SCALAR CONTRIBUTION TO THE GRAVITON SELF-ENERGY DURING INFLATION

By

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To my parents and my fiancé

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We use dimensional regularization to evaluate the one loop contribution to the graviton self-energy from a massless, minimally coupled scalar on a locally de Sitter background. For noncoincident points our result agrees with the stress tensor correlators obtained recently by Perez-Nadal, Roura and Verdaguer. We absorb the ultraviolet divergences using the R^2 and C^2 counterterms first derived by 't Hooft and Veltman, and we take the $D = 4$ limit of the finite remainder. The renormalized result is expressed as the sum of two transverse, 4th order differential operators acting on nonlocal, de Sitter invariant structure functions. In this form it can be used to quantum-correct the linearized Einstein equations so that one can study how the inflationary production of infrared scalars affects the propagation of dynamical gravitons and the force of gravity. We have seen that they have no effect on the propagation of dynamical gravitons. Our computation motivates a conjecture for the first correction to the vacuum state wave functional of gravitons. We comment as well on performing the same analysis for the more interesting contribution from inflationary gravitons, and on inferring one loop corrections to the force of gravity.

CHAPTER 1
INTRODUCTION

My research involves quantum effects during primordial inflation. Primordial inflation is a phase of accelerated expansion during the very early universe which explains why the current universe is so homogeneous and isotropic on large scales, and why it is so nearly spatially flat. Quantum effects are vastly strengthened during inflation because the rapid expansion rips quantum fluctuations out of the vacuum so that they become real particles. This is thought to be the source of the observed density perturbations. My work concerns how the ensemble of scalars produced in this way would affect the propagation of gravitational radiation and the force of gravity.

In the following sections, we will review what inflation is, why it enhances quantum effects and how one can understand this enhancement as the classical response to virtual particles. We will also discuss how reliable information from quantum general relativity can be obtained in spite of its nonrenormalizability. This chapter closes with an overview of my project.

1.1 Inflation

On scales above 100Mpc our universe is observed to be homogeneous and isotropic. It also seems to have zero spatial curvature. Based on these three features our universe can be described by the Friedmann-Robertson-Walker (FRW) metric, with the invariant element

$$ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x}. \tag{1–1}$$

Here the coordinate t is physical time and the $a(t)$ is called the scale factor because it converts coordinate distance $\|\vec{x} - \vec{y}\|$ into physical distance $a(t)\|\vec{x} - \vec{y}\|$.

There are three observable cosmological quantities that can be constructed from the scale factor:

$$\text{Red Shift} \quad z(t) \equiv \frac{a(t_0)}{a(t)} - 1, \tag{1–2}$$

$$\text{Hubble Parameter} \quad H(t) \equiv \frac{\dot{a}}{a}, \tag{1–3}$$

$$\text{Deceleration parameter} \quad q(t) \equiv -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} \equiv= -1 + \epsilon(t) \tag{1–4}$$

where t_0 is the current time. The Hubble parameter $H(t)$ tells us the expansion rate of the universe. The deceleration parameter measures the fractional acceleration rate $\ddot{a}/a$ in units of the Hubble parameter $(\dot{a}/a)^2$.

Inflation is defined as accelerated expansion, [1, 2]

$$H(t) > 0 \text{ and } (\ q(t) < 0 \text{ or equivalently } \ \epsilon < 1). \tag{1–5}$$

Their current values are: $H_{now} = (73.8 \pm 2.4)\frac{Km/s}{Mpc} \simeq 2.4 \times 10^{-18} Hz \simeq 10^{-33} eV$ [3] and $\epsilon_{now} \simeq 0.33 \pm 0.13$ [4, 5]. So our universe is currently inflating. However the inflationary epoch of relevance to my work is primordial inflation. Because the effects I study derive from quantum gravity, they contain powers of GH^2, and the current Hubble parameter is just too small for these effects to be observable. In contrast, the latest data [5, 6] plus the assumption of single scalar inflation imply $H_I \lesssim 1.7 \times 10^{38} Hz \sim 10^{13} GeV$ with $\epsilon_I \lesssim 0.011$ [7]. This is only about six orders of magnitude below the Planck scale, $M_{Pl} \sim 10^{19} GeV$ and that makes these effects small, but observable.

Here it is useful to comment that primordial inflation is very close to de Sitter which has a positive constant for H and q exactly -1 (or $\epsilon = 0$). This allows us to take de Sitter space as a paradigm for primordial inflation. All my calculations concerning quantum effects during inflation are done on Sitter background.

1.2 Understanding Quantum Effects

Quantum loop effects can be understood as the classical response to virtual particles. For example, consider the vacuum polarization of quantum electrodynamics.

The energy-time uncertainty principle says that virtual electro-positron pairs are created out of the vacuum and exist for a brief period of time. If we just accept this, the vacuum polarization is completely analogous to the phenomenon of classical polarization in a mediuum full of charged particles. The bottom line is that if whatever increases the number of virtual particles strengthens quantum effects. In the following subsections we will discuss how inflation does this. We consider two aspects, the persistence time and the emergence rate of virtual particles.

1.2.1 Uncertainty Principle during Inflation

The energy-time uncertainty principle of flat space

$$\Delta E \Delta t \gtrsim 1 \; . \tag{1–6}$$

says that to resolve an energy with accuracy ΔE we have to wait at least a time Δt. To reslove the production of a virtual pair of wave number k and mass m we have $\Delta E = 2\sqrt{m^2 + k^2}$. We would not notice a violation of energy conservation provided $\Delta t \lesssim \frac{1}{\Delta E}$. That is, we can take $1/\Delta E$ as the lifetime of a virtual pair.

$$\Delta t = \frac{1}{\Delta E} = \frac{1}{2\sqrt{m^2 + k^2}} \; . \tag{1–7}$$

How would this effect change during inflation? If we consider the homogeneous, isotropic and spatially flat geometry described in (1–1), from its spatial translation invariance one can still label particles by constant wave numbers $\vec{k}$, just as in flat space. However, this "co-moving wave vector" $\vec{k}$ involves an inverse length and hence one must multiply it by the scale factor $a(t)$ to get the "physical wave vector" $\vec{k}/a(t)$. This time dependent wave number implies the expression for (1–7) becomes an integral.

$$\int_t^{t+\triangle t} dt' 2E(t', \vec{k}) \lesssim 1 \; . \tag{1–8}$$

with $E(t, \vec{k}) = \sqrt{m^2 + k^2/a^2(t)}$. Note that spacetime expansion always lengthens the time a virtual pair can exist because $k_{phys} = k/a(t)$ becomes smaller as $a(t)$ grows. Just as in flat space, massless particles of the same wave number live longer than massive ones. Taking $m = 0$ and the de Sitter limit of the scale factor, $a(t) = a_I e^{Ht}$ we have

$$\frac{2k}{Ha(t)}(1 - e^{-H\Delta t}) \lesssim 1 \; . \tag{1–9}$$

This means that massless virtual particles can live forever during inflation if they emerge with $k \lesssim Ha(t)$.

1.2.2 Conformal Invariance

Another important factor for virtual particle creation is the rate at which virtual particles emerge from the vacuum. It turns out the rate depends on the type of particle. In flat space the emergence rate Γ_{flat} is constant by Poincare invariance. In an expanding universe it will become time dependent because time translation invariance is no longer valid. Recall now that virtual particles live longest when they are massless. Unfortunately, almost all massless particles possess conformal invariance, which leads to an exponential suppression of their emergence rate.

To understand this, note first that conformally invariant theories in FRW conformal coordinates are locally identical to flat space. This becomes clear if we express the FRW metric (1–1) in conformal coordinates,

$$\begin{aligned} dt = a(t)d\eta \Longrightarrow ds^2 &= -dt^2 + a^2(t)d\vec{x}\cdot d\vec{x} \\ &= a^2(t)(-d\eta^2 + d\vec{x}\cdot d\vec{x}\,) \; . \end{aligned} \tag{1–10}$$

Here t is physical time and η is conformal time. In $(\eta, \vec{x})$ coordinates the homogeneous and isotropic, spatially flat geometry looks like a conformal rescaling of flat space. One consequence is that the emergence rate per conformal time must be the same as in flat space, $\Gamma_\eta = \Gamma_{flat}$. Now just convert to physical time, we see

$$\frac{dN}{dt} = \frac{d\eta}{dt}\frac{dN}{d\eta} = \frac{d\eta}{dt}\Gamma_\eta = \frac{\Gamma_{flat}}{a(t)} \tag{1–11}$$

This means that the emergence rate of virtual particles possessing conformal symmetry is suppressed by a factor of $1/a$. Therefore any conformal-invariant, massless virtual particles with $k \lesssim Ha(t)$ can live forever but the problem is that they don't have much chance to emerge from the vacuum.

From this and the previous discussions, we have the conditions which leads to big quantum effects:

- Inflationary spacetime
- Massless particles
- No conformal invariance

We have only two kinds of massless and not conformally invariant particles: MMC scalars and gravitons, my thesis concerns the effect of the former on the latter.

1.2.3 Particle Production during Inflation

It is also useful to explicitly show the number of virtual particles grows. In this subsection we compute the particle production rate during inflation for nonconformally invariant particles. Consider the massless, minimally coupled scalar[1] $\varphi(t, \vec{x})$. Its Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} \tag{1–12}$$

Using the metric (1–1) and integrating gives the Lagrangian,

$$L \equiv \int d^3x\mathcal{L} = \frac{1}{2}a^3\int d^3x\dot{\varphi}(t,\vec{x})\dot{\varphi}(t,\vec{x}) - \frac{1}{2}a\int d^3x\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi \tag{1–13}$$

Using Parseval's theorem

$$\int_{-\infty}^{\infty} f(x)g(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\tilde{f}(k)\tilde{g}(-k)\ , \tag{1–14}$$

and the reality condition ($f(x)$ is real) $\tilde{f}(-k) = \tilde{f}^*(k)$, we find

[1] The result is equivalent for the gravitons, as recognized by Grishchuck[8]

$$L = \frac{1}{2}a^3 \int \frac{d^3k}{(2\pi)^3}\, \dot{\tilde{\varphi}}(t,\vec{k})\dot{\tilde{\varphi}}^*(t,\vec{k}) - \frac{1}{2}a \int \frac{d^3k}{(2\pi)^3}\, k^2 \tilde{\varphi}(t,\vec{k})\tilde{\varphi}^*(t,\vec{k}) \ . \tag{1–15}$$

Because there is no coupling between different $\vec{k}$'s, let us consider one mode with $\vec{k}$ and call it q(t). Its Lagrangian is,

$$L = \frac{1}{2}a^3\dot{q}^2 - \frac{1}{2}k^2 a q^2 \ . \tag{1–16}$$

Now we notice this is the Lagrangian of a harmonic oscillator with mass $m(t) = a^3(t)$ and frequency $\omega(t) = k/a(t)$. Because mass and frequency are time dependent there are no stationary states but we can still construct the Hamiltonian

$$H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L = \frac{1}{2}a^3\dot{q}^2 + \frac{1}{2}k^2 a q^2 = \frac{1}{2}m(t)\dot{q}^2 + \frac{1}{2}m(t)\omega^2(t)q^2 \ . \tag{1–17}$$

with the equation of motion

$$\ddot{q} + 3H\dot{q} + \frac{k^2}{a^2}q = 0 \tag{1–18}$$

Solving this equation for general $a(t)$ is not easy but for the special case of de Sitter ($a(t) = e^{Ht}$ and $\dot{H} = 0$) which is relevant in our discussion, the general solution takes the form,

$$q(t) = u(t)\alpha + u^*(t)\alpha^\dagger, \quad u(t,k) = \frac{H}{\sqrt{2k^3}}\left[1 - \frac{ik}{Ha(t)}\right] e^{\frac{ik}{Ha(t)}} \tag{1–19}$$

where α and $\alpha^\dagger$ are operators which we canonically normalize as in normal quantum mechanics,

$$[\alpha, \alpha^\dagger] = 1 \ . \tag{1–20}$$

We define *Bunch-Davies vacuum* $|\Omega\rangle$ as the state with minimum energy in the distant past. The Bunch-Davis vacuum is $|\Omega>$ is annihilated by α, $\alpha|\Omega >= 0$. To find the the number of particles which emerge with wave number k, consider the expectation value of the energy in this state

$$\begin{aligned} < \Omega|H(t)|\Omega > &= \frac{1}{2}a^3(t) < \Omega|\dot{q}^2(t)|\Omega > + \frac{1}{2}k^2 a(t) < \Omega|q^2(t)|\Omega > \\ &= \frac{k}{a}\left[\frac{1}{2} + \left(\frac{Ha}{2k}\right)^2\right] = \frac{k}{a}\left[\frac{1}{2} + \text{“}N(t)\text{”}\right] \end{aligned} \tag{1–21}$$

We recognize that the number of particles with wave number k grows in time as the square of the scale factor,

$$N(k, t) = \left(\frac{Ha(t)}{2k}\right)^2 \tag{1–22}$$

Note that the growth becomes significant for infrared wave numbers, $k \leq Ha$. One consequence of this being an infrared effect is that perturbative general relativity can be used reliably as an effective field theory, even though it is not renormalizable. This issue will be discussed in the next section.

1.3 Using Quantum Gravity as an Effective Field Theory

Quantum gravity is not perturbatively renormalizable [9], however, ultraviolet divergences can always be absorbed in the sense of Bogoliubov, Parasiuk, Hepp and Zimmerman (BPHZ) [10]. A widespread misconception exists that no valid quantum predictions can be extracted from such an exercise. This is false: while nonrenormalizability does preclude being able to compute *everything*, that is not the same thing as being able to compute *nothing*. The problem with a nonrenormalizable theory is that no physical principle fixes the finite parts of the escalating series of BPHZ counterterms needed to absorb ultraviolet divergences, order-by-order in perturbation theory. Hence any prediction of the theory that can be changed by adjusting the finite parts of these counterterms is essentially arbitrary. However, loops of massless particles make nonlocal contributions to the effective action that can never be affected by local counterterms. These nonlocal contributions typically dominate in the infrared. Further, they cannot be affected by whatever modification of ultraviolet physics ultimately results in a completely consistent formalism. As long as the eventual fix introduces no new massless particles, and does not disturb the low energy couplings of the existing ones, the far infrared predictions of a BPHZ-renormalized quantum theory will agree with those of its fully consistent descendant.

To see this issue more specifically, let us first recall the theorem of Bogoliubov, Parasiuk, Hepp and Zimmerman (BPHZ) which constructs the local counterterms needed to absorb the ultraviolet divergences of any quantum field theory at some fixed order in the loop expansion [10]. This applies as well for quantum general relativity plus any matter theory, and the two one loop counterterms R^2 and C^2 required for scalars in $D = 4$ spacetime dimensions have long been known [9]. The problem for quantum general relativity is that these counterterms are not present in the original Lagrangian. We could include R^2; it would add a massive, positive energy scalar particle which poses no essential problem for the theory. However, incorporating C^2 on a nonperturbative level would add a negative energy, spin two particle whose presence would cause the universe to decay instantly. We must therefore treat the one loop counterterms perturbatively, and regard them as proxies for the still unknown ultraviolet completion of the theory.

The remaining problem with these perturbative counterterms is that we don't know their finite parts. Their divergent parts are fixed by the need to subtract off the infinities one encounters in loop corrections, but nothing fixes the finite parts, and these finite parts affect physical results, even when we only use them perturbatively. Of course this ambiguity reflects the fact that we don't know the ultraviolet completion of quantum gravity. What it means is that the only reliable predictions are those for which the arbitrary finite parts of the counterterms are unimportant.

That there are such predictions derives from two things:

- BPHZ counterterms are guaranteed to be local [10]; and
- Massless particles make nonlocal corrections to the effective field equations.

As an example, consider one loop corrections to the quantum gravitational effective action which are quadratic in the graviton field $h_{\mu\nu}$. For simplicity, let the background be flat space, and let us agree not to worry about how the various indices are contracted. The one loop counterterms contribute to the effective action as,

$$\Gamma^{\rm 1loop}_{\rm cterms} \sim \int d^4x \, \partial^2 h \cdot \partial^2 h + O(h^3) \; . \tag{1–23}$$

As stated, the problem with these sorts of terms is that we don't know the numerical coefficients which multiply them. In contrast, one loop effects from massless particles contribute terms of the form,

$$\Gamma^{\rm 1loop}_{\rm finite} \sim \int d^4x \, \partial^2 h \cdot \ln(-\partial^2) \cdot \partial^2 h + O(h^3) \; . \tag{1–24}$$

Perturbative quantum general relativity makes an exact prediction for the coefficients of these terms. Further, in the large distance regime the finite, nonlocal contributions (1–24) dominate over the local counterterms (1–23) owing to their enhancement by the factor of $\ln(-\partial^2)$, which diverges in the infrared. In momentum space $\partial^2 \to -p^2$, and the long wavelength regime is $p^2 \approx 0$. Then the local counterterm goes like p^4, and the nonlocal primitive effects go like $p^4 \ln(p^2)$. For small enough p^2 the nonlocal effects are larger, no matter how big the finite parts of the counterterms are.

It is worthwhile to review the vast body of distinguished work that has employed to derive valid quantum effects in the long distance regime. The oldest example is the solution of the infrared problem in quantum electrodynamics by Bloch and Nordsieck [13], long before that theory's renormalizability was suspected. Weinberg [14] was able to achieve a similar resolution for quantum gravity with zero cosmological constant. The same principle was at work in the Fermi theory computation of the long range force due to loops of massless neutrinos by Feinberg and Sucher [15, 16]. In pure quantum gravity Donoghue and others has applied the principles of low energy effective field theory to compute graviton corrections to the long range gravitational force [17–23].

To summarize, as long as we consider the low energy regime, the finite quantum corrections from the original Lagrangian can be distinguished from those of local counterterms. In the previous subsections I discussed the virtual particles produced during inflation. The two key facts were

1. The number of virtual particles present during a period of accelerated expansion is *vastly* larger than during a phase of deceleration; and

2. The extra virtual particles have cosmological wavelengths.

The first fact means one can get significant quantum effects; the second point means that these effects can be computed reliably without knowing the ultraviolet completion of quantum gravity.

1.4 Overview

The linearized equations for all known force fields do two things:

- They give the linearized force fields induced by sources; and
- They describe the propagation of dynamical particles which carry the force but are, in principle, independent of any source.

This is the classic distinction between the constrained and unconstrained parts of a force field. In electromagnetism it amounts to the Coulomb potential versus photons. In gravity there is the Newtonian potential, plus its three relativistic partners, versus gravitons.

Quantum corrections to the linearized field equations derive from how the 0-point fluctuations of various fields in whatever background is assumed, respond to the linearized force fields. These quantum corrections do not change the dichotomy between constrained and unconstrained fields but they can, of course, modify classical results. Around flat space background there is no effect, after renormalization, on the propagation of dynamical photons or gravitons but there are small corrections to the Coulomb and Newtonian potentials. As might be expected, the long distance effects are greatest for the 0-point fluctuations of massless particles and they take the form required by perturbation theory and dimensional analysis [43, 44],

$$\left(\frac{\Delta\Phi}{\Phi}\right)_{\text{Coul.}} \sim -\frac{e^2}{\hbar c}\ln\!\left(\frac{r}{r_0}\right) \qquad , \qquad \left(\frac{\Delta\Phi}{\Phi}\right)_{\text{Newt.}} \sim -\frac{\hbar G}{c^3 r^2} \;, \tag{1–25}$$

where r is the distance to the source, r_0 is the point at which the renormalized charge is defined, and the other constants have their usual meanings.

Schrödinger was the first to suggest that the expansion of spacetime can lead to particle production by ripping the virtual particles (which are implicit in 0-point fluctuations) out of the vacuum [45]. Following early work by Imamura [46], the first quantitative results were obtained by Parker [47]. He found that the effect is maximized during accelerated expansion, and for massless particles which are not conformally invariant [48], such as massless, minimally coupled (MMC) scalars and (as noted by Grishchuk [8]) gravitons. This result was reviewed in the previous sections.

The de Sitter geometry is the most highly accelerated expansion consistent with classical stability. For de Sitter background with Hubble constant H and scale factor $a(t) = e^{Ht}$ we have shown that the number of MMC scalars, or either polarization of graviton, created with wave vector $\vec{k}$ is [49],

$$N(t, \vec{k}) = \left(\frac{Ha(t)}{2c\|\vec{k}\|}\right)^2 . \tag{1–26}$$

It is these particles which comprise the scalar and tensor perturbations produced by inflation [50], the scalar contribution of which has been imaged [51]. Of course the same particles also enter loop diagrams to cause an enormous strengthening of the quantum effects caused by MMC scalars and gravitons. A number of analytic results have been obtained for one loop corrections to the way various particles propagate on de Sitter background and also to how long range forces act:

- In MMC scalar quantum electrodynamics, infrared photons behave as if they had an increasing mass [52], and the charge screening very quickly becomes nonperturbatively strong [53], but there is no big effect on scalars [54];
- For a MMC scalar which is Yukawa-coupled to a massless fermion, infrared fermions behave as if they had an increasing mass [55] but the associated scalars experience no large correction [56];
- For a MMC scalar with a quartic self-interaction, infrared scalars behave as if they had an increasing mass (which persists to two loop order) [57];
- For quantum gravity minimally coupled to a massless fermion, the fermion field strength grows without bound [58]; and

- For quantum gravity plus a MMC scalar, the scalar shows no secular effect but its field strength may acquire a momentum-dependent enhancement [59].

The great omission from this list is how inflationary scalars and gravitons affect gravity, both as regards the propagation of dynamical gravitons and as regards the force of gravity. My project represents a first step in completing the list.

One includes quantum corrections to the linearized field equation by subtracting the integral of the appropriate one-particle-irreducible (1PI) 2-point function up against the linearized field. For example, a MMC scalar $\varphi(x)$ in a background metric $g_{\mu\nu}(x)$ whose 1PI 2-point function is $-iM^2(x;x')$, would have the linearized effective field equation,

$$\partial_\mu\Big[\sqrt{g}g^{\mu\nu}\partial_\nu\varphi(x)\Big] - \int d^4x'\, M^2(x;x')\varphi(x') = 0 \; . \tag{1–27}$$

To include gravity on the list we must therefore compute the graviton self-energy, either from MMC scalars or from gravitons, and then use it to correct the linearized Einstein equation.

In the first part of my dissertation we evaluate the contribution from MMC scalars which is described in chapters 2 through 4; In chapter 2 we give those of the Feynman rules which are needed for this computation, and we describe the geometry of our D-dimensional, locally de Sitter background. Chapter 3 derives the relatively simple form for the D-dimensional graviton self-energy with noncoincident points. We show that this version of the result agrees with the flat space limit [61] and with the de Sitter stress tensor correlators recently derived by Perez-Nadal, Roura and Verdaguer [99]. Chapter 4 undertakes the vastly more difficult reorganization which must be done to isolate the local divergences for renormalization. At the end we subtract off the divergences with the same counterterms originally computed for this model in 1974 by 't Hooft and Veltman [9], and we take the unregulated limit of $D = 4$.

The second part of dissertation solves the linearized effective field equations to determine quantum corrections to the propagation of gravitons. In chapter 5 we carry out the same calculations for flat space which also serve the correspondence limit for the vastly more complicated de Sitter case. Chapter 6 is dedicated for the scalar one loop correction to dynamical gravitons. Our conclusion comprises chapter 7.

CHAPTER 2
FEYNMAN RULES

In this chapter we derive Feynman rules for the computation. We start by expressing the full metric as

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \kappa h_{\mu\nu} \ , \tag{2–1}$$

where $\overline{g}_{\mu\nu}$ is the background metric, $h_{\mu\nu}$ is the graviton field whose indices are raised and lowered with the background metric, and $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. Expanding the MMC scalar Lagrangian around the background metric we get interaction vertices between the scalar and dynamical gravitons. We take the D-dimensional locally de Sitter space as our background and introduce de Sitter invariant bi-tensors which will be used throughout the calculation. We close this section by providing the MMC scalar propagator on the de Sitter background.

2.1 Interaction Vertices

The Lagrangian which describes pure gravity and the interaction between gravitons and the MMC scalar is,

$$\mathcal{L} = \frac{1}{16\pi G}\Big[R - (D-1)(D-2)H^2\Big]\sqrt{-g} - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} \ . \tag{2–2}$$

where R is Ricci scalar, G is Newton's constant and H is the Hubble constant.

