in increments of 0.4 cm from $x_1 = 0.4 \text{cm}$ to $x_1 = 4.0 \text{cm}$. We conclude that
for 1–2 GHz, the optimal value of $R$ is 150–200 $\Omega$/square and the larger the value of $x$, the shorter the attenuation length $1/\alpha$. 
References


\( \text{\( \eta = 15 \text{ cm} \) (2 GHz)} \)

**Fig. 4**
$\lambda_{\%2} = 15 \text{ cm} \quad (2 \text{ GHz})$

$\chi_1 = 4 \text{ cm}$

FIG. 5
$\lambda_{0.1} = 20\text{ cm}$

(1.5 GHz)
$\eta_0 = 20\text{ cm (1.5 GHz)}$

$\chi_1 = 4.0\text{ cm}$
\[ \lambda_{\text{I}} = 28 \text{ cm} \]
\[ (\sim 1 \text{ GHz}) \]

**FIG. 8**