Computing the one loop scalar contributions to the graviton self-energy consists of summing the 3 Feynman diagrams depicted in Figure 2-1. The sum of these three

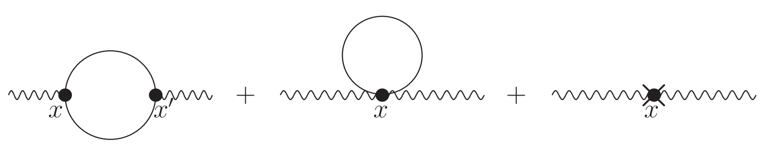

Figure 2-1. The one loop graviton self-energy from MMC scalars.

diagrams has the following analytic form:

$$
\begin{aligned}
&-i\left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right](x;x') \\
&\quad = \frac{1}{2}\sum_{I=1}^{2} T_I^{\mu\nu\alpha\beta}(x)\sum_{J=1}^{2} T_J^{\rho\sigma\gamma\delta}(x') \times \partial_\alpha \partial'_\gamma i\triangle(x;x') \times \partial_\beta \partial'_\delta i\triangle(x;x') \\
&\qquad\qquad + \frac{1}{2}\sum_{I=1}^{4} F_I^{\mu\nu\rho\sigma\alpha\beta}(x) \times \partial_\alpha \partial'_\beta i\triangle(x;x') \times \delta^D(x-x') \\
&\qquad\qquad\qquad + 2\sum_{I=1}^{2} C_I^{\mu\nu\rho\sigma}(x) \times \delta^D(x-x')\;. \qquad (2\text{–}3)
\end{aligned}
$$

The 3-point and 4-point vertex factors $T_I^{\mu\nu\alpha\beta}$ and $F_I^{\mu\nu\rho\sigma\alpha\beta}$ derive from expanding the MMC scalar Lagrangian using (2–1),

$$
-\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} \qquad (2\text{–}4)
$$

$$
\begin{aligned}
&= -\frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi \overline{g}^{\mu\nu}\sqrt{-\overline{g}} - \frac{\kappa}{2}\partial_\mu\varphi\partial_\nu\varphi\Big(\frac{1}{2}h\overline{g}^{\mu\nu} - h^{\mu\nu}\Big)\sqrt{-\overline{g}} \\
&\quad -\frac{\kappa^2}{2}\partial_\mu\varphi\partial_\nu\varphi\bigg\{\Big[\frac{1}{8}h^2 - \frac{1}{4}h^{\rho\sigma}h_{\rho\sigma}\Big]\overline{g}^{\mu\nu} - \frac{1}{2}hh^{\mu\nu} + h^{\mu}_{\ \rho}h^{\rho\nu}\bigg\}\sqrt{-\overline{g}} + O(\kappa^3)\;. \qquad (2\text{–}5)
\end{aligned}
$$

The resulting 3-point and 4-point vertex factors are given in the Tables 1 and 2, respectively. The procedure to get the counterterm vertex operators $C_I^{\mu\nu\rho\sigma}(x)$ is given in section 4.

Table 2-1. 3-point vertices $T_I^{\mu\nu\alpha\beta}$ where $\overline{g}_{\mu\nu}$ is the de Sitter background metric, $\kappa^2 \equiv 16\pi G$ and parenthesized indices are symmetrized.

I	$T_I^{\mu\nu\alpha\beta}$
1	$-\frac{i\kappa}{2}\sqrt{-\overline{g}}\;\overline{g}^{\mu\nu}\overline{g}^{\alpha\beta}$
2	$+i\kappa\sqrt{-\overline{g}}\;\overline{g}^{\mu(\alpha}\overline{g}^{\beta)\nu}$

Table 2-2. 4-point vertices $F_I^{\mu\nu\rho\sigma\alpha\beta}$ where $\overline{g}_{\mu\nu}$ is the de Sitter background metric, $\kappa^2 \equiv 16\pi G$ and parenthesized indices are symmetrized.

I	$F_I^{\mu\nu\rho\sigma\alpha\beta}$
1	$-\frac{i\kappa^2}{4}\sqrt{-\overline{g}}\;\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}\overline{g}^{\alpha\beta}$
2	$+\frac{i\kappa^2}{2}\sqrt{-\overline{g}}\;\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\overline{g}^{\alpha\beta}$
3	$+\frac{i\kappa^2}{2}\sqrt{-\overline{g}}\;\Big[\overline{g}^{\mu(\alpha}\overline{g}^{\beta)\nu}\overline{g}^{\rho\sigma} + \overline{g}^{\mu\nu}\overline{g}^{\rho(\alpha}\overline{g}^{\beta)\sigma}\Big]$
4	$-2i\kappa^2\sqrt{-\overline{g}}\;\overline{g}^{\alpha(\mu}\overline{g}^{\nu)(\rho}\overline{g}^{\sigma)\beta}$

These interaction vertices are valid for any background metric $\overline{g}_{\mu\nu}$. In the next two subsections we specialize to a locally de Sitter background and give the scalar propagator $i\triangle(x;x')$ on it.

2.2 Working on de Sitter Space

We specify our background geometry as the open conformal coordinate submanifold of D-dimensional de Sitter space. A spacetime point $x^\mu = (\eta, x^i)$ takes values in the ranges

$$-\infty < \eta < 0 \qquad \text{and} \qquad -\infty < x^i < +\infty \; . \tag{2–6}$$

In these coordinates the invariant element is,

$$ds^2 \equiv \overline{g}_{\mu\nu} dx^\mu dx^\nu = a^2 \eta_{\mu\nu} dx^\mu dx^\nu \; , \tag{2–7}$$

where $\eta_{\mu\nu}$ is the Lorentz metric and $a = -1/H\eta$ is the scale factor. The Hubble parameter H is constant for the de Sitter space. So in terms of $\eta_{\mu\nu}$ and a our background metric is

$$\overline{g}_{\mu\nu} \equiv a^2 \eta_{\mu\nu} \; . \tag{2–8}$$

De Sitter space has the maximum number of space-time symmetries in a given dimension. For our D-dimensional conformal coordinates the $\frac{1}{2}D(D+1)$ de Sitter transformations can be decomposed as follows:

- Spatial transformations - $(D-1)$ transformations.

$$\eta' = \eta \; , \quad x'^i = x^i + \epsilon^i \; . \tag{2–9}$$

- Rotations - $\frac{1}{2}(D-1)(D-2)$ transformations.

$$\eta' = \eta \; , \quad x'^i = R^{ij} x^j \; . \tag{2–10}$$

- Dilation - 1 transformation.

$$\eta' = k\eta \; , \quad x'^i = kx^j \; . \tag{2–11}$$

- Spatial special conformal transformations - $(D-1)$ transformations.

$$\eta' = \frac{\eta}{1-2\vec{\theta}\cdot\vec{x}+\|\vec{\theta}\|^2\, x\cdot x} \;, \quad x' = \frac{x^i-\theta^i x\cdot x}{1-2\vec{\theta}\cdot\vec{x}+\|\vec{\theta}\|^2\, x\cdot x} \;. \tag{2–12}$$

It turns out that the MMC scalar contribution to the graviton self-energy is de Sitter invariant. This suggests to express it in terms of the de Sitter length function $y(x;x')$,

$$y(x;x') \equiv aa'H^2\left[\left\|\vec{x}-\vec{x}'\right\|^2-\Big(|\eta-\eta'|-i\epsilon\Big)^2\right] . \tag{2–13}$$

Except for the factor of $i\epsilon$ (whose purpose is to enforce Feynman boundary conditions) the function $y(x;x')$ is closely related to the invariant length $\ell(x;x')$ from x^μ to x'^μ,

$$y(x;x') = 4\sin^2\Big(\frac{1}{2}H\ell(x;x')\Big) . \tag{2–14}$$

With this de Sitter invariant quantity $y(x;x')$, we can form a convenient basis of de Sitter invariant bi-tensors. Note that because $y(x;x')$ is de Sitter invariant, so too are covariant derivatives of it. With the metrics $\overline{g}_{\mu\nu}(x)$ and $\overline{g}_{\mu\nu}(x')$, the first three derivatives of $y(x;x')$ furnish a convenient basis of de Sitter invariant bi-tensors [54],

$$\frac{\partial y(x;x')}{\partial x^\mu} = Ha\Big(y\delta^0_\mu+2a'H\Delta x_\mu\Big) \;, \tag{2–15}$$

$$\frac{\partial y(x;x')}{\partial x'^\nu} = Ha'\Big(y\delta^0_\nu-2aH\Delta x_\nu\Big) \;, \tag{2–16}$$

$$\frac{\partial^2 y(x;x')}{\partial x^\mu \partial x'^\nu} = H^2aa'\Big(y\delta^0_\mu\delta^0_\nu+2a'H\Delta x_\mu\delta^0_\nu-2a\delta^0_\mu H\Delta x_\nu-2\eta_{\mu\nu}\Big) . \tag{2–17}$$

Here and subsequently $\Delta x_\mu \equiv \eta_{\mu\nu}(x-x'^\nu)$.

Acting covariant derivatives generates more basis tensors, for example [54],

$$\frac{D^2 y(x;x')}{Dx^\mu Dx^\nu} = H^2(2-y)\overline{g}_{\mu\nu}(x) \;, \tag{2–18}$$

$$\frac{D^2 y(x;x')}{Dx'^\mu Dx'^\nu} = H^2(2-y)\overline{g}_{\mu\nu}(x') \;. \tag{2–19}$$

The contraction of any pair of the basis tensors also produces more basis tensors [54],

$$\overline{g}^{\mu\nu}(x)\frac{\partial y}{\partial x^{\mu}}\frac{\partial y}{\partial x^{\nu}} = H^2\Big(4y-y^2\Big) = \overline{g}^{\mu\nu}(x')\frac{\partial y}{\partial x'^{\mu}}\frac{\partial y}{\partial x'^{\nu}} \; , \tag{2–20}$$

$$\overline{g}^{\mu\nu}(x)\frac{\partial y}{\partial x^{\nu}}\frac{\partial^2 y}{\partial x^{\mu}\partial x'^{\sigma}} = H^2(2-y)\frac{\partial y}{\partial x'^{\sigma}} \; , \tag{2–21}$$

$$\overline{g}^{\rho\sigma}(x')\frac{\partial y}{\partial x'^{\sigma}}\frac{\partial^2 y}{\partial x^{\mu}\partial x'^{\rho}} = H^2(2-y)\frac{\partial y}{\partial x^{\mu}} \; , \tag{2–22}$$

$$\overline{g}^{\mu\nu}(x)\frac{\partial^2 y}{\partial x^{\mu}\partial x'^{\rho}}\frac{\partial^2 y}{\partial x^{\nu}\partial x'^{\sigma}} = 4H^4\overline{g}_{\rho\sigma}(x') - H^2\frac{\partial y}{\partial x'^{\rho}}\frac{\partial y}{\partial x'^{\sigma}} \; , \tag{2–23}$$

$$\overline{g}^{\rho\sigma}(x')\frac{\partial^2 y}{\partial x^{\mu}\partial x'^{\rho}}\frac{\partial^2 y}{\partial x^{\nu}\partial x'^{\sigma}} = 4H^4\overline{g}_{\mu\nu}(x) - H^2\frac{\partial y}{\partial x^{\mu}}\frac{\partial y}{\partial x^{\nu}} \; . \tag{2–24}$$

Our basis tensors are naturally covariant, but their indices can of course be raised using the metric at the appropriate point. To save space in writing this out we define the basis tensors with raised indices as differentiation with respect to "covariant" coordinates,

$$\frac{\partial y}{\partial x_{\mu}} \equiv \overline{g}^{\mu\nu}(x)\frac{\partial y}{\partial x^{\nu}} \; , \tag{2–25}$$

$$\frac{\partial y}{\partial x'_{\rho}} \equiv \overline{g}^{\rho\sigma}(x')\frac{\partial y}{\partial x'^{\sigma}} \; , \tag{2–26}$$

$$\frac{\partial^2 y}{\partial x_{\mu}\partial x'_{\rho}} \equiv \overline{g}^{\mu\nu}(x)\overline{g}^{\rho\sigma}(x')\frac{\partial^2 y}{\partial x^{\nu}\partial x'^{\sigma}} \; . \tag{2–27}$$

2.3 Scalar Propagator on de Sitter

From the MMC scalar Lagrangian (2–2) we see that the propagator obeys

$$\partial_{\mu}\Big[\sqrt{-\overline{g}}\;\overline{g}^{\mu\nu}\partial_{\nu}\Big] i\triangle(x;x') = \sqrt{-\overline{g}}\,\square\, i\triangle(x;x') = i\delta^D(x-x') \tag{2–28}$$

Although this equation is de Sitter invariant, there is no de Sitter invariant solution for the propagator [62], hence some of the symmetries (2–9)-(2–12) must be broken. We choose to preserve the homogeneity and isotropy of cosmology — relations (2–9)-(2–10) — which corresponds to what is known as the "E3" vacuum [63]. It can be realized in terms of plane wave mode sums by making the spatial manifold T^{D-1}, rather than R^{D-1}, with coordinate radius H^{-1} in each direction, and then using the integral approximation with the lower limit cut off at $k = H$ [64]. The final result consists of a de Sitter invariant function of $y(x;x')$ plus a de Sitter breaking part which depends upon the scale factors at the two points [41],

$$i\triangle(x;x') = A\Big(y(x;x')\Big) + k\ln(aa') \; . \tag{2–29}$$

Here the constant k is given as,

$$k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \; , \tag{2–30}$$

and the function $A(y)$ has the expansion,

$$A(y) \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \Biggl\{ \frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1} \Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2} \Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} - \pi\cot\Bigl(\frac{\pi D}{2}\Bigr) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \\ + \sum_{n=1}^{\infty} \Biggl[\frac{1}{n} \frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})} \Bigl(\frac{y}{4}\Bigr)^{n} - \frac{1}{n-\frac{D}{2}+2} \frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)} \Bigl(\frac{y}{4}\Bigr)^{n-\frac{D}{2}+2} \Biggr] \Biggr\} . \tag{2–31}$$

The infinite series terms of $A(y)$ vanish for $D = 4$, so they only need to be retained when multiplying a potentially divergent quantity, and even then one only needs to include a handful of them. This makes loop computations manageable.

We note that the MMC scalar propagator (2–29) has a de Sitter breaking term, $k\ln(aa')$. However, the one loop scalar contribution to the graviton self-energy only involves the terms like $\partial_\alpha \partial'_\beta i\triangle(x;x')$, which are de Sitter invariant,

$$\partial_\alpha \partial'_\beta i\Delta(x;x') = \frac{\partial}{\partial x^\alpha} \Biggl\{ A'(y) \frac{\partial y}{\partial x'^\beta} + Ha'\delta^0_\beta \Biggr\} = A''(y) \frac{\partial y}{\partial x^\alpha} \frac{\partial y}{\partial x'^\beta} + A'(y) \frac{\partial^2 y}{\partial x^\alpha \partial x'^\beta} \; . \tag{2–32}$$

Another useful relation follows from the propagator equation,

$$(4y - y^2)A''(y) + D(2-y)A'(y) = (D-1)k \; . \tag{2–33}$$

CHAPTER 3
ONE LOOP GRAVITON SELF-ENERGY

In this chapter we calculate the first two, primitive, diagrams of Figure 1. It turns out that the contribution from the 4-point vertex (the middle diagram) vanishes in $D = 4$ dimensions. The contribution from two 3-point vertices (the leftmost diagram) is nonzero. For noncoincident points it gives a relatively simple form which agrees with the flat space limit [61] and with the de Sitter stress tensor correlator recently derived by Perez-Nadal, Roura and Verdaguer [99].

3.1 Contribution from 4-Point Vertices

The 4-point contribution from the middle diagram of Figure 1 takes the form,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 4pt}(x;x') \equiv \frac{1}{2}\sum_{I=1}^{4} F_I^{\mu\nu\rho\sigma\alpha\beta}(x) \times \partial_\alpha \partial'_\beta i\triangle(x;x' \times \delta^D(x-x') \; . \tag{3–1}$$

Recall that the four 4-point vertices $F_I^{\mu\nu\rho\sigma\alpha\beta}(x)$ are given in Table 2-2. Owing to the delta function, we need the coincidence limit of the doubly differentiated propagator (2–32). The coincidence limits of the various tensor factors follow from setting $a' = a$, $\Delta x^\mu = 0$ and $y = 0$ in relations (2–15)-(2–17),

$$\lim_{x'\to x} \frac{\partial y(x;x')}{\partial x^\mu} = 0 = \lim_{x'\to x} \frac{\partial y(x;x')}{\partial x'^\nu} \; , \tag{3–2}$$

$$\lim_{x'\to x} \frac{\partial^2 y(x;x')}{\partial x^\mu \partial x'^\nu} = -2H^2 \overline{g}_{\mu\nu} \; . \tag{3–3}$$

Hence the coincidence limit of the doubly differentiated propagator can be expressed in terms of $A'(y)$ evaluated at $y = 0$,

$$\lim_{x'\to x} \partial_\alpha \partial'_\beta i\triangle(x;x') = A''(0) \times 0 + A'(0) \times \Big[-2H^2 \overline{g}_{\mu\nu}\Big] \; . \tag{3–4}$$

From the definition (2–31) of $A(y)$, we see that $A'(y)$ is,

$$A'(y) = \frac{1}{4}\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Bigg\{-\Gamma(\frac{D}{2})\Big(\frac{4}{y}\Big)^{\frac{D}{2}} - \Gamma(\frac{D}{2}+1)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \sum_{n=1}\Bigg[\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^{n-1} - \frac{\Gamma(n+\frac{D}{2}-1)}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+1}\Bigg]\Bigg\}. \tag{3–5}$$

Now we recall that, in dimensional regularization, any D-dependent power of zero vanishes. Therefore, only the $n = 1$ term of the infinite series in (3–5) contributes to the coincidence limit,

$$A'(0) = \frac{1}{4}\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)} , \tag{3–6}$$

and we have,

$$\lim_{x'\to x} \partial_\alpha \partial'_\beta i\triangle(x;x') = -\frac{1}{2}\frac{H^{D}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)}\,\overline{g}_{\alpha\beta} . \tag{3–7}$$

Substituting (3–7), and the 4-point vertices from Table 2-2, into expression (3–1) gives,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 4pt}(x;x')$$
$$= -\frac{1}{2}\frac{H^{D}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)}\overline{g}_{\alpha\beta} \times i\kappa^2\sqrt{-\overline{g}}\Bigg\{-\frac{1}{4}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}\overline{g}^{\alpha\beta} + \frac{1}{2}\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\overline{g}^{\alpha\beta} + \frac{1}{2}\Big[\overline{g}^{\mu(\alpha}\overline{g}^{\beta)\nu}\overline{g}^{\rho\sigma} + \overline{g}^{\mu\nu}\overline{g}^{\rho(\alpha}\overline{g}^{\beta)\sigma}\Big] - 2\overline{g}^{\alpha(\mu}\overline{g}^{\nu)(\rho}\overline{g}^{\sigma)\beta}\Bigg\}\delta^D(x-x') , \tag{3–8}$$

$$= \Big(\frac{D-4}{4}\Big)\frac{i\kappa^2 H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)}\sqrt{-\overline{g}}\,\Big\{\frac{1}{2}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma} - \overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\Big\}\delta^D(x-x') . \tag{3–9}$$

Because the Gamma functions are finite for $D = 4$ dimensions so we can dispense with dimensional regularization and set $D = 4$. At that point the net contribution (3–9) vanishes.

3.2 Contribution from 3-Point Vertices

The contribution from the leftmost diagram of Figure 1 takes the form,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 3pt}(x;x') = \frac{1}{2}\sum_{I=1}^{2} T_I^{\mu\nu\alpha\beta}(x)\sum_{J=1}^{2} T_J^{\rho\sigma\gamma\delta}(x') \times \partial_\alpha\partial'_\gamma i\triangle(x;x') \times \partial_\beta\partial'_\delta i\triangle(x;x') \; . \tag{3–10}$$

Recall from chapter 2 (section 2.2) that any de Sitter invariant bitensor can be expressed as a linear combination of functions of $y(x;x')$ times the five basis tensors,

$$\begin{aligned} -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 3pt}(x;x') = \sqrt{-\overline{g}}\sqrt{-\overline{g}'}\Bigg\{ &\frac{\partial^2 y}{\partial x_\mu \partial x'_{(\rho}}\frac{\partial^2 y}{\partial x'_{\sigma)}\partial x_\nu}\times\alpha(y) \\ +\frac{\partial y}{\partial x_{(\mu}}\frac{\partial^2 y}{\partial x_{\nu)}\partial x'_{(\rho}}\frac{\partial y}{\partial x'_{\sigma)}}\times\beta(y) &+ \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma}\times\gamma(y) \\ +\overline{g}^{\mu\nu}\overline{g}'^{\rho\sigma}H^4\times\delta(y) + \Big[\overline{g}^{\mu\nu}\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} &+ \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\overline{g}'^{\rho\sigma}\Big]H^2\times\epsilon(y)\Bigg\} \; . \end{aligned} \tag{3–11}$$

By substituting our result (2–32) for the mixed second derivative of the scalar propagator, along with the vertices from Table 2-1, and then making use of the contraction identities (2–20)-(2–24), it is straightforward to obtain expressions for the five coefficient functions,

$$\alpha(y) = -\frac{1}{2}\kappa^2(A')^2 \; , \tag{3–12}$$

$$\beta(y) = -\kappa^2 A'A'' \; , \tag{3–13}$$

$$\gamma(y) = -\frac{1}{2}\kappa^2(A'')^2 \; , \tag{3–14}$$

$$\begin{aligned}\delta(y) = -\frac{1}{8}\kappa^2\Big\{&(A'')^2(4y-y^2)^2 + 2A'A''(2-y)(4y-y^2) \\ &+(A')^2\Big[4(D-4)-(4y-y^2)\Big]\Big\} \; ,\end{aligned} \tag{3–15}$$

$$\epsilon(y) = \frac{1}{4}\kappa^2\Big[(4y-y^2)(A'')^2 + 2(2-y)A'A'' - (A')^2\Big] \; . \tag{3–16}$$

Expressions (3–12)-(3–16) for the coefficient functions have the advantage of being exact for any dimension D, but the disadvantages of being neither very explicit nor very simple functions of $y(x;x')$. We can obtain expressions which are both simple and explicit, and totally adequate for use in the $D=4$ effective field equations, by noting that each pair of terms in the infinite series part of $A(y)$ (2–31) vanishes for $D=4$ spacetime dimensions. Therefore, it is only neceesary to retain those parts of the infinite series which can potentially multiply a divergence. For our computation that turns out to mean only the $n=1$ terms, and we can write the two derivatives as,

$$A' = \frac{\Gamma(\frac{D}{2})H^{D-2}}{4(4\pi)^{\frac{D}{2}}}\Bigg\{-\Big(\frac{4}{y}\Big)^{\frac{D}{2}} - \frac{D}{2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} - \frac{1}{2}\frac{D}{2}\Big(\frac{D}{2}+1\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} + \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)} + \Big(\text{Irrelevant}\Big)\Bigg\}, \qquad (3–17)$$

$$A'' = \frac{\Gamma(\frac{D}{2})H^{D-2}}{16(4\pi)^{\frac{D}{2}}}\Bigg\{\frac{D}{2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}+1} + \Big(\frac{D}{2}-1\Big)\frac{D}{2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}} + \frac{1}{2}\Big(\frac{D}{2}-2\Big)\frac{D}{2}\Big(\frac{D}{2}+1\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \Big(\text{Irrelevant}\Big)\Bigg\}. \qquad (3–18)$$

Substituting these expansions in (3–12)-(3–16) gives,

$$\alpha = \frac{K}{2^5}\Bigg\{-\Big(\frac{4}{y}\Big)^{D} - D\Big(\frac{4}{y}\Big)^{D-1} - \frac{D(D+1)}{2}\Big(\frac{4}{y}\Big)^{D-2} + \frac{2\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big(\frac{4}{y}\Big)^{\frac{D}{2}} + \Big(\text{Irrelevant}\Big)\Bigg\}, \qquad (3–19)$$

$$\beta = \frac{K}{2^7}\Bigg\{D\Big(\frac{4}{y}\Big)^{D+1} + (D-1)D\Big(\frac{4}{y}\Big)^{D} + \frac{1}{2}(D-2)D(D+1)\Big(\frac{4}{y}\Big)^{D-1} - \frac{D\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big(\frac{4}{y}\Big)^{\frac{D}{2}+1} + \Big(\text{Irrelevant}\Big)\Bigg\}, \qquad (3–20)$$

$$\gamma = \frac{K}{2^{11}}\Bigg\{-D^2\Big(\frac{4}{y}\Big)^{D+2} - (D-2)D^2\Big(\frac{4}{y}\Big)^{D+1} - \frac{1}{2}(D^2-3D-2)D^2\Big(\frac{4}{y}\Big)^{D} + \Big(\text{Irrelevant}\Big)\Bigg\}, \qquad (3–21)$$

$$\delta = \frac{K}{2^5}\Bigg\{-(D^2-D-4)\Big(\frac{4}{y}\Big)^{D} - (D^3-5D^2+4D-4)\Big(\frac{4}{y}\Big)^{D-1} - \frac{1}{2}\Big(D^4-8D^3 + 19D^2-28D+8\Big)\Big(\frac{4}{y}\Big)^{D-2} - \frac{8\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big(\frac{4}{y}\Big)^{\frac{D}{2}} + (\text{Irrelevant})\Bigg\}, \qquad (3–22)$$

$$\epsilon = \frac{K}{2^8}\Bigg\{(D-2)D\Big(\frac{4}{y}\Big)^{D+1} + (D^3-5D^2+6D-4)\Big(\frac{4}{y}\Big)^{D} + \frac{1}{2}D\Big(D^3-7D^2 +12D-12\Big)\Big(\frac{4}{y}\Big)^{D-1} + \frac{D\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big(\frac{4}{y}\Big)^{\frac{D}{2}+1} + (\text{Irrelevant})\Bigg\} . \qquad (3\text{–}23)$$

where the constant K is,

$$K \equiv \frac{\kappa^2 H^{2D-4}\Gamma^2(\frac{D}{2})}{(4\pi)^D} . \qquad (3\text{–}24)$$

3.3 Correspondence with Flat Space

An important and illuminating correspondence limit comes from taking the Hubble constant to zero, with the conformal time going to minus infinity so as to keep the physical time t fixed,

$$\eta = -\frac{1}{H}e^{-Ht} = -\frac{1}{H} + t + O(H) . \qquad (3\text{–}25)$$

When this is done the background geometry degenerates to flat space and we should recover well-known results [43]. We will also see in the next chapter that the flat space limit provides crucial guidance in how to reorganize the de Sitter result for renormalization.

Although each independent conformal time diverges under (3–25), the conformal coordinate separation just goes to the usual temporal separation of flat space,

$$\Delta x^0 \longrightarrow t - t' . \qquad (3\text{–}26)$$

All scale factors approach unity, and the de Sitter length function goes to H^2 times the invariant interval of flat space,

$$y(x;x') \longrightarrow H^2 \Delta x^2 . \qquad (3\text{–}27)$$

In the flat space limit the leading behaviors of the various basis tensors are,

$$\frac{\partial y}{\partial x_\mu} \longrightarrow 2H^2\Delta x^\mu \;\; , \;\; \frac{\partial y}{\partial x'_\nu} \longrightarrow -2H^2\Delta x^\nu \;\; , \;\; \frac{\partial y^2}{\partial x_\mu \partial x'_\nu} \longrightarrow -2H^2\eta^{\mu\nu} . \qquad (3\text{–}28)$$

And the leading behaviors for derivatives of the function $A(y)$ are,

$$H^2 A'(y) \longrightarrow -\frac{1}{4\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2})}{(\Delta x^2)^{\frac{D}{2}}} \equiv -\frac{1}{4\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2})}{\Delta x^D} \; , \tag{3–29}$$

$$H^4 A''(y) \longrightarrow \frac{1}{4\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}+1)}{(\Delta x^2)^{\frac{D}{2}+1}} \equiv \frac{1}{4\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}+1)}{\Delta x^{D+2}} \; . \tag{3–30}$$

The 4-point contribution (3–9) to the graviton self-energy vanishes in the flat space limit, even for $D \neq 4$. We can take the flat space limit of the 3-point contribution (3–11) in two steps. First, substitute the leading behaviors (3–27) for $y(x;x')$ and (3–28) for the basis tensors. Then use expressions (3–29)-(3–30) on the derivatives of $A(y)$. The result is,

$$\begin{aligned}
-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\text{flat}}(x;x') = \lim_{H\to 0} \kappa^2 \Bigg\{ & 4H^4 \eta^{\mu(\rho}\eta^{\sigma)\nu} \times -\frac{1}{2}(A')^2 \\
& +8H^6 \Delta x^{(\mu}\eta^{\nu)(\rho}\Delta x^{\sigma)} \times -A'A'' + 16H^8 \Delta x^\mu \Delta x^\nu \Delta x^\rho \Delta x^\sigma \times -\frac{1}{2}(A'')^2 \\
& +H^4 \eta^{\mu\nu}\eta^{\rho\sigma} \times -\frac{1}{8}\Big[16H^4\Delta x^4 (A'')^2 + 16H^2 \Delta x^2 A'A'' + 4(D-4)(A')^2\Big] \\
& +4H^6 \Big[\eta^{\mu\nu}\Delta x^\rho \Delta x^\sigma + \Delta x^\mu \Delta x^\nu \eta^{\rho\sigma}\Big] \times \frac{1}{4}\Big[4H^2\Delta x^2 (A'')^2 + 4A'A''\Big] \Bigg\} \; ,
\end{aligned} \tag{3–31}$$

$$\begin{aligned}
= \frac{\kappa^2 \Gamma^2(\frac{D}{2})}{16\pi^D} \Bigg\{ & \eta^{\mu(\rho}\eta^{\sigma)\nu} \times \Big[-\frac{2}{\Delta x^{2D}}\Big] + \Delta x^{(\mu}\eta^{\nu)(\rho}\Delta x^{\sigma)} \times \Big[\frac{4D}{\Delta x^{2D+2}}\Big] \\
& +\Delta x^\mu \Delta x^\nu \Delta x^\rho \Delta x^\sigma \times \Big[-\frac{2D^2}{\Delta x^{2D+4}}\Big] + \eta^{\mu\nu}\eta^{\rho\sigma} \times \Big[-\frac{1}{2}\frac{(D^2-D-4)}{\Delta x^{2D}}\Big] \\
& +\Big[\eta^{\mu\nu}\Delta x^\rho \Delta x^\sigma + \Delta x^\mu \Delta x^\nu \eta^{\rho\sigma}\Big] \times \Big[\frac{D(D-2)}{\Delta x^{2D+2}}\Big] \Bigg\} \; .
\end{aligned} \tag{3–32}$$

Our result (3–32) agrees with equation (26) of [61].

3.4 Correspondence with Stress Tensor Correlators

Although the flat space limit (3–32) will prove a useful guide when we renormalize in the next section, it does not check the purely de Sitter parts of (3–11). A true de Sitter check is provided by the stress tensor correlators recently derived by Perez-Nadal, Roura and Verdaguer [99]. To exploit their result we first elucidate the relation between the graviton 2-point 1PI function and correlators of the stress tensor. Then we convert their notation to ours.

The Heisenberg equation for the metric field operator coupled to a matter stress tensor $T^{\mu\nu}$ is,

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \frac{1}{2}(D-2)(D-1) H^2 g^{\mu\nu} = \frac{1}{2} \kappa^2 T^{\mu\nu} \; . \tag{3–33}$$

Perturbation theory is implemented by expressing the full metric $g_{\mu\nu} = \overline{g}_{\mu\nu} + \kappa h_{\mu\nu}$ as the sum of a vacuum solution $\overline{g}_{\mu\nu}$ plus κ times the graviton field $h_{\mu\nu}$. Expanding the left hand side of (3–33) in powers of the graviton field gives,

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \frac{1}{2}(D-2)(D-1) H^2 g^{\mu\nu} = \kappa \mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{1}{2} \kappa^2 \Delta \mathcal{T}^{\mu\nu} \; , \tag{3–34}$$

where the nonlinear terms comprise the graviton pseudo-stress tensor $\Delta\mathcal{T}^{\mu\nu}$. The Lichnerowicz operator of the linear term is,

$$\begin{aligned} \mathcal{D}^{\mu\nu\rho\sigma} \equiv D^{(\rho} \overline{g}^{\sigma)(\mu} D^{\nu)} - \frac{1}{2} \Big[\overline{g}^{\rho\sigma} D^{\mu} D^{\nu} + \overline{g}^{\mu\nu} D^{\rho} D^{\sigma} \Big] \\ + \frac{1}{2} \Big[\overline{g}^{\mu\nu} \overline{g}^{\rho\sigma} - \overline{g}^{\mu(\rho} \overline{g}^{\sigma)\nu} \Big] D^2 + (D-1) \Big[\frac{1}{2} \overline{g}^{\mu\nu} \overline{g}^{\rho\sigma} - \overline{g}^{\mu(\rho} \overline{g}^{\sigma)\nu} \Big] H^2 \; , \end{aligned} \tag{3–35}$$

where D^{μ} is the covariant derivative operator in the background geometry. Substituting these expansions in (3–33) and rearranging gives,

$$\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma} = \frac{1}{2} \kappa \Big(T^{\mu\nu} + \Delta \mathcal{T}^{\mu\nu} \Big) \equiv \frac{1}{2} \kappa \mathcal{T}^{\mu\nu} \; . \tag{3–36}$$

We are computing the 1PI graviton 2-point function, which can be obtained from the full graviton 2-point function by eliminating the one particle reducible parts and amputating the external leg propagators. At the one loop order we are working, the one particle reducible part drops out if one computes the correlator of the field minus its expectation value,

$$\delta h_{\mu\nu}(x) \equiv h_{\mu\nu}(x) - \Big\langle \Omega \Big| h_{\mu\nu}(x) \Big| \Omega \Big\rangle \; , \tag{3–37}$$

$$\delta \mathcal{T}^{\mu\nu}(x) \equiv \mathcal{T}^{\mu\nu}(x) - \Big\langle \Omega \Big| \mathcal{T}^{\mu\nu}(x) \Big| \Omega \Big\rangle \; . \tag{3–38}$$

To amputate, recall that the graviton propagator obeys,

$$\sqrt{-\overline{g}(x)}\mathcal{D}^{\mu\nu\alpha\beta} i\Big[{}_{\alpha\beta}\Delta_{\rho\sigma}\Big](x;x') = \delta^{\mu}_{(\rho}\delta^{\nu}_{\sigma)} i\delta^D(x-x') + \Big(\text{Gauge Terms}\Big) \; , \qquad (3\text{–}39)$$

where "Gauge Terms" refers to the extra pieces needed to complete the projection operator onto whatever gauge condition is employed. (For example, the projection operator for de Donder gauge is given in equation (120) of [65].) This means that external leg propagators are amputated by $-i\sqrt{-\overline{g}}$ times the Lichnerowicz operator. Hence the desired relation between the 2-point graviton 1PI function and a 2-point correlator of the stress tensor is,

$$\begin{aligned} -i\Big[{}^{\mu\nu}\Delta^{\rho\sigma}\Big](x;x') & \\ = \Big\langle\Omega\Big|\Big(-i\sqrt{-\overline{g}}\mathcal{D}^{\mu\nu\alpha\beta}\delta h_{\alpha\beta}(x)\Big)\Big(-i\sqrt{-\overline{g}}\mathcal{D}^{\rho\sigma\gamma\delta}\delta h_{\gamma\delta}(x')\Big)\Big|\Omega\Big\rangle + O(\kappa^4) \; , & \qquad (3\text{–}40) \\ = -\frac{1}{4}\kappa^2\sqrt{-\overline{g}(x)}\,\sqrt{-\overline{g}(x')}\Big\langle\Omega\Big|\delta\mathcal{T}^{\mu\nu}(x)\delta\mathcal{T}^{\rho\sigma}(x')\Big|\Omega\Big\rangle + O(\kappa^4) \; . & \qquad (3\text{–}41) \end{aligned}$$

The expectation value on the right hand side of (3–41) is the stress tensor correlator $F^{\mu\nu\rho\sigma}$ of Perez-Nadal, Roura and Verdaguer [99].

Perez-Nadal, Roura and Verdaguer actually derived $F^{\mu\nu\rho\sigma}$ for a scalar with arbitrary mass, but we can compare our result (3–11) for the massless case with their equation (28) [99]

$$\begin{aligned} F_{\mu\nu\rho\sigma} = {} & P(\mu)n_\mu n_\nu n_\rho n_\sigma + Q(\mu)(n_\mu n_\nu \overline{g}_{\rho\sigma} + n_\rho n_\sigma \overline{g}_{\mu\nu}) \\ & + R(\mu)(n_\mu n_\rho \overline{g}_{\nu\sigma} + n_\nu n_\sigma \overline{g}_{\mu\rho} + n_\mu n_\sigma \overline{g}_{\nu\rho} + n_\nu n_\rho \overline{g}_{\mu\sigma}) \\ & + S(\mu)(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma} + \overline{g}_{\nu\rho}\overline{g}_{\mu\sigma}) + T(\mu)\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma} \; . \qquad (3\text{–}42) \end{aligned}$$

Note that here they expressed the stress tensor correlator in terms of five basis tensors which are different from ours given in equation (3–11). Each of these five bitensors are formed as a linear combination of products of the de Sitter invariant bitensors, n_a, $n_{a'}$, $\overline{g}_{ab}$, $\overline{g}_{a'b'}$ and $\overline{g}_{ab'}$. The variable μ and bitensors are defined as [99]:

- $\mu(x,x')$: the distance along the shortest geodesic joining x and x', also called the geodesic distance;
- n_a and $n_{a'}$: the unit vectors tangent to the geodesic at the points x and x' respectively, pointing outward from it;
- $\overline{g}_{ab'}$: the parallel propagator which parallel-transports a vector from x to x' along the geodesic;
- $\overline{g}_{ab}$ and $\overline{g}_{a'b'}$: the metric tensors at the points x and x' respectively.

The distance $\mu(x,x')$ (in our notation $\mu(x,x') = H\ell(x;x')$ which is given in section 2) corresponds to our de Sitter invariant function $y(x,x')$ with the relation,

$$\cos(\mu) \equiv Z = 1 - \frac{y}{2} \,. \tag{3–43}$$

In comparing their results with ours it is also useful to note the relations between their basis tensors and ours,

$$\begin{aligned}
n_a &= \frac{1}{H\sqrt{y(4-y)}}\frac{\partial y}{\partial x^a} \,, &(3–44)\\
n_{b'} &= \frac{1}{H\sqrt{y(4-y)}}\frac{\partial y}{\partial x'^{b'}} \,, &(3–45)\\
\overline{g}_{ab'} &= -\frac{1}{2H^2}\left\{\frac{\partial^2 y}{\partial x^a \partial x'^{b'}} + \frac{1}{4-y}\frac{\partial y}{\partial x^a}\frac{\partial y}{\partial x'^{b'}}\right\} . &(3–46)
\end{aligned}$$

Thus the five basis tensors given in (3–42) are converted into our basis tensors as,

$$\begin{aligned}
n_a n_b n_{c'} n_{d'} &= \frac{1}{H^4(4y-y^2)^2}\frac{\partial y}{\partial x^a}\frac{\partial y}{\partial x^b}\frac{\partial y}{\partial x'^{c'}}\frac{\partial y}{\partial x'^{d'}} \,, &(3–47)\\
n_a n_b \overline{g}_{c'd'} + n_{c'} n_{d'} \overline{g}_{ab} &= \frac{1}{H^2(4y-y^2)}\left[\overline{g}_{ab}\frac{\partial y}{\partial x'^{c'}}\frac{\partial y}{\partial x'^{d'}} + \frac{\partial y}{\partial x^a}\frac{\partial y}{\partial x^b}\overline{g}_{c'd'}\right] , &(3–48)\\
4n_{(a}\overline{g}_{b)(c'} n_{d')} &= -\frac{2}{H^4(4y-y^2)}\frac{\partial y}{\partial x^{(a}}\frac{\partial^2 y}{\partial x^{b)}\partial x'^{(c'}}\frac{\partial y}{\partial x'^{d')}} \\
&\quad -\frac{2}{H^4(4y-y^2)(4-y)}\frac{\partial y}{\partial x^a}\frac{\partial y}{\partial x^b}\frac{\partial y}{\partial x'^{c'}}\frac{\partial y}{\partial x'^{d'}} \,, &(3–49)\\
2\overline{g}_{a(c'}\overline{g}_{d')b} &= \frac{1}{2H^4}\frac{\partial^2 y}{\partial x^a \partial x'^{(c'}}\frac{\partial^2 y}{\partial x^{d')}\partial x'^{b}} + \frac{1}{H^4(4-y)}\frac{\partial y}{\partial x^{(a}}\frac{\partial^2 y}{\partial x^{b)}\partial x'^{(c'}}\frac{\partial y}{\partial x'^{d')}} \\
&\quad + \frac{1}{2H^4}\frac{1}{(4-y)^2}\frac{\partial y}{\partial x^a}\frac{\partial y}{\partial x^b}\frac{\partial y}{\partial x'^{c'}}\frac{\partial y}{\partial x'^{d'}} \,, &(3–50)\\
\overline{g}_{ab}\overline{g}_{c'd'} &= \overline{g}_{ab}\overline{g}_{c'd'} \,. &(3–51)
\end{aligned}$$

(Note that we have restored the factor of H which Perez-Nadal, Roura and Veraguer set to unity.)

For a massless, minimally coupled scalar field, the μ-dependent coefficients are [99],

$$\begin{aligned} P &= 2G_1^2 \; , \\ Q &= -G_1^2 + 2G_1 G_2 \; , \\ R &= G_1 G_2 \; , \\ S &= G_2^2 \; , \\ T &= \frac{1}{2} G_1^2 - G_1 G_2 + \frac{D-4}{2} G_2^2 \; . \end{aligned} \tag{3–52}$$

Here the G_1 and G_2 are defined as

$$\begin{aligned} G_1(\mu) &= G''(\mu) - G'(\mu) \csc(\mu) \; , \\ G_2(\mu) &= -G'(\mu) \csc(\mu) \; , \end{aligned} \tag{3–53}$$

where prime stands for derivative with respect to μ.

The comparison can be completed by noting that the Wightman function $G(\mu)$ becomes almost the same as our $A(y)$ for the case of MMC scalar. In the massless limit, their propagator has the formal expansion,

$$G(\mu) = \frac{H^{D-2}}{(4\pi)^{D/2}} \sum_{n=0}^{\infty} \frac{\Gamma(D-1+n)\Gamma(n)}{\Gamma(\frac{D}{2}+n)} \frac{1}{n!} \Big(\frac{1+Z}{2}\Big)^n \; . \tag{3–54}$$

(Note that we have restored the factor of H^{D-2} which Perez-Nadal, Roura and Veraguer set to unity.) Recalling the hypergeometric function,

$${}_2F_1\Big(\alpha, \beta; \gamma; z\Big) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} \frac{z^n}{n!} \; , \tag{3–55}$$

we see that $G(Z)$ can be written as,

$$G(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)\Gamma(0)}{\Gamma(\frac{D}{2})} \, {}_2F_1\Big(D-1, 0; \frac{D}{2}; 1-\frac{y}{4}\Big) \; . \tag{3–56}$$

Now we use one of the transformation formulae for hypergeometric functions (See for example, 9.131 of [66]) to expand G in powers of $y/4$:

$$G(y) = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Bigg\{\frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1}+\frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2}-\Gamma(0)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} + \sum_{n=1}^{\infty}\Bigg[\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^{n} - \frac{1}{n-\frac{D}{2}+2}\frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}\Bigg]\Bigg\}. \tag{3–57}$$

So we see that $G(y)$ is the same as the function $A(y)$ except for the replacement,

$$\Gamma(0)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \longrightarrow \pi\cot\Big(\frac{\pi D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\ . \tag{3–58}$$

This makes no difference because $G(y)$ only enters the stress tensor correlator (3–42) differentiated (See equations (3–52)-(3–53)). Thus for comparison, we replace the derivatives of G by the ones of A:

$$\begin{aligned}\frac{\partial G}{\partial\mu} &= \sqrt{4y-y^2}\,G' \equiv \sqrt{4y-y^2}\,A'\ ,\\ \frac{\partial^2 G}{\partial\mu^2} &= (4y-y^2)G''+(2-y)G' \equiv (4y-y^2)A''+(2-y)A'\ .\end{aligned} \tag{3–59}$$

Here the prime stand for derivative with respect to y. Then the coefficients P, Q, R, S and T given in equation (3–52) are written in terms of y as

$$\begin{aligned}P &= 2(4y-y^2)^2(A'')^2 - 4y(4y-y^2)A''A' + 2y^2(A')^2\ ,\\ Q &= -(4y-y^2)^2(A'')^2 - 2(2-y)(4y-y^2)A''A' + (4y-y^2)(A')^2\ .\\ R &= -2(4y-y^2)A''A' + 2y(A')^2\ ,\\ S &= 4(A')^2\ ,\\ T &= \frac{1}{2}\Big[(4y-y^2)^2(A'')^2 + 2(2-y)(4y-y^2)A''A' \\ &\qquad + \{4(D-4)-(4y-y^2)\}(A')^2\Big]\ .\end{aligned} \tag{3–60}$$

With this equation (3–60) and the conversion of basis given in equations (3–47)-(3–51) we can arrange $F_{\mu\nu\rho\sigma}$ for the MMC scalar in terms of our basis tensors,

$$F_{\mu\nu\rho\sigma} = -\frac{4}{\kappa^2}\Bigg\{ \frac{\partial^2 y}{\partial x^\mu \partial x'^{(\rho}} \frac{\partial^2 y}{\partial x'^{\sigma)} \partial x^\nu} \times \alpha(y)$$
$$+ \frac{\partial y}{\partial x^{(\mu}} \frac{\partial^2 y}{\partial x^{\nu)} \partial x'^{(\rho}} \frac{\partial y}{\partial x'^{\sigma)}} \times \beta(y) + \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu} \frac{\partial y}{\partial x'^\rho} \frac{\partial y}{\partial x'^\sigma} \times \gamma(y)$$
$$+ \overline{g}^{\mu\nu} \overline{g}'^{\rho\sigma} H^4 \times \delta(y) + \Big[\overline{g}^{\mu\nu} \frac{\partial y}{\partial x'^\rho} \frac{\partial y}{\partial x'^\sigma} + \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x^\nu} \overline{g}'^{\rho\sigma} \Big] H^2 \times \epsilon(y) \Bigg\} . \tag{3–61}$$

$$= -\frac{4}{\kappa^2} \times \frac{1}{\sqrt{-\overline{g}(x)}\sqrt{-\overline{g}(x')}} \times -i \Big[{}_{\mu\nu}\Sigma_{\rho\sigma} \Big]_{\text{3pt}}(x; x') . \tag{3–62}$$

CHAPTER 4
RENORMALIZATION

Our result (3–11) is valid as long as $x'^{\mu} \neq x^{\mu}$, either with the exact coefficient functions (3–12)-(3–16) or with the relevant expansions (3–19)-(3–23) for $D = 4$. However, it is not immediately usable in the quantum-corrected, linearized Einstein equations because they involve an integration over x'^{μ},

$$\sqrt{-g}\,\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) - \int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm ren}(x;x') h_{\rho\sigma}(x') = \frac{1}{2}\kappa\sqrt{-g}\; T^{\mu\nu}_{\rm lin}(x)\;. \tag{4–1}$$

To obtain a usable form we must express (3–11) as a product of up to six differential operators acting upon a function of $y(x;x')$ which is integrable in $D = 4$ spacetime dimensions. The derivatives with respect to x^{μ} can be pulled outside the integral, and those with respect to x'^{μ} can be partially integrated to act back on the $h_{\rho\sigma}(x')$,[1] leaving an expression for which the $D = 4$ limit could be taken were it not for some factors of $1/(D-4)$. At this stage one adds zero in the form of identities such as,

$$\left[\square - \frac{D}{2}\Big(\frac{D}{2}-1\Big)H^2\right]\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} - \frac{(4\pi)^{\frac{D}{2}}\, i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1)H^{D-2}\sqrt{-g}} = 0\;. \tag{4–2}$$

We combine (4–2) with terms which arise from extracting derivatives to segregate the divergences on local, delta function terms, for example,

$$\frac{1}{D-4}\left[\square - \frac{D}{2}\Big(\frac{D}{2}-1\Big)H^2\right]\Big(\frac{4}{y}\Big)^{D-3}$$
$$= \left[\square - \frac{D}{2}\Big(\frac{D}{2}-1\Big)H^2\right]\left\{\frac{(\frac{4}{y})^{D-3} - (\frac{4}{y})^{\frac{D}{2}-1}}{D-4}\right\} + \frac{(4\pi)^{\frac{D}{2}}\, i\delta^D(x-x')/\sqrt{-g}}{(D-4)\Gamma(\frac{D}{2}-1)H^{D-2}}\;, \tag{4–3}$$
$$= -\frac{1}{2}\Big[\square - 2H^2\Big]\left\{\frac{4}{y}\ln\Big(\frac{y}{4}\Big)\right\} + O(D-4) + \frac{(4\pi)^{\frac{D}{2}}\, i\delta^D(x-x')/\sqrt{-g}}{(D-4)\Gamma(\frac{D}{2}-1)H^{D-2}}\;. \tag{4–4}$$

Renormalization consists of subtracting off the divergent delta functions with counterterms. In section 4.1 we exhibit the one loop counterterms for quantum gravity. We review how

[1] The resulting surface terms can be absorbed by correcting the initial state [67].

to renormalize the flat space limit (3–32) in section 4.2. That suggests a convenient way of organizing the tensor algebra into two transverse, 4th order differential operators, one with spin zero and the other with spin two. In section 4.3 we implement this for de Sitter. The spin zero part is renormalized in section 4.4, and the spin two part in section 4.5.

4.1 One Loop Counterterms

Gravity + Scalar is not renormalizable in $D = 4$ dimensions [9]. However, the theorem of Bogoliubov, Parasiuk, Hepp and Zimmerman (BPHZ) shows us how to construct local counterterms which absorb the ultraviolet divergences of any quantum field theory to any fixed order in the loop expansion [10]. For quantum gravity at one loop order the necessary counterterms can be taken to be the squares of the Ricci scalar and the Weyl tensor [9]. The problem of quantum gravity is that the Weyl counterterm would destabilize the universe if it were regarded as a fundamental, nonperturbative interaction [68]. We shall therefore consider it only perturbatively, in the sense of effective field theory, as a proxy for the yet unknown ultraviolet completion of quantum gravity. The quantum effects we seek to study derive from infrared virtual scalars with wavelengths on the order of the Hubble radius, and they will manifest as nonlocal and ultraviolet finite contributions to the graviton self-energy which are not affected by how nature resolves the ultraviolet problem of quantum gravity.

Because the background Ricci scalar is nonzero it is useful to reorganize R^2 into a part which is quadratic in the graviton field,

$$R^2 = \Bigl[R - D(D-1)H^2\Bigr]^2 + 2D(D-1)H^2 R - D^2(D-1)^2 H^4 \; . \tag{4–5}$$

So we will employ four counterterms,

$$\Delta\mathcal{L}_1 \equiv c_1 \Bigl[R - D(D-1)H^2\Bigr]^2 \sqrt{-g} \; , \tag{4–6}$$

$$\Delta\mathcal{L}_2 \equiv c_2 C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \sqrt{-g} \; , \tag{4–7}$$

$$\Delta\mathcal{L}_3 \equiv c_3 H^2 \Big[R - (D-1)(D-2)H^2\Big]\sqrt{-g} \; , \tag{4–8}$$

$$\Delta\mathcal{L}_4 \equiv c_4 H^4 \sqrt{-g} \; . \tag{4–9}$$

Of course the divergences can really be eliminated with just $\Delta\mathcal{L}_2$ and the particular linear combination of $\Delta\mathcal{L}_1$, $\Delta\mathcal{L}_3$ and $\Delta\mathcal{L}_4$ which is proportional to just $R^2\sqrt{-g}$. It must therefore be that two linear combinations of the coefficients are finite,

$$\lim_{D\to 4}\Big[-2D(D-1)c_1 + c_3\Big] = \text{Finite} \; , \tag{4–10}$$

$$\lim_{D\to 4}\Big[D^2(D-1)^2 c_1 - (D-1)(D-2)c_3 + c_4\Big] = \text{Finite} \; . \tag{4–11}$$

And the divergent parts of c_1 and c_2 must agree with the values obtained long ago by 't Hooft and Veltman [9].

At this point we digress to define two 2nd order differential operators of great importance to our subsequent analysis. They come from expanding the scalar and Weyl curvatures around de Sitter background,

$$R - D(D-1)H^2 \equiv \mathcal{P}^{\mu\nu}\kappa h_{\mu\nu} + O(\kappa^2 h^2) \; , \tag{4–12}$$

$$C_{\alpha\beta\gamma\delta} \equiv \mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}\kappa h_{\mu\nu} + O(\kappa^2 h^2) \; . \tag{4–13}$$

From (4–12) we have,

$$\mathcal{P}^{\mu\nu} = D^\mu D^\nu - \overline{g}^{\mu\nu}\Big[D^2 + (D-1)H^2\Big] \; , \tag{4–14}$$

where D^μ is the covariant derivative operator in de Sitter background. The more difficult expansion of the Weyl tensor gives,

$$\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta} = \mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} + \frac{1}{D-2}\Big[\overline{g}_{\alpha\delta}\mathcal{D}^{\mu\nu}_{\beta\gamma} - \overline{g}_{\beta\delta}\mathcal{D}^{\mu\nu}_{\alpha\gamma} - \overline{g}_{\alpha\gamma}\mathcal{D}^{\mu\nu}_{\beta\delta} + \overline{g}_{\beta\gamma}\mathcal{D}^{\mu\nu}_{\alpha\delta}\Big] + \frac{1}{(D-1)(D-2)}\Big[\overline{g}_{\alpha\gamma}\overline{g}_{\beta\delta} - \overline{g}_{\alpha\delta}\overline{g}_{\beta\gamma}\Big]\mathcal{D}^{\mu\nu} \; , \tag{4–15}$$

where we define,

$$\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} \equiv \frac{1}{2}\Big[\delta^{(\mu}_{\alpha}\delta^{\nu)}_{\delta} D_\gamma D_\beta - \delta^{(\mu}_{\beta}\delta^{\nu)}_{\delta} D_\gamma D_\alpha - \delta^{(\mu}_{\alpha}\delta^{\nu)}_{\gamma} D_\delta D_\beta + \delta^{(\mu}_{\beta}\delta^{\nu)}_{\gamma} D_\delta D_\alpha\Big] , \tag{4–16}$$

$$\mathcal{D}^{\mu\nu}_{\beta\delta} \equiv \overline{g}^{\alpha\gamma}\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} = \frac{1}{2}\Big[\delta^{(\mu}_{\delta} D^{\nu)} D_\beta - \delta^{(\mu}_{\beta}\delta^{\nu)}_{\delta} D^2 - \overline{g}^{\mu\nu} D_\delta D_\beta + \delta^{(\mu}_{\beta} D_\delta D^{\nu)}\Big] , \tag{4–17}$$

$$\mathcal{D}^{\mu\nu} \equiv \overline{g}^{\alpha\gamma}\overline{g}^{\beta\delta}\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} = D^{(\mu}D^{\nu)} - \overline{g}^{\mu\nu} D^2 . \tag{4–18}$$

One obtains the counterterm vertices by functionally differentiating i times each counterterm action twice, and then setting the graviton field to zero. They are,

$$\frac{i\delta\Delta S_1}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigg|_{h=0} = 2c_1\kappa^2\sqrt{-\overline{g}}\,\mathcal{P}^{\mu\nu}\mathcal{P}^{\rho\sigma} i\delta^D(x-x') , \tag{4–19}$$

$$\frac{i\delta\Delta S_2}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigg|_{h=0} = 2c_2\kappa^2\sqrt{-\overline{g}}\,\overline{g}^{\alpha\kappa}\overline{g}^{\beta\lambda}\overline{g}^{\gamma\theta}\overline{g}^{\delta\phi}\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi} i\delta^D(x-x') , \tag{4–20}$$

$$\frac{i\delta\Delta S_3}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigg|_{h=0} = -c_3\kappa^2 H^2\sqrt{-\overline{g}}\,\mathcal{D}^{\mu\nu\rho\sigma} i\delta^D(x-x') , \tag{4–21}$$

$$\frac{i\delta\Delta S_4}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigg|_{h=0} = c_4\kappa^2 H^4\sqrt{-\overline{g}}\,\Big[\frac{1}{4}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma} - \frac{1}{2}\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\Big] i\delta^D(x-x') . \tag{4–22}$$

Recall that the Lichnerowicz operator in expression (4–21) was defined in expression (3–35). Also note the flat space limits,

$$\frac{i\delta\Delta S_1}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigg|_{h=0} \longrightarrow 2c_1\kappa^2\Pi^{\mu\nu}\Pi^{\rho\sigma} i\delta^D(x-x') , \tag{4–23}$$

$$\frac{i\delta\Delta S_2}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigg|_{h=0} \longrightarrow 2c_2\kappa^2\Big(\frac{D-3}{D-2}\Big)\Big[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{D-1}\Big] i\delta^D(x-x') , \tag{4–24}$$

$$\frac{i\delta\Delta S_3}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigg|_{h=0} \longrightarrow 0 , \tag{4–25}$$

$$\frac{i\delta\Delta S_4}{\delta h_{\mu\nu}(x)\delta h_{\rho\sigma}(x')}\Bigg|_{h=0} \longrightarrow 0 , \tag{4–26}$$

where we define,

$$\Pi^{\mu\nu} \equiv \partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2 . \tag{4–27}$$

4.2 Renormalizing the Flat Space Result

Renormalizing the flat space result (3–32) provides an excellent guide for the vastly more complicated reduction required on de Sitter background. We begin by extracting a 4th order differential operator from each term using the identities,

$$\frac{1}{\Delta x^{2D}} = \frac{\partial^4}{4(D-2)^2(D-1)D} \frac{1}{\Delta x^{2D-4}} , \tag{4–28}$$

$$\frac{\Delta x^\mu \Delta x^\nu}{\Delta x^{2D+2}} = \frac{1}{8(D-2)^2(D-1)D} \left\{ \partial^\mu \partial^\nu \partial^2 + \frac{\eta^{\mu\nu}\partial^4}{D} \right\} \frac{1}{\Delta x^{2D-4}} , \tag{4–29}$$

$$\frac{\Delta x^\mu \Delta x^\nu \Delta x^\rho \Delta x^\sigma}{\Delta x^{2D+4}} = \frac{1}{16(D-2)(D-1)D(D+1)} \Biggl\{ \partial^\mu \partial^\nu \partial^\rho \partial^\sigma + \frac{6}{D-2} \eta^{(\mu\nu} \partial^\rho \partial^{\sigma)} \partial^2 + \frac{3}{(D-2)D} \eta^{(\mu\nu} \eta^{\rho\sigma)} \partial^4 \Biggr\} \frac{1}{\Delta x^{2D-4}} . \tag{4–30}$$

Substituting these relations into (3–32), and then organizing the various derivatives into factors of the transverse operator $\Pi^{\mu\nu}$ of expression (4–27), gives a manifestly transverse form,

$$-i \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\text{flat}}(x;x') = \frac{\kappa^2 \Gamma^2(\frac{D}{2})}{16\pi^D} \left\{ -\frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{8(D-1)^2} - \frac{[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{D-1}\Pi^{\mu\nu}\Pi^{\rho\sigma}]}{4(D-2)^2(D-1)(D+1)} \right\} \frac{1}{\Delta x^{2D-4}} . \tag{4–31}$$

Let us pause at this point to note that we could have guessed most of the form of expression (4–31). Gauge invariance implies transversality. We also have Poincaré invariance, symmetry under the interchanges $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$, and symmetry under interchange of the primed and unprimed coordinates and indices. All this implies the form,

$$-i \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\text{flat}}(x;x') = \Pi^{\mu\nu}\Pi^{\rho\sigma} F_1(\Delta x^2) + \left[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{D-1} \right] F_2(\Delta x^2) . \tag{4–32}$$

Taking the trace of this and our result (3–32) against $\eta_{\mu\nu}\eta_{\rho\sigma}$ gives an equation for the spin zero structure function $F_1(\Delta x^2)$,

$$\eta_{\mu\nu}\eta_{\rho\sigma} \times -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\text{flat}} = (D-1)^2\partial^4 F_1(\Delta x^2) = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D} \times -\frac{(D-2)^2(D-1)D}{2\Delta x^{2D}} \; . \qquad (4\text{–}33)$$

Of course the solution is just what we found in (4–31) by direct computation,

$$F_1(\Delta x^2) = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D} \times -\frac{1}{8(D-1)^2}\left(\frac{1}{\Delta x^2}\right)^{D-2} . \qquad (4\text{–}34)$$

Determining the spin two structure function $F_2(\Delta x^2)$ is done by first acting the derivatives on the spin zero structure function,

$$\begin{aligned}\Pi^{\mu\nu}\Pi^{\rho\sigma}F_1 &= \eta^{\mu(\rho}\eta^{\sigma)\nu} \times 8F_1'' + \Delta x^{(\mu}\eta^{\nu)(\rho}\Delta x^{\sigma)} \times 32F_1''' + \Delta x^\mu\Delta x^\nu\Delta x^\rho\Delta x^\sigma \\ &\quad \times 16F_1'''' + \eta^{\mu\nu}\eta^{\rho\sigma} \times \Big[4(D^2-3)F_1'' + 16(D+1)\Delta x^2F_1''' + 16\Delta x^4F_1''''\Big] \\ &\quad + \Big[\eta^{\mu\nu}\Delta x^\rho\Delta x^\sigma + \Delta x^\mu\Delta x^\nu\eta^{\rho\sigma}\Big] \times \Big[-8(D+3)F_1''' - 16\Delta x^2F_1''''\Big] \; . \qquad (4\text{–}35)\end{aligned}$$

We subtract these from each tensor factor in (3–32) and then act the spin two operator $[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{D-1}\Pi^{\mu\nu}\Pi^{\rho\sigma}]$ on $F_2(\Delta x^2)$ to read off an equation for each of the five tensor factors,

$$\begin{aligned}\eta^{\mu(\rho}\eta^{\sigma)\nu} \Rightarrow \frac{4(D-2)D(D+1)}{D-1}\,F_2'' + 16(D+1)\Delta x^2F_2''' + 16\Delta x^4F_2'''' \\ = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\left\{-\frac{D}{D-1}\frac{1}{\Delta x^{2D}}\right\} , \qquad (4\text{–}36)\end{aligned}$$

$$\begin{aligned}\Delta x^{(\mu}\eta^{\nu)(\rho}\Delta x^{\sigma)} \Rightarrow -\frac{16D(D+1)}{D-1}\,F_2''' - 32\Delta x^2F_2'''' \\ = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\left\{\frac{4D}{D-1}\frac{1}{\Delta x^{2D}}\right\} , \qquad (4\text{–}37)\end{aligned}$$

$$\Delta x^\mu\Delta x^\nu\Delta x^\rho\Delta x^\sigma \Rightarrow 16\Big(\frac{D-2}{D-1}\Big)F_2'''' = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\left\{-\frac{4D}{D-1}\frac{1}{\Delta x^{2D}}\right\} , \qquad (4\text{–}38)$$

$$\eta^{\mu\nu}\eta^{\rho\sigma} \Rightarrow -\frac{4}{D-1}\Big[(D-2)(D+1)F_2'' + 4(D+1)\Delta x^2 F_2''' + 4\Delta x^4 F_2''''\Big]$$
$$= \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\left\{\frac{1}{D-1}\frac{1}{\Delta x^{2D}}\right\} , \qquad (4\text{–}39)$$
$$\Big[\eta^{\mu\nu}\Delta x^\rho\Delta x^\sigma + \Delta x^\mu\Delta x^\nu\eta^{\rho\sigma}\Big] \Rightarrow \frac{16}{D-1}\Big[(D+1)F_2''' + \Delta x^2 F_2''''\Big] = 0 \; . \qquad (4\text{–}40)$$

Each of these equations has the same solution, which of course agrees with (4–31),

$$F_2(\Delta x^2) = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D} \times -\frac{1}{4(D-2)^2(D-1)(D+1)}\Big(\frac{1}{\Delta x^2}\Big)^{D-2} . \qquad (4\text{–}41)$$

We note for future reference that a particular linear combination of the five relations (4–36)-(4–40) gives a second order equation for $F_2(\Delta x^2)$,

$$(4-39) + \Delta x^2(4-40) = -\frac{4}{D-1}(D-2)(D+1)F_2'' = \frac{\kappa^2\Gamma^2(\frac{D}{2})}{16\pi^D}\left\{\frac{1}{D-1}\frac{1}{\Delta x^{2D}}\right\} . \qquad (4\text{–}42)$$

Even after extracting the 4th order differential operators from the integration of (4–1), the factor of $1/\Delta x^{2D-4}$ is logarithmically divergent. We must therefore extract one more d'Alembertian,

$$\Big(\frac{1}{\Delta x^2}\Big)^{D-2} = \frac{\partial^2}{2(D-3)(D-4)}\Big(\frac{1}{\Delta x^2}\Big)^{D-3} . \qquad (4\text{–}43)$$

After this final derivative is extracted the integrand converges, however, we still cannot take the $D = 4$ limit owing to the factor of $1/(D-4)$. The solution is to add zero in the form of the identity,

$$\partial^2\Big(\frac{1}{\Delta x^2}\Big)^{\frac{D}{2}-1} - \frac{4\pi^{\frac{D}{2}}\, i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1)} = 0 \; . \qquad (4\text{–}44)$$

To make this dimensionally consistent with (4–43) we must multiply by the dimensional regualrization mass scale μ raised to the $(D-4)$ power,

$$\Big(\frac{1}{\Delta x^2}\Big)^{D-2} = \frac{\partial^2}{2(D-3)(D-4)}\left\{\frac{1}{\Delta x^{2D-6}} - \frac{\mu^{D-4}}{\Delta x^{D-2}}\right\} + \frac{4\pi^{\frac{D}{2}}\mu^{D-4} i\delta^D(x-x')}{2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} ,$$
$$= -\frac{1}{4}\partial^2\left\{\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2} + O(D-4)\right\} + \frac{4\pi^{\frac{D}{2}}\mu^{D-4} i\delta^D(x-x')}{2(D-3)(D-4)\Gamma(\frac{D}{2}-1)} . \qquad (4\text{–}45)$$

The divergences have now been segregated on delta function terms which can be removed with local counterterms. From expressions (4–23)-(4–26) we see that the counterterms make the following contribution to the graviton self-energy,

$$-i\Big[{}^{\mu\nu}\Delta\Sigma^{\rho\sigma}\Big]_{\rm flat}(x;x') = \Pi^{\mu\nu}\Pi^{\rho\sigma}\Big\{2c_1\kappa^2 i\delta^D(x-x')\Big\}$$
$$+\Big[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{D-1}\Big]\Big\{2\Big(\frac{D-3}{D-2}\Big)c_2\kappa^2 i\delta^D(x-x')\Big\} . \tag{4–46}$$

The delta function terms will be entirely absorbed by choosing the constants c_1 and c_2 as,

$$c_1 = \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{2^8\pi^{\frac{D}{2}}}\frac{(D-2)}{(D-1)^2(D-3)(D-4)} , \tag{4–47}$$
$$c_2 = \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{2^8\pi^{\frac{D}{2}}}\frac{2}{(D+1)(D-1)(D-3)^2(D-4)} . \tag{4–48}$$

Of course the divergent parts agree with the results obtained long ago by 't Hooft and Veltman [9], with the arbitrary finite parts represented by μ. The fully renormalized graviton self-energy (for flat space background) is,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\substack{\rm ren\\ \rm flat}} = \lim_{D\to 4}\Big\{-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm flat}(x;x') - i\Big[{}^{\mu\nu}\Delta\Sigma^{\rho\sigma}\Big]_{\rm flat}(x;x')\Big\} , \tag{4–49}$$
$$= \Pi^{\mu\nu}\Pi^{\rho\sigma}\partial^2\Big\{\frac{\kappa^2}{2^9 3^2\pi^4}\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big\}$$
$$+\Big[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{3}\Pi^{\mu\nu}\Pi^{\rho\sigma}\Big]\partial^2\Big\{\frac{\kappa^2}{2^{10}3^1 5^1\pi^4}\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big\} . \tag{4–50}$$

4.3 The de Sitter Structure Functions

We must now extend the flat space ansatz (4–32) to de Sitter and determine the resulting structure functions by comparison with the explicit result (3–11) of section 3. As before, gauge invariance implies transversality, which suggests that we make use of the differential operators $\mathcal{P}^{\mu\nu}$ and $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}$ which were defined in expressions (4–14) and (4–15), respectively. In place of Poincaré invariance we now have de Sitter invariance. We also have symmetry under the interchanges $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$, and under interchange of the primed and unprimed coordinates and indices. A simple generalization is,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = \sqrt{-\overline{g}(x)}\,\mathcal{P}^{\mu\nu}(x)\sqrt{-\overline{g}(x')}\,\mathcal{P}^{\rho\sigma}(x')\Big\{\mathcal{F}_1(y)\Big\}$$
$$+\sqrt{-\overline{g}(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\sqrt{-\overline{g}(x')}\,\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')\Bigg\{\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big(\frac{D-2}{D-3}\Big)\mathcal{F}_2(y)\Bigg\}, \qquad (4–51)$$

where the bitensor $\mathcal{T}^{\alpha\kappa}$ is,[2]

$$\mathcal{T}^{\alpha\kappa}(x;x') \equiv -\frac{1}{2H^2}\,\frac{\partial^2 y(x;x')}{\partial x_\alpha \partial x'_\kappa}\;. \qquad (4–52)$$

As in flat space, the second term is traceless.

Note the flat space limits of the bitensor and the two structure functions,

$$\lim_{H\to 0}\mathcal{T}^{\alpha\kappa} = \eta^{\kappa\lambda} \quad , \quad \lim_{H\to 0}\mathcal{F}_1(y) = F_1(\Delta x^2) \quad , \quad \lim_{H\to 0}\mathcal{F}_2(y) = F_2(\Delta x^2)\;. \qquad (4–53)$$

These limits mean one can immediately read off the most singular parts of the expansions for each structure function from the corresponding flat space result,

$$\mathcal{F}_1(y) = \frac{\kappa^2 H^{2D-4}\Gamma^2(\frac{D}{2})}{(4\pi)^D}\Bigg\{\frac{-1}{8(D-1)^2}\Big(\frac{4}{y}\Big)^{D-2} + \dots\Bigg\}, \qquad (4–54)$$
$$\mathcal{F}_2(y) = \frac{\kappa^2 H^{2D-4}\Gamma^2(\frac{D}{2})}{(4\pi)^D}\Bigg\{\frac{-1}{4(D-3)(D-2)(D-1)(D+1)}\Big(\frac{4}{y}\Big)^{D-2} + \dots\Bigg\}. \qquad (4–55)$$

The interesting de Sitter physics we seek to elucidate derives from the subdominant terms.

Just as for the flat space limit, we can obtain an equation for the spin zero structure function by tracing (4–51) and then comparing with the trace of the explicit computation

[2] One could actually employ any bitensor — for example, the parallel transport matrix (3–46) — which reduces to $\eta^{\alpha\kappa}$ in the flat space limit. Different choices for $\mathcal{T}^{\alpha\kappa}(x;x')$ make corresponding changes in the subdominant parts of the spin two structure function $\mathcal{F}_2(y)$. We have not troubled to determine the "simplest" choice.

(3–11). Tracing the ansatz gives,

$$\frac{\overline{g}_{\mu\nu}(x)}{\sqrt{-\overline{g}(x)}} \times \frac{\overline{g}_{\rho\sigma}(x')}{\sqrt{-\overline{g}(x')}} \times -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = (D-1)^2 \Big[\square + DH^2\Big]\Big[\square' + DH^2\Big]\mathcal{F}_1(y)\; . \quad (4\text{–}56)$$

Tracing the explicit result (3–11), substituting (3–12)-(3–16), and then making use of (2–33) gives,

$$\frac{\overline{g}_{\mu\nu}(x)}{\sqrt{-\overline{g}(x)}} \times \frac{\overline{g}_{\rho\sigma}(x')}{\sqrt{-\overline{g}(x')}} \times -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm 3pt}(x;x') = H^4 \Bigg\{ \Big[4D - (4y-y^2)\Big]\alpha$$
$$+(2-y)(4y-y^2)\beta + (4y-y^2)^2\gamma + D^2\delta + 2D(4y-y^2)\epsilon \Bigg\}, \quad (4\text{–}57)$$
$$= \frac{1}{8}(D-2)^2\kappa^2 H^4 \Bigg\{ \Big[(4y-y^2) - 4D\Big](A')^2$$
$$-2(2-y)(4y-y^2)A'A'' - (4y-y^2)^2(A'')^2 \Bigg\}, \quad (4\text{–}58)$$
$$= -\frac{1}{8}(D-1)^2(D-2)^2\kappa^2 H^4 \Bigg\{ \frac{4}{D-1}(A')^2 + \Big[(2-y)A' - k\Big]^2 \Bigg\}\; . \quad (4\text{–}59)$$

Now note that the primed and unprimed scalar d'Alembertian's agree when acting on any function of only $y(x;x')$. Equating (4–56) and (4–59) and expanding implies,

$$\Big[\frac{\square}{H^2} + D\Big]^2 \mathcal{F}_1(y) = -\frac{1}{8}(D-2)^2\kappa^2 \Bigg\{ \frac{4}{D-1}(A')^2 + \Big[(2-y)A' - k\Big]^2 \Bigg\}\; . \quad (4\text{–}60)$$
$$= -\frac{K}{32}\frac{(D-2)^2}{(D-1)} \Bigg\{ D\Big(\frac{4}{y}\Big)^D + (D-2)^2\Big(\frac{4}{y}\Big)^{D-1}$$
$$+\frac{1}{2}(D^3 - 7D^2 + 16D - 8)\Big(\frac{4}{y}\Big)^{D-2} + \Big(\text{Irrelevant}\Big) \Bigg\}, \quad (4\text{–}61)$$

where the constant K was defined in (3–24) and "Irrelevant" means terms which are both integrable at coincidence, and which vanish in $D = 4$ dimensions.

Let us first note that we can find a Green's function for the differential operator $[\square/H^2 + D]$. To see this, act the operator on some function $f(y)$ which is free of the unique power $y^{\frac{D}{2}-1}$ which produces a delta function,

$$\Big[\frac{\square}{H^2} + D\Big] f(y) = (4y-y^2)f'' + D(2-y)f' + Df\; . \quad (4\text{–}62)$$

Now note that $f_1(y) = (2 - y)$ is a homogeneous solution, which means we can factor to obtain a first order equation (and hence solvable) for the second solution,

$$f_1(y) = (2{-}y) \Longrightarrow f_2(y) \equiv f_1(y)g(y) \quad \text{with} \quad g'(y) = \frac{1}{(4y{-}y^2)^{\frac{D}{2}} f_1^2(y)} \; . \tag{4–63}$$

With the two, linearly independent solutions one can construct a Green's function,

$$G_1(y;y') = \theta(y{-}y')\Big[f_2(y)f_1(y'){-}f_1(y)f_2(y')\Big](4y'{-}y'^2)^{\frac{D}{2}-1} \; . \tag{4–64}$$

Hence we can solve (4–61) to obtain on integral epxression for the spin zero structure function,

$$\mathcal{F}_1(y) = \left[\frac{1}{\frac{\Box}{H^2}{+}D}\right]^2 \left\{ \text{Right hand side of } (4{-}61) \right\} \tag{4–65}$$

Although we will eventually make use of the Green's function (4–64), it is best to delay this until the point at which one can set $D = 4$. For the more singular terms the best strategy is to exploit the fact that the "source" terms on the right hand side of (4–61) upon which we wish to act the inverse of $[\Box/H^2 + D]^2$ are just powers of y. Consider acting the operator upon a power $p - 2 \neq \frac{D}{2} - 1$ or $\frac{D}{2} - 2$ (those powers produce delta functions),

$$\begin{aligned}\left[\frac{\Box}{H^2}{+}D\right]^2 \left(\frac{4}{y}\right)^{p-2} &= (p{-}2)(p{-}1)(p{-}1{-}\frac{D}{2})(p{-}\frac{D}{2})\left(\frac{4}{y}\right)^p + (p{-}2)(p{-}1{-}\frac{D}{2}) \\ &\times\Big[D(2p{-}1){-}2(p{-}1)^2\Big]\left(\frac{4}{y}\right)^{p-1} + (p{-}1)^2(D{-}p{+}2)^2\left(\frac{4}{y}\right)^{p-2} .\end{aligned} \tag{4–66}$$

We can therefore develop a recursive procedure for reducing the power of the source,

$$\begin{aligned}\left[\frac{1}{\frac{\Box}{H^2}{+}D}\right]^2 \left(\frac{4}{y}\right)^p &= \frac{1}{(p{-}2)(p{-}1)(p{-}1{-}\frac{D}{2})(p{-}\frac{D}{2})}\left(\frac{4}{y}\right)^{p-2} - \left[\frac{1}{\frac{\Box}{H^2}{+}D}\right]^2 \\ &\times\left\{\frac{[D(2p{-}1){-}2(p{-}1)^2]}{(p{-}1)(p{-}\frac{D}{2})}\left(\frac{4}{y}\right)^{p-1} + \frac{(p{-}1)(D{+}2{-}p)^2}{(p{-}2)(p{-}1{-}\frac{D}{2})(p{-}\frac{D}{2})}\left(\frac{4}{y}\right)^{p-2}\right\} .\end{aligned} \tag{4–67}$$

The strategy is to apply this until the source is integrable, at which point the dimension can be set to $D = 4$ (unless there are factors of $1/(D - 4)$) and the $D = 4$ Green's function can be used to obtain the full solution for $\mathcal{F}_1(y)$.

It is useful to examine the sorts of terms generated when this recursive procedure is applied to the source terms on the right hand side of (4–61). The most singular term introduces no factors of $1/(D-4)$, nor does it produce remainder terms different from those in the original source term (4–61),

$$\Big[\frac{1}{\frac{\square}{H^2}+D}\Big]^2 \Big(\frac{4}{y}\Big)^{D} = \frac{4}{(D-2)D(D-2)(D-1)}\Big(\frac{4}{y}\Big)^{D-2} - \Big[\frac{1}{\frac{\square}{H^2}+D}\Big]^2 \Big\{\frac{2(3D-2)}{D(D-1)}\Big(\frac{4}{y}\Big)^{D-1} + \frac{16(D-1)}{(D-2)D(D-2)}\Big(\frac{4}{y}\Big)^{D-2}\Big\}. \qquad (4\text{–}68)$$

Neither statement is true for the remaining two source terms,

$$\Big[\frac{1}{\frac{\square}{H^2}+D}\Big]^2 \Big(\frac{4}{y}\Big)^{D-1} = \frac{4}{(D-4)(D-2)(D-3)(D-2)}\Big(\frac{4}{y}\Big)^{D-3} - \Big[\frac{1}{\frac{\square}{H^2}+D}\Big]^2 \Big\{\frac{2(5D-8)}{(D-2)(D-2)}\Big(\frac{4}{y}\Big)^{D-2} + \frac{36(D-2)}{(D-4)(D-2)(D-3)}\Big(\frac{4}{y}\Big)^{D-3}\Big\}, \qquad (4\text{–}69)$$

$$\Big[\frac{1}{\frac{\square}{H^2}+D}\Big]^2 \Big(\frac{4}{y}\Big)^{D-2} = \frac{4}{(D-6)(D-4)(D-4)(D-3)}\Big(\frac{4}{y}\Big)^{D-4} - \Big[\frac{1}{\frac{\square}{H^2}+D}\Big]^2 \Big\{\frac{2(7D-18)}{(D-4)(D-3)}\Big(\frac{4}{y}\Big)^{D-3} + \frac{64(D-3)}{(D-6)(D-4)(D-4)}\Big(\frac{4}{y}\Big)^{D-4}\Big\}. \qquad (4\text{–}70)$$

These relations allow the the spin zero structure function to be expressed as a “quotient” and a “remainder” of the form,

$$\mathcal{F}_1(y) = \mathcal{Q}_1(y) + \Big[\frac{1}{\frac{\square}{H^2}+D}\Big]^2 \mathcal{R}_1(y)\ , \qquad (4\text{–}71)$$

$$\mathcal{Q}_1(y) = -K\Big\{f_{1a}\Big(\frac{4}{y}\Big)^{D-2} + \frac{f_{1b}}{D-4}\Big(\frac{4}{y}\Big)^{D-3} + \frac{f_{1c}}{(D-4)^2}\Big(\frac{4}{y}\Big)^{D-4}\Big\}\ , \qquad (4\text{–}72)$$

$$\mathcal{R}_1(y) = -K\Big\{\frac{f_{1d}}{D-4}\Big(\frac{4}{y}\Big)^{D-3} + \frac{f_{1e}}{(D-4)^2}\Big(\frac{4}{y}\Big)^{D-4} + \Big(\text{Irrelevant}\Big)\Big\}\ , \qquad (4\text{–}73)$$

where the coefficients are,

$$f_{1a} = \frac{1}{8(D-1)^2}\ , \qquad (4\text{–}74)$$

$$f_{1b} = \frac{D(D^2-5D+2)}{8(D-3)(D-1)^2} , \quad (4–75)$$

$$f_{1c} = \frac{D^2(D^4-12D^3+39D^2-16D-36)}{16(D-6)(D-3)(D-1)^2} , \quad (4–76)$$

$$f_{1d} = -\frac{8}{3} + \frac{79}{9}(D-4) + O\Bigl((D-4)^2\Bigr) , \quad (4–77)$$

$$f_{1e} = \frac{32}{3} - \frac{64}{9}(D-4) - \frac{274}{9}(D-4)^2 + O\Bigl((D-4)^3\Bigr) . \quad (4–78)$$

Although the powers y^{D-3} and y^{D-4} in the remainder term of (4–71) are integrable, the factors of $1/(D-4)$ they carry preclude us setting $D = 4$ and then obtaining an explicit form using the $D = 4$ Green's function. In the next section we will see how to add zero so as to localize the divergences, and then absorb them into counterterms. For now, let us assume $\mathcal{F}_1(y)$ has been derived and explain the procedure for computing the spin two structure function $\mathcal{F}_2(y)$.

The spin zero part of the graviton self-energy can be expressed as a sum of the five de Sitter invariant bitensors times functions of y,

$$\mathcal{P}^{\mu\nu}(x)\times\mathcal{P}^{\rho\sigma}(x')\times\mathcal{F}_1(y) = \frac{\partial^2 y}{\partial x_\mu \partial x'_{(\rho}}\frac{\partial^2 y}{\partial x'_{\sigma)}\partial x_\nu}\times\alpha_1(y) + \frac{\partial y}{\partial x_{(\mu}}\frac{\partial^2 y}{\partial x_{\nu)}\partial x'_{(\rho}}\frac{\partial y}{\partial x'_{\sigma)}}$$
$$\times\beta_1(y) + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma}\times\gamma_1(y) + H^4\overline{g}^{\mu\nu}(x)\overline{g}^{\rho\sigma}(x')\times\delta_1(y)$$
$$+H^2\Bigl[\overline{g}^{\mu\nu}(x)\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\overline{g}^{\rho\sigma}(x')\Bigr]\times\epsilon_1(y) , \quad (4–79)$$

Here the spin zero coefficient functions are,

$$\alpha_1 = 2\mathcal{F}_1'' , \quad (4–80)$$

$$\beta_1 = 4\mathcal{F}_1''' , \quad (4–81)$$

$$\gamma_1 = \mathcal{F}_1'''' , \quad (4–82)$$

$$\delta_1 = (4y-y^2)^2\mathcal{F}_1'''' + 2(D+1)(2-y)(4y-y^2)\mathcal{F}_1''' - 4(4y-y^2)\mathcal{F}_1''$$
$$+(D^2-3)(2-y)^2\mathcal{F}_1'' + (D-1)^2(2-y)\mathcal{F}_1' + (D-1)^2\mathcal{F}_1 , \quad (4–83)$$

$$\epsilon_1 = -(4y-y^2)\mathcal{F}_1'''' - (D+3)(2-y)\mathcal{F}_1''' + (D+1)\mathcal{F}_1'' . \quad (4–84)$$

Of course the spin two contribution can be reduced to the same form,

$$\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x) \times \mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x') \times \left\{\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big(\frac{D-2}{D-3}\Big)\mathcal{F}_2(y)\right\}$$
$$= \frac{\partial^2 y}{\partial x_\mu \partial x'_{(\rho}} \frac{\partial^2 y}{\partial x'_{\sigma)}\partial x_\nu} \times \alpha_2(y) + \frac{\partial y}{\partial x_{(\mu}} \frac{\partial^2 y}{\partial x_{\nu)}\partial x'_{(\rho}} \frac{\partial y}{\partial x'_{\sigma)}} \times \beta_2(y)$$
$$+ \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} \times \gamma_2(y) + H^4 \overline{g}^{\mu\nu}(x)\overline{g}^{\rho\sigma}(x') \times \delta_2(y)$$
$$+ H^2\Big[\overline{g}^{\mu\nu}(x)\frac{\partial y}{\partial x'_\rho}\frac{\partial y}{\partial x'_\sigma} + \frac{\partial y}{\partial x_\mu}\frac{\partial y}{\partial x_\nu}\overline{g}^{\rho\sigma}(x')\Big] \times \epsilon_2(y) \; , \qquad (4\text{–}85)$$

Determining the coefficient functions is an extremely tedious exercise that was done by computer. The results for each coefficient function are expressed as an expansion in powers of derivatives of the spin two structure function, for example,

$$\alpha_2 = \sum_{k=0}^{4} \alpha_{2k} \frac{d^k \mathcal{F}_2}{dy^k} \; . \qquad (4\text{–}86)$$

The various coefficients, which are functions of D and y, are reported in Tables 4-1-4-5.

Table 4-1. Coefficient of F_2: each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

	Coefficient of F_2
α_{20}	$-(D-3)D^2(D+1)^2\Big[-4(D-2) + (D-1)(4y-y^2)\Big]$
β_{20}	$2(D-3)(D-1)D^2(D+1)^2(2-y)$
γ_{20}	$(D-3)(D-1)D^2(D+1)^2$
δ_{20}	$4(D-3)D(D+1)^2\Big[-4(D-2) + D(4y-y^2)\Big]$
ϵ_{20}	$-4(D-3)D^2(D+1)^2$

Table 4-2. Coefficient of F_2': each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

	Coefficient of F_2
α_{21}	$4(D-3)(D+1)^2(2-y)\Big[-2(D-2)D + (D-1)(D+1)(4y-y^2)\Big]$
β_{21}	$8(D-3)(D+1)^2\Big[-3D^2 + (D-1)(D+1)(4y-y^2)\Big]$
γ_{21}	$-4(D-3)(D-1)(D+1)^3(2-y)$
δ_{21}	$-16(D-3)(D+1)^2(2-y)\Big[-2(D-2) + (D+1)(4y-y^2)\Big]$
ϵ_{21}	$16(D-3)(D+1)^3(2-y)$

Now recall the second order equation (4–42) we were able to find for the flat space structure function $F_2(\Delta x^2)$ by adding δ and $\Delta x^2\epsilon$. After long contemplation of

Table 4-3. Coefficient of F_2'': each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

	Coefficient of F_2''
α_{22}	$2\Big[8(D-2)^2D(D+1) - 4(D+1)(3D^3-8D^2-6D+12)(4y-y^2)$ $+(D-3)(D-1)(3D^2+9D+7)(4y-y^2)^2\Big]$
β_{22}	$-4(2-y)\Big[-2D(D+1)(3D^2-5D-10)$ $+(D-3)(D-1)(3D^2+9D+7)(4y-y^2)\Big]$
γ_{22}	$-2\Big[-12(D^4-D^3-7D^2+D+10)$ $+(D-3)(D-1)(3D^2+9D+72)(4y-y^2)\Big]$
δ_{22}	$-8\Big[8(D-2)^2(D+1) - 2(D+1)(6D^2-11D-18)(4y-y^2)$ $+(D-3)(3D^2+9D+7)(4y-y^2)^2\Big]$
ϵ_{22}	$8\Big[-2(D+1)(5D^2-6D-24) + (D-3)(3D^2+9D+7)(4y-y^2)\Big]$

Table 4-4. Coefficient of F_2''': each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

	Coefficient of F_2'''
α_{23}	$-4(D-1)(2-y)(4y-y^2)\Big[-2(D-2)(D+1)$ $+(D-3)(D+2)(4y-y^2)\Big]$
β_{23}	$-8\Big[4(D-2)D(D+1) - (5D^3-8D^2-23D+22)(4y-y^2)$ $+(D-3)(D-1)(D+2)(4y-y^2)^2\Big]$
γ_{23}	$4(2-y)\Big[-4(D-2)(D^2-5) + (D-3)(D-1)(D+2)(4y-y^2)\Big]$
δ_{23}	$16(2-y)(4y-y^2)\Big[-2(D-2)(D+1) + (D-3)(D+2)(4y-y^2)\Big]$
ϵ_{23}	$-16(2-y)\Big[-2(D-2)(D+1) + (D-3)(D+2)(4y-y^2)\Big]$

Table 4-5. Coefficient of F_2'''': each term is multiplied by $\frac{1}{16(D-2)(D-1)}$

	Coefficient of F_2''''
α_{24}	$-(D-1)(4y-y^2)^2\Big[-4(D-2) + (D-3)(4y-y^2)\Big]$
β_{24}	$2(D-1)(2-y)(4y-y^2)\Big[-4(D-2) + (D-3)(4y-y^2)\Big]$
γ_{24}	$\Big[4(D-2) - (D-3)(4y-y^2)\Big]\Big[4(D-2) - (D-1)(4y-y^2)\Big]$
δ_{24}	$4(4y-y^2)^2\Big[-4(D-2) + (D-3)(4y-y^2)\Big]$
ϵ_{24}	$-4(4y-y^2)\Big[-4(D-2) + (D-3)(4y-y^2)\Big]$

the bewildering data in Tables 4-1-4-5 it becomes apparent that a similar second order equation for $\mathcal{F}_2(y)$ derives from the combination,

$$\delta_2(y) + (4y-y^2)\epsilon_2(y) = \Big[\delta(y)-\delta_1(y)\Big] + (4y-y^2)\Big[\epsilon(y)-\epsilon_1(y)\Big] \; , \tag{4–87}$$

$$= -\Big(\frac{D+1}{D-1}\Big)\Bigg\{(D-2)\mathcal{F}_2'' - (D-3)\Bigg[(4y-y^2)\mathcal{F}_2'' + 2(D+1)(2-y)\mathcal{F}_2' - D(D+1)\mathcal{F}_2\Bigg\}. \tag{4–88}$$

Hence we can express the equation for $\mathcal{F}_2(y)$ as,

$$\mathcal{D}\mathcal{F}_2 = -\Big(\frac{D-1}{D+1}\Big)\Bigg\{\Big[\delta(y)-\delta_1(y)\Big] + (4y-y^2)\Big[\epsilon(y)-\epsilon_1(y)\Big]\Bigg\}, \tag{4–89}$$

where the second order operator $\mathcal{D}$ is,

$$\mathcal{D} \equiv 4(D-2)\Big(\frac{d}{dy}\Big)^2 -(D-3)\Bigg[(4y-y^2)\Big(\frac{d}{dy}\Big)^2 + 2(D+1)(2-y)\frac{d}{dy} - D(D+1)\Bigg] \; , \tag{4–90}$$

$$= 4\Big(\frac{d}{dy}\Big)^2 + (D-3)\Bigg[(2-y)^2\Big(\frac{d}{dy}\Big)^2 - 2(D+1)(2-y)\frac{d}{dy} + D(D+1)\Bigg] \; . \tag{4–91}$$

The source term on the right hand side of (4–89) has the form,

$$-\Big(\frac{D-1}{D+1}\Big)\Bigg\{\Big[\delta(y)-\delta_1(y)\Big] + (4y-y^2)\Big[\epsilon(y)-\epsilon_1(y)\Big]\Bigg\}$$

$$= K\Bigg\{s_a\Big(\frac{4}{y}\Big)^D + \frac{s_b}{D-4}\Big(\frac{4}{y}\Big)^{D-1} + \frac{s_c}{D-4}\Big(\frac{4}{y}\Big)^{D-2} + s_{c'}\Big(\frac{4}{y}\Big)^{\frac{D}{2}} + \frac{s_d}{D-4}\Big(\frac{4}{y}\Big)^{D-3} + \frac{s_e}{(D-4)^2}\Big(\frac{4}{y}\Big)^{D-4} + \Big(\text{Irrelevant}\Big)\Bigg\} + \mathcal{R} \; , \tag{4–92}$$

where the remainder term $\mathcal{R}$ derives from the remainder $\mathcal{R}_1$ of $\mathcal{F}_1$,

$$\mathcal{R} = \Big(\frac{D-1}{D+1}\Big)\Bigg\{(D-1)(2-y)(4y-y^2)\Big(\frac{\partial}{\partial y}\Big)^3 - D(D-1)(4y-y^2)\Big(\frac{\partial}{\partial y}\Big)^2 + 4(D^2-3)\Big(\frac{\partial}{\partial y}\Big)^2 + (D-1)^2(2-y)\Big(\frac{\partial}{\partial y}\Big) + (D-1)^2\Bigg\}\Bigg[\frac{1}{\frac{\square}{H^2}+D}\Bigg]^2 \mathcal{R}_1 \; . \tag{4–93}$$

The coefficients in (4–92) are,

$$s_a = -\frac{1}{16(D+1)} , \tag{4–94}$$
$$s_b = -\frac{(D-2)D}{16(D-1)} , \tag{4–95}$$
$$s_c = -\frac{(D-4)(D-2)D(D+3)}{32(D-6)(D-1)} , \tag{4–96}$$
$$s_{c'} = -\frac{(D-4)(D-1)\Gamma(D)}{16(D+1)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)} , \tag{4–97}$$
$$s_d = -\frac{7}{5} + \frac{263}{100}(D-4) + O\Bigl((D-4)^2\Bigr) , \tag{4–98}$$
$$s_e = \frac{18}{5} - \frac{18}{25}(D-4) - \frac{11331}{1000}(D-4)^2 + O\Bigl((D-4)^3\Bigr) . \tag{4–99}$$

Just as for the differential operator $(\frac{\square}{H^2} + D)$, it is straightforward to construct a Green's function to invert $\mathcal{D}$. The first step is to change variables in the second form (4–91),

$$w \equiv \sqrt{\frac{D-3}{4}}\,(2-y) \Longrightarrow \mathcal{D} = (D-3)\Bigl[(1+w^2)\Bigl(\frac{d}{dw}\Bigr)^2 + 2(D+1)w\frac{d}{dw} + D(D+1)\Bigr] . \tag{4–100}$$

The homogeneous equation $\mathcal{D}f(w) = 0$ gives rise to a simple, 2-term recursion relation which generates even and odd solutions. These series solutions can be expressed as hypergeometric functions that reduce to elementary functions for $D = 4$,

$$f_e(w) = {}_2F_1\Bigl(\frac{D}{2}, \frac{D+1}{2}; \frac{1}{2}; w^2\Bigr) \longrightarrow \frac{(1-6w^2+w^4)}{(1+w^2)^4} , \tag{4–101}$$
$$f_o(w) = w \times {}_2F_1\Bigl(\frac{D+1}{2}, \frac{D+2}{2}; \frac{3}{2}; w^2\Bigr) \longrightarrow \frac{(w-w^3)}{(1+w^2)^4} . \tag{4–102}$$

Because we again have both homogeneous solutions it is simple to write down a Green's function,

$$G_2(w; w') = \frac{\theta(w-w')}{D-3}\Bigl[f_o(w)f_e(w') - f_e(w)f_o(w')\Bigr](1+w'^2)^D . \tag{4–103}$$

As was the case for it spin zero cousin (4–64), the spin two Green's function (4–103) is not simple to use for arbitrary D. We therefore adopt the same strategy we used for $\mathcal{F}_1$, of recursively extracting powers until the remainder is integrable and the $D=4$ forms can be employed. Acting $\mathcal{D}$ on a power gives,

$$\mathcal{D}\Bigl(\frac{4}{y}\Bigr)^{p-2} = \frac{1}{4}(D-2)(p-2)(p-1)\Bigl(\frac{4}{y}\Bigr)^{p} \\ +(D-3)(p-2)(D+2-p)\Bigl(\frac{4}{y}\Bigr)^{p-1} + (D-3)(D+2-p)(D+3-p)\Bigl(\frac{4}{y}\Bigr)^{p-2}. \qquad (4–104)$$

Hence we conclude,

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{p} = \frac{4}{(D-2)(p-2)(p-1)}\Bigl(\frac{4}{y}\Bigr)^{p-2} \\ -\frac{4}{\mathcal{D}}\Biggl\{\frac{(D-3)(D+2-p)}{(D-2)(p-1)}\Bigl(\frac{4}{y}\Bigr)^{p-1} + \frac{(D-3)(D+2-p)(D+3-p)}{(D-2)(p-2)(p-1)}\Bigl(\frac{4}{y}\Bigr)^{p-2}\Biggr\}. \qquad (4–105)$$

For the four powers of relevance expression (4–105) gives,

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{D} = \frac{4}{(D-2)^2(D-1)}\Bigl(\frac{4}{y}\Bigr)^{D-2} \\ -\frac{1}{\mathcal{D}}\Biggl\{\frac{8(D-3)}{(D-2)(D-1)}\Bigl(\frac{4}{y}\Bigr)^{D-1} + \frac{24(D-3)}{(D-2)^2(D-1)}\Bigl(\frac{4}{y}\Bigr)^{D-2}\Biggr\}, \qquad (4–106)$$

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{D-1} = \frac{4}{(D-3)(D-2)^2}\Bigl(\frac{4}{y}\Bigr)^{D-3} \\ -\frac{1}{\mathcal{D}}\Biggl\{\frac{12(D-3)}{(D-2)^2}\Bigl(\frac{4}{y}\Bigr)^{D-2} + \frac{48}{(D-2)^2}\Bigl(\frac{4}{y}\Bigr)^{D-3}\Biggr\}, \qquad (4–107)$$

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{D-2} = \frac{4}{(D-4)(D-3)(D-2)}\Bigl(\frac{4}{y}\Bigr)^{D-4} \\ -\frac{1}{\mathcal{D}}\Biggl\{\frac{16}{(D-2)}\Bigl(\frac{4}{y}\Bigr)^{D-3} + \frac{80}{(D-4)(D-2)}\Bigl(\frac{4}{y}\Bigr)^{D-4}\Biggr\}, \qquad (4–108)$$

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}} = \frac{16}{(D-4)(D-2)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} \\ -\frac{4}{\mathcal{D}}\Biggl\{\frac{(D-3)(D+4)}{(D-2)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{(D-3)(D+4)(D+6)}{(D-4)(D-2)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2}\Biggr\}. \qquad (4–109)$$

These relations allow the spin two structure function to be expressed as a "quotient" and "remainder" of the form,

$$\mathcal{F}_2 = \mathcal{Q}_2(y) + \frac{1}{\mathcal{D}}\mathcal{R}_2(y) \; , \tag{4–110}$$

$$\mathcal{Q}_2 = -K\left\{ f_{2a}\left(\frac{4}{y}\right)^{D-2} + \frac{f_{2b}}{D-4}\left(\frac{4}{y}\right)^{D-3} + \frac{f_{2c}}{(D-4)^2}\left(\frac{4}{y}\right)^{D-4} + \frac{f_{2c'}}{D-4}\left(\frac{4}{y}\right)^{\frac{D}{2}-2}\right\}, \tag{4–111}$$

$$\mathcal{R}_2 = -K\left\{ \frac{f_{2d}}{D-4}\left(\frac{4}{y}\right)^{D-3} + \frac{f_{2e}}{(D-4)^2}\left(\frac{4}{y}\right)^{D-4} + \Bigl(\text{Irrelevant}\Bigr)\right\} + \mathcal{R} \; , \tag{4–112}$$

where the coefficients are,

$$f_{2a} = \frac{1}{4(D-2)^2(D-1)(D+1)} \; , \tag{4–113}$$

$$f_{2b} = \frac{D^4-3D^3-8D^2+60D-96}{4(D-3)(D-2)^3(D-1)(D+1)} \; , \tag{4–114}$$

$$f_{2c} = \frac{D^8-8D^7-13D^6+348D^5-1136D^4-2^{10}D^3+15056D^2-38208D+34560}{8(D-6)(D-3)(D-2)^4(D-1)(D+1)}, \tag{4–115}$$

$$f_{2c'} = \frac{(D-4)(D-1)\Gamma(D)}{(D-2)^2(D+1)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)} \; , \tag{4–116}$$

$$f_{2d} = \frac{17}{5} + \frac{161}{300}(D-4) + O\Bigl((D-4)^2\Bigr) \; ,$$

$$f_{2e} = \frac{82}{5} + \frac{243}{25}(D-4) + \frac{13343}{3000}(D-4)^2 + O\Bigl((D-4)^3\Bigr) \; . \tag{4–117}$$

4.4 Renormalizing the Spin Zero Structure Function

Recall the form (4–71) we obtained for the spin zero struncture function from taking the trace of the graviton self-energy,

$$\mathcal{F}_1(y) = \mathcal{Q}_1(y) + \left[\frac{1}{\frac{\square}{H^2}+D}\right]^2 \mathcal{R}_1(y) \; . \tag{4–118}$$

Recall also that the quotient $\mathcal{Q}_1(y)$ and the remainder $\mathcal{R}_1(y)$ are given in relations (4–72)-(4–78). From these expressions we perceive three sorts of ultraviolet divergences:

- The factor of $(\frac{4}{y})^{D-2}$ in $\mathcal{Q}_1$, which has a finite coefficient but is still not integrable in $D = 4$ dimensions;

- The factors of $\frac{1}{D-4}(\frac{4}{y})^{D-3}$ in $\mathcal{Q}_1$ and $\mathcal{R}_1$ which are integrable in $D = 4$ dimensions but have divergent coefficients that preclude taking the unregulated limits; and

- The factors of $(\frac{1}{D-4})^2(\frac{4}{y})^{D-4}$ in $\mathcal{Q}_1$ and $\mathcal{R}_1$ which are integrable in $D = 4$ dimensions but have even more divergent coefficients.

In this section we will explain how to localize all three divergences onto delta function terms which can be absorbed by the counterterms (4–19), (4–21) and (4–22). We will also take the unregulated limits of the remaining, finite parts, and use the $D = 4$ Green's function (4–64) to obtain an explicit result for the renormalized structure function.

In dealing with the factor of $(\frac{4}{y})^{D-2}$ in $\mathcal{Q}_1$, the first step is to extract a d'Alembertian,

$$\Big(\frac{4}{y}\Big)^{D-2} = \frac{2}{(D-4)(D-3)}\left[\frac{\square}{H^2}\Big(\frac{4}{y}\Big)^{D-3} - 2(D-3)\Big(\frac{4}{y}\Big)^{D-3}\right] . \tag{4–119}$$

The resulting factors of $(\frac{4}{y})^{D-3}$ are integrable in $D = 4$ dimensions, at which point we could take the unregulated limit except for the factor of $1/(D-4)$ in (4–119). We can localize the divergence on a delta function by adding zero in the form of the identity (4–2),

$$\Big(\frac{4}{y}\Big)^{D-2} = \frac{2}{(D-4)(D-3)}\left\{\frac{\square}{H^2}\left[\Big(\frac{4}{y}\Big)^{D-3} - \Big(\frac{4}{y}\Big)^{\frac{D}{2}-1}\right]\right.$$
$$\left. -2(D-3)\Big(\frac{4}{y}\Big)^{D-3} + \frac{D}{2}\Big(\frac{D}{2}-1\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2}-1)}\frac{i\delta^D(x-x')}{H^D\sqrt{-g}}\right\} , \tag{4–120}$$
$$= -\Big[\frac{\square}{H^2} - 2\Big]\left\{\frac{4}{y}\ln\Big(\frac{y}{4}\Big)\right\} - \frac{4}{y} + O(D-4) + \frac{2(4\pi)^{\frac{D}{2}} i\delta^D(x-x')/\sqrt{-g}}{(D-4)(D-3)\Gamma(\frac{D}{2}-1)H^D} . \tag{4–121}$$

We turn now to the factors of $\frac{1}{D-4}(\frac{4}{y})^{D-3}$ and $(\frac{1}{D-4})^2(\frac{4}{y})^{D-4}$ in $\mathcal{Q}_1$ and $\mathcal{R}_1$. The key relations for resolving these terms follow from (4–2),

$$\Big[\frac{\square}{H^2} + D\Big]^2\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} = \frac{1}{16}D^2(D+2)^2\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1}$$
$$+ \frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2}-1)H^D\sqrt{-g}}\left[\frac{\square}{H^2} + D + \frac{1}{4}D(D+2)\right] i\delta^D(x-x') , \tag{4–122}$$

$$\Big[\frac{\square}{H^2}+D\Big]^2\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} = -\frac{1}{4}(D-4)(D^2+2D-4)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \frac{1}{16}(D-2)^2(D+4)^2\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} - \frac{(D-4)(4\pi)^{\frac{D}{2}}}{2\Gamma(\frac{D}{2}-1)H^D\sqrt{-g}}i\delta^D(x-x') \; , \tag{4–123}$$

$$\Big[\frac{\square}{H^2}+D\Big]^2 1 = D^2 \; . \tag{4–124}$$

One adds zero using these relations so as to resolve the problematic terms in $\mathcal{Q}_1$, and the remainder automatically resolves the problematic terms in $\mathcal{R}_1$,

$$\begin{aligned}
&\frac{f_{1b}}{D-4}\Big(\frac{4}{y}\Big)^{D-3} + \frac{f_{1c}}{(D-4)^2}\Big(\frac{4}{y}\Big)^{D-4} + \Bigg[\frac{1}{\frac{\square}{H^2}+D}\Bigg]^2\Bigg\{\frac{f_{1d}}{D-4}\Big(\frac{4}{y}\Big)^{D-3} + \frac{f_{1e}}{(D-4)^2}\Big(\frac{4}{y}\Big)^{D-4}\Bigg\} \\
&= \frac{f_{1b}}{D-4}\Bigg\{\Big(\frac{4}{y}\Big)^{D-3} - \Big(\frac{4}{y}\Big)^{\frac{D}{2}-1}\Bigg\} + \frac{f_{1c}}{(D-4)^2}\Bigg\{\Big(\frac{4}{y}\Big)^{D-4} - 2\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} + 1\Bigg\} \\
&\quad + \Bigg[\frac{1}{\frac{\square}{H^2}+D}\Bigg]^2\Bigg\{\frac{f_{1d}}{D-4}\Big(\frac{4}{y}\Big)^{D-3} + \frac{[D^2(D+2)^2 f_{1b} - 8(D^2+2D-4)f_{1c}]}{16(D-4)}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} \\
&\quad + \frac{f_{1e}}{(D-4)^2}\Big(\frac{4}{y}\Big)^{D-4} + \frac{(D-2)^2(D+4)^2 f_{1c}}{8(D-4)^2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} - \frac{D^2 f_{1c}}{(D-4)^2} \\
&\quad + \frac{(4\pi)^{\frac{D}{2}}/\sqrt{-g}}{\Gamma(\frac{D}{2}-1)H^D}\Bigg[\frac{f_{1b}}{D-4}\Big[\frac{\square}{H^2}+D\Big] + \frac{D(D+2)f_{1b}-4f_{1c}}{4(D-4)}\Bigg]i\delta^D(x-x')\Bigg\},
\end{aligned} \tag{4–125}$$

$$\begin{aligned}
&= \frac{1}{18}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big) - \frac{1}{6}\times\ln^2\Big(\frac{y}{4}\Big) + O(D-4) + \Bigg[\frac{1}{\frac{\square}{H^2}+4}\Bigg]^2\Bigg\{\frac{4}{3}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big) \\
&\quad + \frac{8}{3}\times\frac{4}{y} + \frac{8}{3}\ln^2\Big(\frac{y}{4}\Big) - 8\ln\Big(\frac{y}{4}\Big) + \frac{1}{3}\Bigg\} + \Bigg[\frac{1}{\frac{\square}{H^2}+D}\Bigg]^2\Bigg\{\frac{(4\pi)^{\frac{D}{2}}/\sqrt{-g}}{\Gamma(\frac{D}{2}-1)H^D} \\
&\quad \times\Bigg[\frac{f_{1b}}{D-4}\Big[\frac{\square}{H^2}+D\Big] + \frac{D(D+2)f_{1b}-4f_{1c}}{4(D-4)}\Bigg]i\delta^D(x-x')\Bigg\}.
\end{aligned} \tag{4–126}$$

Employing expressions (4–121) and (4–126) in (4–71) allows us to separate the spin zero structure function into a finite part and a divergent part,

$$\mathcal{F}_1 = \mathcal{F}_{1R} + O(D-4) + \Delta\mathcal{F}_1 \; . \tag{4–127}$$

The finite part consists of the renormalized spin zero structure function,

$$\mathcal{F}_{1R} = \frac{\kappa^2 H^4}{(4\pi)^4}\Biggl\{\frac{\square}{H^2}\Biggl[\frac{1}{72}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)\Biggr] - \frac{1}{12}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr) + \frac{1}{72}\times\frac{4}{y} + \frac{1}{6}\ln^2\Bigl(\frac{y}{4}\Bigr)\Biggr\}$$
$$+\frac{\kappa^2 H^4}{(4\pi)^4}\Biggl[\frac{1}{\frac{\square}{H^2}+4}\Biggr]^2\Biggl\{-\frac{4}{3}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr) - \frac{8}{3}\times\frac{4}{y} - \frac{8}{3}\ln^2\Bigl(\frac{y}{4}\Bigr) + 8\ln\Bigl(\frac{y}{4}\Bigr) - \frac{1}{3}\Biggr\}. \quad (4–128)$$

The divergent part consists of $[\frac{\square}{H^2} + D]^{-2}$ acting on a sum of three local terms,

$$\Delta\mathcal{F}_1 = \frac{\kappa^2 H^{D-4}(\frac{D}{2}-1)\Gamma(\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\Biggl[\frac{1}{\frac{\square}{H^2}+D}\Biggr]^2\Biggl\{\frac{-2f_{1a}}{(D-4)(D-3)}\Biggl[\frac{\square}{H^2}+D\Biggr]^2\frac{i\delta^D(x-x')}{\sqrt{-\overline{g}}}$$
$$-\frac{f_{1b}}{D-4}\Biggl[\frac{\square}{H^2}+D\Biggr]\frac{i\delta^D(x-x')}{\sqrt{-\overline{g}}} - \Biggl[\frac{D(D+2)f_{1b}-4f_{1c}}{4(D-4)}\Biggr]\frac{i\delta^D(x-x')}{\sqrt{-\overline{g}}}\Biggr\}. \quad (4–129)$$

Of course one cancels $\Delta\mathcal{F}_1$ with counterterms. From expressions (4–19)-(4–22) we see that the four counterterms contribute to the graviton self-energy as,

$$-i\Bigl[^{\mu\nu}\Delta\Sigma^{\rho\sigma}\Bigr](x;x') = \sqrt{-\overline{g}}\Biggl[2c_1\kappa^2\mathcal{P}^{\mu\nu}\mathcal{P}^{\rho\sigma} + 2c_2\kappa^2\overline{g}^{\alpha\kappa}\overline{g}^{\beta\lambda}\overline{g}^{\gamma\theta}\overline{g}^{\delta\phi}\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}$$
$$-c_3\kappa^2 H^2\mathcal{D}^{\mu\nu\rho\sigma} + c_4\kappa^2 H^4\sqrt{-\overline{g}}\Bigl[\frac{1}{4}\overline{g}^{\mu\nu}\overline{g}^{\rho\sigma} - \frac{1}{2}\overline{g}^{\mu(\rho}\overline{g}^{\sigma)\nu}\Bigr]\Biggr]i\delta^D(x-x'). \quad (4–130)$$

Tracing as we did in (4–56) gives,

$$\frac{\overline{g}_{\mu\nu}(x)}{\sqrt{-\overline{g}(x)}}\times\frac{\overline{g}_{\rho\sigma}(x')}{\sqrt{-\overline{g}(x')}}\times -i\Bigl[^{\mu\nu}\Delta\Sigma^{\rho\sigma}\Bigr](x;x') = (D-1)^2H^4\Biggl[2c_1\kappa^2\Bigl[\frac{\square}{H^2}+D\Bigr]^2$$
$$+0-\frac{1}{2}\Bigl(\frac{D-2}{D-1}\Bigr)c_3\kappa^2\Bigl[\frac{\square}{H^2}+D\Bigr] + \frac{D(D-2)}{4(D-1)^2}c_4\kappa^2\Biggr]\frac{i\delta^D(x-x')}{\sqrt{-\overline{g}}}. \quad (4–131)$$

We can entirely absorb $\Delta\mathcal{F}_1$ by making the choices,

$$c_1 = \frac{H^{D-4}(\frac{D}{2}-1)\Gamma(\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\times\frac{f_{1a}}{(D-4)(D-3)}$$
$$= \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}}\times\frac{(D-2)}{(D-4)(D-3)(D-1)^2}, \quad (4–132)$$

$$c_3 = \frac{H^{D-4}(\frac{D}{2}-1)\Gamma(\frac{D}{2})}{(4\pi)^{\frac{D}{2}}} \times -2\Big(\frac{D-1}{D-2}\Big) \times \frac{f_{1b}}{D-4}$$
$$= \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times -\frac{2D(D^2-5D+2)}{(D-4)(D-3)(D-1)} \,, \qquad (4\text{–}133)$$
$$c_4 = \frac{H^{D-4}(\frac{D}{2}-1)\Gamma(\frac{D}{2})}{(4\pi)^{\frac{D}{2}}} \times \frac{4(D-1)^2}{D(D-2)} \times \Big[\frac{D(D+2)f_{1b}-4f_{1c}}{4(D-4)}\Big]$$
$$= \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times -\frac{D(D^3-11D^2+24D+12)}{(D-6)(D-3)(D-2)} \,. \qquad (4\text{–}134)$$

The linear combinations (4–10) and (4–11) are finite,

$$-2(D-1)Dc_1 + c_3 = \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times \frac{-2D^2}{(D-3)(D-1)} \,, \qquad (4\text{–}135)$$
$$(D-1)^2D^2c_1 - (D-2)(D-1)c_3 + c_4$$
$$= \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}} \times \frac{D(D^3-6D^2+8D-24)}{(D-6)(D-3)} \,. \qquad (4\text{–}136)$$

Therefore neither the Newton constant nor the cosmological constant requires a divergent renormalization, although we are free to continue making the finite renormalizations of these constants which are implied by equations (4–132)-(4–134).

It remains to act the $D = 4$ Green's function (4–64) twice on the renormalized remainder term in expression (4–128). The result is,

$$\Big[\frac{1}{\frac{\square}{H^2}+4}\Big]^2 \Big\{-\frac{4}{3}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big)-\frac{8}{3}\times\frac{4}{y}-\frac{8}{3}\ln^2\Big(\frac{y}{4}\Big)+8\ln\Big(\frac{y}{4}\Big)-\frac{1}{3}\Big\}$$
$$= -\frac{1}{3}\times\frac{y}{4}\ln^2\Big(\frac{y}{4}\Big)+\frac{1}{3}\times\frac{y}{4}\ln\Big(\frac{y}{4}\Big)$$
$$-\frac{7}{540}(12\pi^2+265)\times\frac{y}{4}+\frac{84\pi^2-131}{1080}$$
$$+\frac{1}{9}\times\frac{y}{4}\ln(\frac{y}{4})-\frac{1}{45}\ln(\frac{y}{4})+\frac{1}{45}\times\frac{4}{4-y}\ln(\frac{y}{4})$$
$$-\frac{1}{30}(2-y)\Big[7\mathrm{Li}_2(1-\frac{y}{4})-2\mathrm{Li}_2(\frac{y}{4})+5\ln(1-\frac{y}{4})\ln(\frac{y}{4})\Big]$$
$$+\frac{43}{216}\times\frac{4}{4-y}-\frac{5}{6}\times\frac{y}{4}\ln(1-\frac{y}{4})-\frac{1}{20}\ln(1-\frac{y}{4})+\frac{7}{90}\times\frac{4}{y}\ln(1-\frac{y}{4}) \,. \qquad (4\text{–}137)$$

Here $\text{Li}_2(z)$ is the dilogarithm function,

$$\text{Li}_2(z) \equiv -\int_0^z dt \, \frac{\ln(1-t)}{t} = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \, . \tag{4–138}$$

Hence our final result for the renormalized spin zero structure function is,

$$\begin{aligned}\mathcal{F}_{1R} = \frac{\kappa^2 H^4}{(4\pi)^4} \Bigg\{ & \frac{\square}{H^2} \left[\frac{1}{72} \times \frac{4}{y} \ln\Big(\frac{y}{4}\Big) \right] - \frac{1}{12} \times \frac{4}{y} \ln\Big(\frac{y}{4}\Big) + \frac{1}{72} \times \frac{4}{y} + \frac{1}{6} \ln^2\Big(\frac{y}{4}\Big) \\ & + \frac{1}{45} \times \frac{4}{4-y} \ln(\frac{y}{4}) - \frac{1}{45} \ln(\frac{y}{4}) + \frac{43}{216} \times \frac{4}{4-y} - \frac{5}{6} \times \frac{y}{4} \ln(1 - \frac{y}{4}) \\ & + \frac{7}{90} \times \frac{4}{y} \ln(1 - \frac{y}{4}) - \frac{1}{20} \ln(1 - \frac{y}{4}) - \frac{7(12\pi^2 + 265)}{540} \times \frac{y}{4} \\ & + \frac{84\pi^2 - 131}{1080} - \frac{1}{3} \times \frac{y}{4} \ln^2\Big(\frac{y}{4}\Big) + \frac{4}{9} \times \frac{y}{4} \ln\Big(\frac{y}{4}\Big) \\ & - \frac{1}{30}(2-y) \Big[7\text{Li}_2(1 - \frac{y}{4}) - 2\text{Li}_2(\frac{y}{4}) + 5 \ln(1 - \frac{y}{4}) \ln(\frac{y}{4}) \Big] \Bigg\} \, . \end{aligned} \tag{4–139}$$

4.5 Renormalizing the Spin Two Structure Function

Recall the form (4–110) we obtained for the spin two structure function,

$$\mathcal{F}_2(y) = \mathcal{Q}_2(y) + \frac{1}{\mathcal{D}} \mathcal{R}_2(y) \, , \tag{4–140}$$

where the second order differential operator $\mathcal{D}$ was defined in (4–91). Recall also that the quotient $\mathcal{Q}_2(y)$ and the remainder $\mathcal{R}_2(y)$ are given in relations (4–111)-(4–117). These expression imply that $\mathcal{F}_2$ harbors the same sort of ultraviolet divergences as $\mathcal{F}_1$:

- The factor of $(\frac{4}{y})^{D-2}$ in $\mathcal{Q}_2$, which has a finite coefficient but is still not integrable in $D = 4$ dimensions;
- The factors of $\frac{1}{D-4}(\frac{4}{y})^{D-3}$ in $\mathcal{Q}_2$ and $\mathcal{R}_2$ which are integrable in $D = 4$ dimensions but have divergent coefficients that preclude taking the unregulated limits; and
- The factors of $(\frac{1}{D-4})^2(\frac{4}{y})^{D-4}$ in $\mathcal{Q}_2$ and $\mathcal{R}_2$ which are integrable in $D = 4$ dimensions but have even more divergent coefficients.

Only the leading divergence requires a new counterterm. It is handled by first extracting another derivative and then adding zero in the form (4–2), just as we did in equations (4–119) and (4–121). The final result is,

$$-Kf_{2a}\Big(\frac{4}{y}\Big)^{D-2} = \frac{\kappa^2 H^4}{(4\pi)^4}\Bigg\{\frac{\square}{H^2}\Bigg[\frac{1}{240}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big)\Bigg]-\frac{1}{120}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big)+\frac{1}{240}\times\frac{4}{y}\Bigg\}$$
$$+O(D-4)-\frac{\kappa^2 H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}}\times\frac{4i\delta^D(x-x')/\sqrt{-g}}{(D-4)(D-3)(D-2)(D-1)(D+1)}\;. \qquad (4\text{–}141)$$

Comparing expressions (4–20) and (4–51) implies that the divergent part can be entirely absorbed by choosing the coefficient c_2 of the Weyl counterterm (4–7) to be,

$$c_2 = \frac{H^{D-4}\Gamma(\frac{D}{2})}{16(4\pi)^{\frac{D}{2}}}\times\frac{2}{(D-4)(D-3)^2(D-1)(D+1)}\;. \qquad (4\text{–}142)$$

Of course the divergent part agrees with [9].

It turns out that the lower divergences of $\mathcal{F}_2$ are canceled by the three factors we added to $\mathcal{Q}_1$ to cancel its lower divergences,

$$\delta\mathcal{Q}_1 = K\Bigg\{\frac{f_{1b}}{D-4}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1}+\frac{2f_{1c}}{(D-4)^2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2}-\frac{f_{1c}}{(D-4)^2}\Bigg\}. \qquad (4\text{–}143)$$

These changes in $\mathcal{Q}_1$ induce changes in the source term upon which we act $\mathcal{D}^{-1}$ to get $\mathcal{F}_2$,

$$\delta S \equiv \Big(\frac{D-1}{D+1}\Big)\Bigg\{(D-1)(2-y)(4y-y^2)\delta\mathcal{Q}_1''' - D(D-1)(4y-y^2)\delta\mathcal{Q}_1''$$
$$+4(D^2-3)\delta\mathcal{Q}_1''+(D-1)^2(2-y)\delta\mathcal{Q}_1'+(D-1)^2\delta\mathcal{Q}_1\Bigg\}\;, \qquad (4\text{–}144)$$

$$= K\Bigg\{\frac{\delta s_b}{D-4}\Big(\frac{4}{y}\Big)^{\frac{D}{2}+1}+\frac{\delta s_c}{D-4}\Big(\frac{4}{y}\Big)^{\frac{D}{2}}+\frac{\delta s_d}{D-4}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1}+\frac{\delta s_e}{(D-4)^2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2}+\frac{\delta s_{e'}}{(D-4)^2}\Bigg\}. \qquad (4\text{–}145)$$

Here the coefficients are,

$$\delta s_b = -\frac{1}{16}(D-2)(D-1)Df_{1b}\;, \qquad (4\text{–}146)$$
$$\delta s_c = \frac{(D-2)(D-1)}{16(D+1)}\Big[-(D-1)(D^2-2D-4)f_{1b}+2(D-3)f_{1c}\Big]\;, \qquad (4\text{–}147)$$
$$\delta s_d = \frac{(D-1)^2}{8(D+1)}\Big[D^3 f_{1b}-(D^2+2D-4)f_{1c}\Big]\;, \qquad (4\text{–}148)$$

$$\delta s_e = \frac{(D-2)^2(D-1)^2(D+2)}{4(D+1)} f_{1c} \; , \tag{4–149}$$

$$\delta s_{e'} = -\frac{(D-1)^3}{(D+1)} f_{1c} \; . \tag{4–150}$$

To infer the corresponding changes in the spin two quotient and remainder we need to invert $\mathcal{D}$ on $(\frac{4}{y})^{\frac{D}{2}+1}$, $(\frac{4}{y})^{\frac{D}{2}}$ and 1. The second one was given in (4–109). From expression (4–105) we find,

$$\frac{1}{\mathcal{D}}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}+1} = \frac{16}{(D-2)^2 D}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} - \frac{4}{\mathcal{D}}\Biggl\{\frac{(D-3)(D+2)}{(D-2)D}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}} + \frac{(D-3)(D+2)(D+4)}{(D-2)^2 D}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1}\Biggr\}, \tag{4–151}$$

$$\frac{1}{\mathcal{D}}\Bigl(1\Bigr) = \frac{1}{(D-3)D(D+1)} \; . \tag{4–152}$$

Although we want to move all the $(\frac{4}{y})^{\frac{D}{2}+1}$ and $(\frac{4}{y})^{\frac{D}{2}}$ terms from the remainder to the quotient, we must allow for an arbitrary amount $\delta f_{2c'}$ of the 1 term. Hence the changes in the quotient and the remainder take the form,

$$\delta\mathcal{Q}_2 = K\Biggl\{\frac{\delta f_{2b}}{D-4}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{\delta f_{2c}}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} + \frac{\delta f_{2c'}}{(D-4)^2}\Biggr\}, \tag{4–153}$$

$$\delta\mathcal{R}_2 = K\Biggl\{\frac{\delta f_{2d}}{D-4}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{\delta f_{2e}}{(D-4)^2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} + \frac{\delta f_{e'}}{(D-4)^2}\Biggr\}. \tag{4–154}$$

The various coefficients are,

$$\delta f_{2b} = \frac{16}{(D-2)^2 D} \times \delta s_b \; , \tag{4–155}$$

$$\delta f_{2c} = -\frac{64(D-3)(D+2)}{(D-2)^3 D} \times \delta s_b + \frac{16}{(D-2)^2} \times \delta s_c \; , \tag{4–156}$$

$$\delta f_{2d} = \frac{4(D-3)(D+2)(D+4)(3D-10)}{(D-2)^3 D} \times \delta s_b - \frac{4(D-3)(D+4)}{(D-2)^2} \times \delta s_c + \delta s_d \; , \tag{4–157}$$

$$\delta f_{2e} = \frac{16(D-3)^2(D+2)(D+4)(D+6)}{(D-2)^3 D} \times \delta s_b - \frac{4(D-3)(D+4)(D+6)}{(D-2)^2} \times \delta s_c + \delta s_e \; , \qquad (4\text{–}158)$$

$$\delta f_{2e'} = -(D-3)D(D+1)\delta f_{2c'} + \delta s_{e'} \; . \qquad (4\text{–}159)$$

It is possible to make the combination $\mathcal{Q}_2 + \delta\mathcal{Q}_2$ possess a finite unregulated limit by choosing,

$$\delta f_{2c'} = 1 - \frac{271}{60}(D-4) + \frac{11057}{3600}(D-4)^2 \; . \qquad (4\text{–}160)$$

With this choice the renormalized spin two quotient is,

$$\mathcal{Q}_{2R} = \frac{\kappa^2 H^4}{(4\pi)^4}\Biggl\{\frac{\Box}{H^2}\left[\frac{1}{240}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)\right] - \frac{1}{120}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr) + \frac{1}{240}\times\frac{4}{y} + \frac{1}{12}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr) - \frac{7}{30}\times\frac{4}{y} + \frac{1}{4}\ln^2\Bigl(\frac{y}{4}\Bigr) - \frac{119}{60}\ln\Bigl(\frac{y}{4}\Bigr)\Biggr\}. \qquad (4\text{–}161)$$

Choosing (4–160) also produces a finite result for the spin two remainder term,

$$\mathcal{R}_{2R} = \frac{\kappa^2 H^4}{(4\pi)^4}\Biggl\{\frac{17}{10}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr) - \frac{149}{30}\times\frac{4}{y} - \frac{41}{10}\ln^2\Bigl(\frac{y}{4}\Bigr) + \frac{193}{6}\ln\Bigl(\frac{y}{4}\Bigr) + \frac{359}{20} + \frac{32}{15(4-y)^3}\left[90\Bigl(\frac{y}{4}\Bigr)^4 - 291\Bigl(\frac{y}{4}\Bigr)^3 + 333\Bigl(\frac{y}{4}\Bigr)^2 - 152\Bigl(\frac{y}{4}\Bigr) + 21\right]\Bigl(\frac{4}{y}\Bigr)\ln(\frac{y}{4}) + \frac{4}{45(4-y)^3}\left[432\Bigl(\frac{y}{4}\Bigr)^3 - 792\Bigl(\frac{y}{4}\Bigr)^2 - 288\Bigl(\frac{y}{4}\Bigr) + 991 - 474\Bigl(\frac{4}{y}\Bigr) - 84\Bigl(\frac{4}{y}\Bigr)^2\right] - \frac{7}{60}(\frac{4}{y})^3\ln(1-\frac{y}{4}) - \frac{9}{10}\ln^2(\frac{y}{4})\Biggr\} \; . \qquad (4\text{–}162)$$

Acting the $D = 4$ Green's function (4–103) on the remainder and adding the result to the quotient gives our final result for the renormalized spin two structure function (recall the definition (4–138) of the dilogarithm function),

$$
\begin{aligned}
\mathcal{F}_{2R} = \frac{\kappa^2 H^4}{(4\pi)^4}\Bigg\{ & \frac{\Box}{H^2}\left[\frac{1}{240}\times\frac{4}{y}\ln(\Big(\frac{y}{4}\Big)\right] + \frac{3}{40}\times\frac{4}{y}\ln\Big(\frac{y}{4}\Big) - \frac{11}{48}\times\frac{4}{y} + \frac{1}{4}\ln^2\Big(\frac{y}{4}\Big) \\
& -\frac{119}{60}\ln\Big(\frac{y}{4}\Big) + \frac{4096}{(4y-y^2-8)^4}\Bigg[\Bigg[-\frac{47}{15}\Big(\frac{y}{4}\Big)^8 + \frac{141}{10}\Big(\frac{y}{4}\Big)^7 \\
& -\frac{2471}{90}\Big(\frac{y}{4}\Big)^6 + \frac{34523}{720}\Big(\frac{y}{4}\Big)^5 - \frac{132749}{1440}\Big(\frac{y}{4}\Big)^4 + \frac{38927}{320}\Big(\frac{y}{4}\Big)^3 \\
& -\frac{10607}{120}\Big(\frac{y}{4}\Big)^2 + \frac{22399}{720}\Big(\frac{y}{4}\Big) - \frac{3779}{960}\Bigg]\frac{4}{4-y} + \Bigg[\frac{193}{30}\Big(\frac{y}{4}\Big)^4 - \frac{131}{10}\Big(\frac{y}{4}\Big)^3 \\
& +\frac{7}{20}\Big(\frac{y}{4}\Big)^2 + \frac{379}{60}\Big(\frac{y}{4}\Big) - \frac{193}{120}\Bigg]\ln(2-\frac{y}{2}) + \Bigg[-\frac{14}{15}\Big(\frac{y}{4}\Big)^5 - \frac{1}{5}\Big(\frac{y}{4}\Big)^4 \\
& +\frac{19}{2}\Big(\frac{y}{4}\Big)^3 - \frac{889}{60}\Big(\frac{y}{4}\Big)^2 + \frac{143}{20}\Big(\frac{y}{4}\Big) - \frac{13}{20} - \frac{7}{60}\Big(\frac{4}{y}\Big)\Bigg]\ln(1-\frac{y}{4}) \\
& +\Bigg[-\frac{476}{15}\Big(\frac{y}{4}\Big)^9 + 160\Big(\frac{y}{4}\Big)^8 - \frac{5812}{15}\Big(\frac{y}{4}\Big)^7 + \frac{8794}{15}\Big(\frac{y}{4}\Big)^6 \\
& \quad -\frac{18271}{30}\Big(\frac{y}{4}\Big)^5 + \frac{54499}{120}\Big(\frac{y}{4}\Big)^4 - \frac{59219}{240}\Big(\frac{y}{4}\Big)^3 + \frac{1917}{20}\Big(\frac{y}{4}\Big)^2 \\
& \quad -\frac{1951}{80}\Big(\frac{y}{4}\Big) + \frac{367}{120}\Bigg]\frac{4}{4-y}\ln(\frac{y}{4}) + \Bigg[4\Big(\frac{y}{4}\Big)^7 - 12\Big(\frac{y}{4}\Big)^6 + 20\Big(\frac{y}{4}\Big)^5 \\
& \quad -20\Big(\frac{y}{4}\Big)^4 + 15\Big(\frac{y}{4}\Big)^3 - 7\Big(\frac{y}{4}\Big)^2 + \Big(\frac{y}{4}\Big)\Bigg]\frac{4-y}{4}\ln^2(\frac{y}{4}) \\
& +\Bigg[\frac{367}{30}\Big(\frac{y}{4}\Big)^4 - \frac{4121}{120}\Big(\frac{y}{4}\Big)^3 + \frac{237}{16}\Big(\frac{y}{4}\Big)^2 + \frac{1751}{240}\Big(\frac{y}{4}\Big) - \frac{367}{120}\Bigg]\ln(\frac{y}{2}) \\
& +\frac{1}{64}(y^2-8)\Big[4(2-y)-(4y-y^2)\Big]\Bigg[\frac{1}{5}\mathrm{Li}_2(1-\frac{y}{4}) + \frac{7}{10}\mathrm{Li}_2(\frac{y}{4})\Bigg]\Bigg]\Bigg\}\,.
\end{aligned}
\tag{4–163}
$$

CHAPTER 5
FLAT SPACE RESULT

5.1 Schwinger-Keldysh Effective Field Eqns

The graviton self-energy $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$ is the one-particle-irreducible (1PI) 2-point function for the graviton field $h_{\mu\nu}(t,\vec{x})$. It serves to quantum correct the linearized Einstein equation.

$$\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) + \int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')\, h_{\rho\sigma}(x') = \frac{8\pi G}{c^2} M \delta^{\mu}_{0} \delta^{\nu}_{0} \delta^3(\vec{x}) \; . \tag{5–1}$$

However, this equation suffers from two embarrassments:

- It isn't causal because the in-out self-energy is nonzero for points x'^{μ} which are spacelike separated from x^{μ}, or lie to its future; and
- It doesn't produce real potentials $h_{\mu\nu}$ because the in-out self-energy has an imaginary part.

One can get the right result for a static potential by simply ignoring the imaginary part [76, 77, 79], but circumventing the limitations of the in-out formalism becomes more and more difficult as time dependent sources and higher order corrections are included, and these techniques break down entirely for the case of cosmology in which there may not even be asymptotic vacua. It is not that the in-out self-energy is somehow "wrong". In fact, it is exactly the right thing to correct the Feynman propagator for asymptotic scattering computations in flat space. The point is rather that equation (5–1) doesn't provide the generalization we seek of the classical field equation.

The better technique is known as the Schwinger-Keldysh formalism [84]. It provides a way of computing true expectation values that is almost as simple as the Feynman diagrams which produce in-out matrix elements. The Schwinger-Keldysh rules are best stated in the context of a scalar field $\varphi(x)$ whose Lagrangian (the space integral of its Lagrangian density) at time t is $L[\varphi(t)]$. Suppose we are given a Heisenberg state $|\Psi\rangle$ whose wave functional in terms of the operator eigenkets at time t_0 is $\Psi[\varphi(t_0)]$, and we wish to take the expectation value, in the presence of this state, of a product of two functionals of the field operator: $A[\varphi]$, which is anti-time-ordered, and $B[\varphi]$, which is time-ordered. The Schwinger-Keldysh functional integral for this is [61],

$$\Big\langle \Psi \Big| A[\varphi] B[\varphi] \Big| \Psi \Big\rangle = \int [d\varphi_+][d\varphi_-] \, \delta\Big[\varphi_-(t_1) - \varphi_+(t_1)\Big] \times A[\varphi_-] B[\varphi_+] \Psi^*[\varphi_-(t_0)] e^{i \int_{t_0}^{t_1} dt \Big\{ L[\varphi_+(t)] - L[\varphi_-(t)] \Big\}} \Psi[\varphi_+(t_0)] \; . \tag{5–2}$$

The time $t_1 > t_0$ is arbitrary as long as it is later than the latest operator which is contained in either $A[\varphi]$ or $B[\varphi]$.

The Schwinger-Keldysh rules can be read off from its functional representation (5–2). Because the same field operator is represented by two different dummy functional variables, $\varphi_\pm(x)$, the endpoints of lines carry a $\pm$ polarity. External lines associated with the anti-time-ordered operator $A[\varphi]$ have the $-$ polarity whereas those associated with the time-ordered operator $B[\varphi]$ have the $+$ polarity. Interaction vertices are either all $+$ or all $-$. Vertices with $+$ polarity are the same as in the usual Feynman rules whereas vertices with the $-$ polarity have an additional minus sign. If the state $|\Psi\rangle$ is something other than free vacuum then it contributes additional interaction vertices on the initial value surface [67].

Propagators can be $++$, $+-$, $-+$, or $--$. All four polarity variations can be read off from the fundamental relation (5–2) when the free Lagrangian is substituted for the full one. It is useful to denote canonical expectation values in the free theory with a subscript 0. With this convention we see that the $++$ propagator is just the ordinary Feynman propagator,

$$i\Delta_{++}(x;x') = \Big\langle \Omega \Big| T\Big(\varphi(x)\varphi(x')\Big) \Big| \Omega \Big\rangle_0 = i\Delta(x;x') \; , \tag{5–3}$$

where T stands for time-ordering and $\overline{T}$ denotes anti-time-ordering. The other polarity variations are simple to read off and to relate to the Feynman propagator,

$$i\Delta_{-+}(x;x') = \Big\langle \Omega \Big| \varphi(x)\varphi(x') \Big| \Omega \Big\rangle_0 = \theta(t-t')i\Delta(x;x') + \theta(t'-t)\Big[i\Delta(x;x')\Big]^*, \tag{5–4}$$

$$i\Delta_{+-}(x;x') = \Big\langle \Omega \Big| \varphi(x')\varphi(x) \Big| \Omega \Big\rangle_0 = \theta(t-t')\Big[i\Delta(x;x')\Big]^* + \theta(t'-t)i\Delta(x;x'), \tag{5–5}$$

$$i\Delta_{--}(x;x') = \Big\langle \Omega \Big| \overline{T}\Big(\varphi(x)\varphi(x')\Big) \Big| \Omega \Big\rangle_0 = \Big[i\Delta(x;x')\Big]^*. \tag{5–6}$$

Therefore we can get the four propagators of the Schwinger-Keldysh formalism from the Feynman propagator once that is known.

Because external lines can be either $_+$ or $_-$ in the Schwinger-Keldysh formalism, every 1PI N-point function of the in-out formalism gives rise to 2^N 1PI N-point functions in the Schwinger-Keldysh formalism. For every classical field $\phi(x)$ of an in-out effective action, the coorresponding Schwinger-Keldysh effective action must depend upon two fields — call them $\phi_+(x)$ and $\phi_-(x)$ — in order to access the appropriate 1PI function [85]. For the scalar paradigm we have been considering this effective action takes the form,

$$\Gamma[\phi_+,\phi_-] = S[\phi_+] - S[\phi_-] - \frac{1}{2}\int d^4x \int d^4x' \\ \times \left\{ \begin{array}{c} \phi_+(x) M^2_{++}(x;x')\phi_+(x') + \phi_+(x) M^2_{+-}(x;x')\phi_-(x') \\ +\phi_-(x) M^2_{-+}(x;x')\phi_+(x') + \phi_-(x) M^2_{--}(x;x')\phi_-(x') \end{array} \right\} + O(\phi^3_\pm), \tag{5–7}$$

where S is the classical action. The effective field equations are obtained by varying with respect to ϕ_+ and then setting both fields equal [85],

$$\frac{\delta\Gamma[\phi_+,\phi_-]}{\delta\phi_+(x)}\bigg|_{\phi_\pm=\phi} = \Big[\partial^2 - m^2\Big]\phi(x) - \int d^4x' \Big[M^2_{++}(x;x') + M^2_{+-}(x;x')\Big]\phi(x') + O(\phi^2). \tag{5–8}$$

The two 1PI 2-point functions we would need to quantum correct the linearized scalar field equation are $M^2_{++}(x;x')$ and $M^2_{+-}(x;x')$. Their sum in (5–8) gives effective field equations which are causal in the sense that the two 1PI functions cancel unless x'^μ lies on or within the past light-cone of x^μ. Their sum is also real, which neither 1PI function is separately.

As mentioned before, the point of the present paper is to lay the groundwork for a computation of the one loop correction to the force of gravity on de Sitter background. The graviton contribution to the self-energy was computed some years ago [86] but has never been used in the effective field equations. A computation of the scalar contribution is underway.

Although the current computation will be the first to explore corrections to a force law, the linearized effective field equations have been studied on de Sitter background for many simpler models. In scalar quantum electrodynamics the one loop vacuum polarization was computed and used to correct for the propagation of dynamical photons [49, 52], but not yet for the Coulomb force. The one loop scalar self-mass-squared has also been used to correct for the propagation of charged, massless, minimally coupled scalars [87]. Both the fermion [55] and scalar [88] 1PI 2-point functions of Yukawa theory have been computed and used to correct the mode functions. The one and two loop scalar self-mass-squared of $\lambda\varphi^4$ theory has been computed and used to correct for the propagation of massless, minimally coupled scalars [57]. In Einstein + Dirac the one loop fermion self-energy has been computed and used to correct the fermion mode function [58]. And the same thing has been done for scalars in Scalar + Einstein [59].

5.2 Solving for the Potentials

We begin by expressing the linearized effective field equations in a form which is both manifestly real and causal. We then explain how these equations can be solved perturbatively. The hardest step is integrating the one loop self-energy against the tree order solution. The section closes by working out the two one loop potentials.

5.2.1 Achieving A Manifestly Real and Causal Form

The basis for our work is a position space result for the one loop contribution to the 1PI graviton 2-point function from a loop of massless, minimally coupled scalars, using dimensional regularization and a minimal choice for the higher derivative counterterms [61]. (Previous Schwinger-Keldysh computations of this quantity had been given in momentum space [89] which is not as useful for us.) All four polarization variations take the form,

$$\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\pm\pm}\Big](x;x') = \mathrm{D}^{\mu\nu\rho\sigma}\Sigma_{\pm\pm}(x;x') \; , \qquad (5\text{–}9)$$

where the 4th-order, tensor-differential operator is,

$$\mathrm{D}^{\mu\nu\rho\sigma} \equiv \Big[\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\Big]\Big[\eta^{\rho\sigma}\partial^2 - \partial^\rho\partial^\sigma\Big] + \frac{1}{3}\Big[\eta^{\mu(\rho}\eta^{\sigma)\nu}\partial^4 - 2\partial^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)} + \partial^\mu\partial^\nu\partial^\sigma\partial^\rho\Big] \; . \quad (5\text{–}10)$$

The four bi-scalars are,

$$\Sigma_{\pm\pm}(x;x') = (\pm)(\pm)\frac{i\kappa^2\partial^2}{5120\pi^4}\left[\frac{\ln(\mu^2\Delta x^2_{\pm\pm})}{\Delta x^2_{\pm\pm}}\right] \; , \qquad (5\text{–}11)$$

where μ^2 is the usual scale of dimensional regularization, $\kappa^2 \equiv 16\pi\hbar G/c^3$ is the loop-counting parameter of quantum gravity and the four Schwinger-Keldysh length functions are,

$$\Delta x^2_{++}(x;x') \equiv \Big\|\vec{x}-\vec{x}'\Big\|^2 - c^2\Big(|t-t'|-i\epsilon\Big)^2 \; , \qquad (5\text{–}12)$$

$$\Delta x^2_{+-}(x;x') \equiv \Big\|\vec{x}-\vec{x}'\Big\|^2 - c^2\Big(t-t'+i\epsilon\Big)^2 \; , \qquad (5\text{–}13)$$

$$\Delta x^2_{-+}(x;x') \equiv \Big\|\vec{x}-\vec{x}'\Big\|^2 - c^2\Big(t-t'-i\epsilon\Big)^2 \; , \qquad (5\text{–}14)$$

$$\Delta x^2_{--}(x;x') \equiv \Big\|\vec{x}-\vec{x}'\Big\|^2 - c^2\Big(|t-t'|+i\epsilon\Big)^2 \; . \qquad (5\text{–}15)$$

Although the divergent parts of (5–9) have been subtracted off [61], it should be noted that they agree exactly with those originally found by ’t Hooft and Veltman [9].

We can achieve a significant simplification by first extracting another d’Alembertian from (5–11),

$$\Sigma_{\pm\pm}(x;x') = (\pm)(\pm)\frac{i\kappa^2\partial^4}{40960\pi^4}\Big[\ln^2(\mu^2\Delta x^2_{\pm\pm}) - 2\ln(\mu^2\Delta x^2_{\pm\pm})\Big] \; . \qquad (5\text{–}16)$$

Now define the position and temporal separations, and the associated invariant length-squared,

$$\Delta r \equiv \|\vec{x}-\vec{x}'\| \qquad , \qquad \Delta t \equiv t-t' \qquad , \qquad \Delta x^2 \equiv \Delta r^2 - c^2\Delta t^2 \; . \qquad (5\text{–}17)$$

The $_{++}$ and $_{+-}$ logarithms can be expanded in terms of their real and imaginary parts,

$$\ln(\mu^2 \Delta x^2_{++}) = \ln(\mu^2 |\Delta x^2|) + i\pi\,\theta(-\Delta x^2) \; , \tag{5–18}$$

$$\ln(\mu^2 \Delta x^2_{+-}) = \ln(\mu^2 |\Delta x^2|) - i\pi\,\mathrm{sgn}(\Delta t)\theta(-\Delta x^2) \; . \tag{5–19}$$

The $_{++}$ and $_{+-}$ logarithms agree for spacelike separation ($\Delta x^2 > 0$), and for $t' > t$, whereas they are complex conjugates of one another for $x'^{\mu} = (ct', \vec{x'})$ in the past light-cone of $x^{\mu} = (ct, \vec{x})$. Hence the sum of $\Sigma_{++}(x; x')$ and $\Sigma_{+-}(x; x')$ is both causal and real,

$$\Sigma_{++}(x; x') + \Sigma_{+-}(x; x') = -\frac{\kappa^2 \partial^4}{10240\pi^3}\,\theta(c\Delta t - \Delta r)\Big[\ln(-\mu^2 \Delta x^2) - 1\Big] \; . \tag{5–20}$$

Let us assume that the state is released in free vacuum at time $t = 0$. Our final result for the linearized, one loop effective field equations is,

$$\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}(t, \vec{x}) - \frac{\kappa^2 \mathrm{D}^{\mu\nu\rho\sigma} \partial^4}{10240\pi^3} \int_0^t dct' \int d^3x'\, \theta(c\Delta t - \Delta r) \times\Big[\ln(-\mu^2 \Delta x^2) - 1\Big] h_{\rho\sigma}(t', \vec{x'}) = \frac{8\pi G M}{c^2}\, \delta^{\mu}_0 \delta^{\nu}_0 \delta^3(\vec{x}) \; . \tag{5–21}$$

Recall that $\kappa^2 \equiv 16\pi\hbar G/c^3$ is the loop counting parameter of quantum gravity, the Lichnerowitz operator $\mathcal{D}^{\mu\nu\rho\sigma}$ was given in (3–35) and the 4th order differential operator $\mathrm{D}^{\mu\nu\rho\sigma}$ was given in (5–10).

5.2.2 Solving the Equation Perturbatively

There is no point in trying to solve equation (5–21) exactly because it only includes the one loop graviton self-energy. A better approach is to seek a perturbative solution in powers of the loop counting parameter κ^2,

$$h_{\mu\nu}(t, \vec{x}) = \sum_{\ell=0}^{\infty} \kappa^{2\ell} h^{(\ell)}_{\mu\nu}(t, \vec{x}) \; . \tag{5–22}$$

Of course the $\ell = 0$ term obeys the linearized Einstein equation whose solution in Schwarzschild coordinates is,

$$\mathcal{D}^{\mu\nu\rho\sigma} h^{(0)}_{\rho\sigma}(t,\vec{x}) = \frac{8\pi GM}{c^2}\,\delta^\mu_0\delta^\nu_0\delta^3(\vec{x}) \implies h^{(0)}_{00} = \frac{2GM}{c^2 r}\;,\; h^{(0)}_{ij} = \frac{2GM}{c^2 r}\,\widehat{r}^i\widehat{r}^j\;. \tag{5–23}$$

The one loop correction $h^{(1)}_{\mu\nu}$ obeys the equation,

$$\mathcal{D}^{\mu\nu\rho\sigma} h^{(1)}_{\rho\sigma}(t,\vec{x}) = \frac{\mathrm{D}^{\mu\nu\rho\sigma}\partial^4}{10240\pi^3}\int_0^t dt'\!\int d^3x'\,\theta(\Delta t - \Delta r)\Big[\ln(-\mu^2\Delta x^2) - 1\Big]\,h^{(0)}_{\rho\sigma}(t',\vec{x'})\;. \tag{5–24}$$

Finding the two loop correction $h^{(2)}_{\mu\nu}$ would require the two loop self-energy, which we do not have, so $h^{(1)}_{\mu\nu}$ is as high as we can go.

5.2.3 Correction to Dynamical Gravitons in Flat Space

The one loop contribution to the graviton self-energy from MMC scalars in a flat background was first computed by ‘t Hooft and Veltman in 1974 [9]. When renormalized and expressed in position space using the Schwinger-Keldysh formalism the result takes the form [61], (this result was reviewed in section 4.2)

$$\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x') = \Pi^{\mu\nu}\Pi^{\rho\sigma}F_0(\Delta x^2) + \Big[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{3}\Pi^{\mu\nu}\Pi^{\rho\sigma}\Big]F_2(\Delta x^2)\;. \tag{5–25}$$

Here $\Pi^{\mu\nu} \equiv \partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2$ and the two structure functions are,

$$F_0(\Delta x^2) \;=\; \frac{i\kappa^2}{(4\pi)^4}\frac{\partial^2}{9}\left[\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\right]\;, \tag{5–26}$$

$$F_2(\Delta x^2) \;=\; \frac{i\kappa^2}{(4\pi)^4}\frac{\partial^2}{60}\left[\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\right] \tag{5–27}$$

The two coordinate intervals are,

$$\Delta x^2_{++} \;\equiv\; \Big\|\vec{x} - \vec{x'}\Big\|^2 - \Big(|x^0 - x'^0| - i\epsilon\Big)^2\;, \tag{5–28}$$

$$\Delta x^2_{+-} \;\equiv\; \Big\|\vec{x} - \vec{x'}\Big\|^2 - \Big(x^0 - x'^0 + i\epsilon\Big)^2\;. \tag{5–29}$$

Of course this same form follows from taking the flat space limit of the de Sitter result summarized in the previous section.

In flat space, the mode function for a plane wave graviton with wave vector $\vec{k}$ is,

$$h^{\rm flat}_{\mu\nu}(x) = \epsilon_{\rho\sigma}(\vec{k}) \frac{1}{\sqrt{2k}}\, e^{-ikx^0 + i\vec{k}\cdot\vec{x}} \; . \tag{5–30}$$

The one loop correction to this (from MMC scalars) is sourced by,

$$\Bigl({\rm Source}\Bigr)^{\mu\nu}(x) = \int dx^4 x' \, \Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Bigr](x;x') h^{\rm flat}_{\rho\sigma}(x') \; . \tag{5–31}$$

It might seem natural to extract the various derivatives with respect to x^μ from the integration, for example,

$$\int d^4x' \, \Pi^{\mu\nu} \Pi^{\rho\sigma} F_0(\Delta x^2) \times h^{\rm flat}_{\rho\sigma}(x') \\ = \frac{i\kappa^2}{(4\pi)^4} \Pi^{\mu\nu} \Pi^{\rho\sigma} \frac{\partial^2}{9} \int d^4x' \left[\frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \right] \times h^{\rm flat}_{\rho\sigma}(x') \; . \tag{5–32}$$

That would reduce the source (5–31) to a tedious set of integrations, followed by some equally tedious differentiations.

The point of this sub-section is that a more efficient strategy is to first convert all the x^μ derivatives to x'^μ derivatives — which can be done because they act on functions of Δx^2. Then ignore surface terms and partially integrate the x'^μ derivatives to act upon $h^{\rm flat}_{\rho\sigma}(x')$. For example, doing this for the spin zero contribution (5–32) gives,

$$\int d^4x' \, \Pi^{\mu\nu} \Pi^{\rho\sigma} F_0(\Delta x^2) \times h^{\rm flat}_{\rho\sigma}(x') \\ \longrightarrow \frac{i\kappa^2}{(4\pi)^4} \int d^4x' \left[\frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \right] \times \frac{\partial'^2}{9} \Pi'^{\mu\nu} \Pi'^{\rho\sigma} h^{\rm flat}_{\rho\sigma}(x') \; . \tag{5–33}$$

Because the graviton mode function is both transverse and traceless, we have $\Pi'^{\rho\sigma} h^{\rm flat}_{\rho\sigma}(x') = 0$. The spin two contribution is only a little more complicated,

$$\int d^4x' \left[\Pi^{\mu(\rho} \Pi^{\sigma)\nu} - \frac{1}{3} \Pi^{\mu\nu} \Pi^{\rho\sigma} \right] F_2(\Delta x^2) \times h^{\rm flat}_{\rho\sigma}(x') \\ \longrightarrow \frac{i\kappa^2}{(4\pi)^4} \int d^4x' \left[\frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \right] \times \frac{\partial'^6}{60} h^{\mu\nu}_{\rm flat}(x') \; . \tag{5–34}$$

This also vanishes because $\partial'^2 h^{\rm flat}_{\rho\sigma}(x') = 0$.

In expressions (5–33) and (5–34) we have employed a rightarrow, rather than an equals sign, because the surface terms produce by partial integration were ignored. There are no surface terms at spatial infinity in the Schwinger-Keldysh formalism because the $++$ and $+-$ terms cancel for spacelike separation. The $++$ and $+-$ contributions also cancel when $x'^0 > x^0$, so there are no future surface terms. However, there are nonzero contributions from the initial value surface.[1] We assume that all such contributions are absorbed into perturbative corrections to the initial state, such as has recently been worked out for a MMC scalar with quartic self-interaction [102].

5.2.4 The One Loop Source Term

In this subsection we evaluate the right hand side of equation(5–24), which sources the one loop correction $h^{(1)}_{\mu\nu}(t,\vec{x})$. This is done in three steps: we first perform the integral, then act the ∂^4, and finally act the $\mathrm{D}^{\mu\nu\rho\sigma}$.

From the form of the tree order potentials (5–23) it is apparent that we need two integrals. The first comes from $h^{(0)}_{00}$,

$$\int_0^t dt' \int d^3x' \, \theta(\Delta t - \Delta r) \Big[\ln(-\mu^2 \Delta x^2) - 1\Big] \times \frac{1}{\|\vec{x'}\|} \equiv F(t,r) \; . \tag{5–35}$$

The second integral derives from the other nonzero potential $h^{(0)}_{ij}$. Its trace part is obviously the same as $F(t,r)$, and we shall call its traceless part $G(t,r)$,

$$\begin{aligned} \int_0^t dt' \int d^3x' \, \theta(\Delta t - \Delta r) \Big[\ln(-\mu^2 \Delta x^2) - 1\Big] \times \frac{\widehat{r}'^i \widehat{r}'^j}{\|\vec{x'}\|} \\ \equiv \frac{1}{2}\Big[\delta^{ij} - \widehat{r}^i \widehat{r}^j\Big] F(t,r) - \frac{1}{2}\Big[\delta^{ij} - 3\widehat{r}^i \widehat{r}^j\Big] G(t,r) \; , \qquad \text{(5–36)} \\ = \frac{1}{3}\delta^{ij} F(t,r) + \frac{1}{2}\Big[3\widehat{r}^i \widehat{r}^j - \delta^{ij}\Big]\Big[G(t,r) - \frac{1}{3}F(t,r)\Big] \; . \qquad \text{(5–37)} \end{aligned}$$

[1] For a two loop example, see [101].

The integrals are tedious but straightforward and give the following results for $F(t,r)$ and the combination $G(t,r) - \frac{1}{3}F(t,r)$,

$$F(t,r) = \frac{4\pi}{r}\Bigg\{\frac{r^4}{6}\ln(2\mu r) - \frac{25}{72}r^4 + \frac{11}{18}r^3 ct - \frac{11}{18}rc^3t^3 + \Big[\frac{1}{12}(ct+r)^4 - \frac{r}{6}(r+ct)^3\Big]\ln\Big[\mu(ct+r)\Big] - \Big[\frac{1}{12}(ct-r)^4 + \frac{r}{6}(ct-r)^3\Big]\ln\Big[\mu(ct-r)\Big]\Bigg\}, \tag{5–38}$$

$$G(t,r) - \frac{F(t,r)}{3} = \frac{4\pi}{r}\Bigg\{-\frac{r^4}{9}\ln(2\mu r) + \frac{23}{108}r^4 - \frac{199}{675}r^3 ct - \frac{13}{135}rc^3t^3 + \frac{2c^5t^5}{45r} + \Big[-\frac{(ct+r)^6}{45r^2} + \frac{2}{15}\frac{(ct+r)^5}{r} - \frac{5}{18}(ct+r)^4 + \frac{2}{9}r(ct+r)^3\Big]\ln\Big[\mu(ct+r)\Big] + \Big[\frac{(ct-r)^6}{45r^2} + \frac{2}{15}\frac{(ct-r)^5}{r} + \frac{5}{18}(ct-r)^4 + \frac{2}{9}r(ct-r)^3\Big]\ln\Big[\mu(ct-r)\Big]\Bigg\}. \tag{5–39}$$

The next step is acting the two d'Alembertians. This purges all the time dependent terms,

$$\partial^4 F(t,r) = \frac{4\pi}{r} \times 4\ln(2\mu r)\ , \tag{5–40}$$

$$\partial^4\Bigg\{\frac{1}{3}\delta^{ij}F(t,r) + \frac{1}{2}\Big[3\widehat{r}^i\widehat{r}^j - \delta^{ij}\Big]\Big[G(t,r) - \frac{1}{3}F(t,r)\Big]\Bigg\} = \frac{4\pi}{r}\Bigg\{\frac{4}{3}\delta^{ij}\ln(2\mu r) + \Big[3\widehat{r}^i\widehat{r}^j - \delta^{ij}\Big]\Big[\frac{4}{3}\ln(2\mu r) - 2\Big]\Bigg\}\ . \tag{5–41}$$

At this stage the linearized, one loop effective field equations (5–21) take the form,

$$\mathcal{D}^{\mu\nu\rho\sigma}h^{(1)}_{\rho\sigma}(t,\vec{x}) = \frac{GM}{1280\pi^2c^2}\,\mathrm{D}^{\mu\nu\rho\sigma}f_{\rho\sigma}(\vec{x})\ , \tag{5–42}$$

where the nonzero components of the tensor $f_{\rho\sigma}(\vec{x})$ are,

$$f_{00}(\vec{x}) = \frac{4}{r}\ln(2\mu r)\ , \tag{5–43}$$

$$f_{ij}(\vec{x}) = \delta^{ij} \times \frac{4}{3}\frac{\ln(2\mu r)}{r} + \Big[3\widehat{r}^i\widehat{r}^j - \delta^{ij}\Big]\Big[\frac{4}{3}\frac{\ln(2\mu r)}{r} - \frac{2}{r}\Big]\ . \tag{5–44}$$

It remains only to act the operator $\mathrm{D}^{\mu\nu\rho\sigma}$ on $f_{\rho\sigma}(\vec{x})$. The first two derivatives give,

$$\partial^\rho \partial^\sigma f_{\rho\sigma} = \frac{4}{r^3} \;\; , \;\; \partial_j f_{ij} = \widehat{r}^i \, \frac{[4\ln(2\mu r) - 4]}{r^2} \; , \tag{5–45}$$

$$\partial^2 f_{00} = -\frac{4}{r^3} \;\; , \;\; \partial^2 f_{ij} = \delta^{ij} \frac{[8\ln(2\mu r) - 12]}{r^3} - \widehat{r}^i \widehat{r}^j \frac{[24\ln(2\mu r) - 32]}{r^3} \; , \tag{5–46}$$

$$\partial_i \partial_j f_{00} = \partial_k \partial_i f_{jk} \;\; , \;\; \partial_i \partial_k f_{jk} = \delta^{ij} \, \frac{[4\ln(2\mu r) - 4]}{r^3} - \widehat{r}^i \widehat{r}^j \, \frac{[12\ln(2\mu r) - 16]}{r^3} \; . \tag{5–47}$$

The source term can then be expressed in terms of two more derivatives of the quantities $g_{\mu\nu}(\vec{x}) \equiv \nabla^2 f_{\mu\nu}(\vec{x})$ and $g(\vec{x}) \equiv \partial_i \partial_j f_{ij}(\vec{x})$,

$$\mathcal{D}^{\mu\nu\rho\sigma} h^{(1)}_{\rho\sigma}(t, \vec{x}) = \frac{GM}{1280\pi^2 c^2} \Big\{ -\eta^{\mu\nu} \nabla^2 g + \frac{4}{3} \partial^\mu \partial^\nu g + \frac{1}{3} \nabla^2 g^{\mu\nu} - \frac{2}{3} \partial_\rho \partial^{(\mu} g^{\nu)\rho} \Big\} \; . \tag{5–48}$$

The final reduction employs the identities,

$$\nabla^2 g = \frac{24}{r^5} \quad , \quad \partial_i \partial_j g = -\frac{12}{r^5} \, \delta^{ij} + \frac{60}{r^5} \, \widehat{r}^i \widehat{r}^j \; , \tag{5–49}$$

$$\nabla^2 g_{00} = -\frac{24}{r^5} \quad , \quad \nabla^2 g_{ij} = -\frac{48}{r^5} \, \delta^{ij} + \frac{120}{r^5} \, \widehat{r}^i \widehat{r}^j \; , \tag{5–50}$$

$$\partial_k g_{jk} = -\frac{12}{r^4} \, \widehat{r}^j \quad , \quad \partial_i \partial_k g_{jk} = -\frac{12}{r^5} \, \delta^{ij} + \frac{60}{r^5} \, \widehat{r}^i \widehat{r}^j \; . \tag{5–51}$$

The nontrivial components of the effective field equations are,

$$\mathcal{D}^{00\rho\sigma} h^{(1)}_{\rho\sigma}(t, \vec{x}) \;\; = \;\; \frac{GM}{80\pi^2 c^2} \times \frac{1}{r^5} \; , \tag{5–52}$$

$$\mathcal{D}^{ij\rho\sigma} h^{(1)}_{\rho\sigma}(t, \vec{x}) \;\; = \;\; \frac{GM}{80\pi^2 c^2} \times \Big\{ -\frac{3\delta^{ij}}{r^5} + \frac{5\widehat{r}^i \widehat{r}^j}{r^5} \Big\} \; . \tag{5–53}$$

5.2.5 The One Loop Potentials

We wish to express the one loop potentials in Schwarzschild coordinates so their nonzero components take the form,

$$h^{(1)}_{00}(\vec{x}) = a(r) \qquad , \qquad h^{(1)}_{ij}(\vec{x}) = \widehat{r}^i \widehat{r}^j b(r) \; . \tag{5–54}$$

Acting the Lichnerowitz operator (3–35) on these gives,

$$\mathcal{D}^{00\rho\sigma} h^{(1)}_{\rho\sigma} = \frac{b'}{r} + \frac{b}{r^2} \; , \tag{5–55}$$

$$\mathcal{D}^{ij\rho\sigma} h^{(1)}_{\rho\sigma} = \delta^{ij}\Big[-\frac{a''}{2} - \frac{a'}{2r} - \frac{b'}{2r}\Big] + \widehat{r}^i\widehat{r}^j\Big[\frac{a''}{2} - \frac{a'}{2r} + \frac{b'}{2r} - \frac{b}{r^2}\Big] \; . \tag{5–56}$$

Comparing (5–55) with (5–52) implies,

$$b(r) = \frac{GM}{160\pi^2 c^2} \times -\frac{1}{r^3} \; . \tag{5–57}$$

Substituting this in (5–56) and comparing with (5–53) implies,

$$a(r) = \frac{GM}{160\pi^2 c^2} \times \frac{1}{r^3} \; . \tag{5–58}$$

Combining the classical and quantum corrections gives the following total results for the potentials,

$$h_{00}(\vec{x}) = \frac{2GM}{c^2 r}\Bigg\{1 + \frac{\hbar G}{20\pi c^3 r^2} + O\Big(\frac{\kappa^4}{r^4}\Big)\Bigg\} \; , \tag{5–59}$$

$$h_{ij}(\vec{x}) = \frac{2GM}{c^2 r}\Bigg\{1 - \frac{\hbar G}{20\pi c^3 r^2} + O\Big(\frac{\kappa^4}{r^4}\Big)\Bigg\}\widehat{r}^i\widehat{r}^j \; . \tag{5–60}$$

Expression (5–59) agrees with equation (3.9) of Hamber and Liu [20], and also with equation (32) of [90]. When transformed to de Donder gauge our results (5–59)-(5–60) give the same trace obtained in equation (59) by Dalvit and Mazzitelli [91].

CHAPTER 6
QUANTUM CORRECTIONS TO DYNAMICAL GRAVITONS

6.1 The Effective Field Equations

The purpose of this section is to present the effective field equation which we solve in the next section. We begin by reviewing some useful facts about the background geometry. We then give our recently derived result for the one loop MMC scalar contribution to the graviton self-energy [98]. The section closes with a discussion of the Schwinger-Keldysh effective field equations and how one solves them perturbatively.

6.1.1 The Schwinger-Keldysh Effective Field Equations

Because the graviton self-energy is the 1PI graviton 2-point function, it gives the quantum correction to the linearized Einstein equation,

$$\sqrt{-g}\,\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}(x) - \int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')h_{\rho\sigma}(x') = \frac{1}{2}\kappa\sqrt{-g}\; T^{\mu\nu}_{\rm lin}(x) \; , \qquad (6\text{–}1)$$

Here $\mathcal{D}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator, (3–35) specialized to de Sitter background

$$\mathcal{D}^{\mu\nu\rho\sigma} \equiv D^{(\rho}g^{\sigma)(\mu}D^{\nu)} - \frac{1}{2}\Big[g^{\rho\sigma}D^{\mu}D^{\nu} + g^{\mu\nu}D^{\rho}D^{\sigma}\Big] + \frac{1}{2}\Big[g^{\mu\nu}g^{\rho\sigma} - g^{\mu(\rho}g^{\sigma)\nu}\Big]D^2 + (D-1)\Big[\frac{1}{2}g^{\mu\nu}g^{\rho\sigma} - g^{\mu(\rho}g^{\sigma)\nu}\Big]H^2 \; , \qquad (6\text{–}2)$$

and D^{μ} is the covariant derivative operator in the background geometry. The point of the Schwinger-Keldysh formalism is explained in Sec 5.1. Here we give the expression for the de Sitter case. At the one loop order we are working $[{}^{\mu\nu}\Sigma^{\rho\sigma}]_{++}(x;x')$ agrees exactly with the in-out result given in the previous sub-section. To get $[{}^{\mu\nu}\Sigma^{\rho\sigma}]_{+-}(x;x')$, at this order, one simply adds a minus sign and replaces the de Sitter length function $y(x;x')$ everywhere with,

$$y(x;x') \longrightarrow y_{+-}(x;x') \equiv H^2 a(\eta)a(\eta')\Big[\|\vec{x}-\vec{x}'\|^2 - (\eta-\eta'+i\epsilon)^2\Big] \; . \qquad (6\text{–}3)$$

It will be seen that the $++$ and $+-$ self-energies cancel unless the point x'^{μ} is on or inside the past light-cone of x^{μ}. That makes the effective field equation (6–1) causal.

When x'^{μ} is on or inside the past light-cone of x^{μ} the $+-$ self-energy is the complex conjugate of the $++$ one, which makes the effective field equation (6–1) real. This also effects a great simplification in the structure functions because only those terms with branch cuts in y can make nonzero contributions, for example,

$$\ln(y_{++}) - \ln(y_{+-}) = 2\pi i \theta\Big(\eta - \eta' - \|\vec{x} - \vec{x'}\|\Big) \; . \tag{6–4}$$

6.1.2 Perturbative Solution

Because we only know the self-energy at one loop order, all we can do is to solve (6–1) perturbatively by expanding the graviton field and the self-energy in powers of κ^2,

$$h_{\mu\nu}(x) = h^{(0)}_{\mu\nu}(x) + \kappa^2 h^{(1)}_{\mu\nu}(x) + O(\kappa^4) \; . \tag{6–5}$$

Of course $h^{(0)}_{\mu\nu}(x)$ obeys the classical, linearized Einstein equation. Given this solution, the corresponding one loop correction is defined by the equation,

$$\sqrt{-g(x)}\, \mathcal{D}^{\mu\nu\rho\sigma} \kappa^2 h^{(1)}_{\rho\sigma}(x) = \int d^4x' \, \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') h^{(0)}_{\rho\sigma}(x') \; . \tag{6–6}$$

The classical solution for a dynamical graviton of wave vector $\vec{k}$ is [70],

$$h^{(0)}_{\rho\sigma}(x) = \epsilon_{\rho\sigma}(\vec{k}) u(\eta, k) e^{i\vec{k}\cdot\vec{x}} \; , \tag{6–7}$$

where the tree order mode function is,

$$u(\eta, k) = \frac{H}{\sqrt{2k^3}} \Big[1 - \frac{ik}{Ha}\Big] \exp\Big[\frac{ik}{Ha}\Big] \; , \tag{6–8}$$

and the polarization tensor obeys all the same relations as in flat space,

$$0 = \epsilon_{0\mu} = k_i \epsilon_{ij} = \epsilon_{jj} \quad \text{and} \quad \epsilon_{ij}\epsilon^*_{ij} = 1 \; . \tag{6–9}$$

6.2 Computing the One Loop Source

The point of this section is to evaluate the one loop source term on the right hand side of equation (6–6) for a dynamical graviton (6–7)-(6–9). We begin by drawing inspiration from what happens in the flat space limit. Our de Sitter analysis commences by partially integrating the projectors. This results in considerable simplification but the plethora of indices is still problematic. To effect further simplification we extract and partially integrate another d'Alembertian, whereupon the x^μ projector can be acted on the residual structure function to eliminate four contractions. At this point we digress to derive some important identities concerning covariant derivatives of the Weyl tensor. The final reduction reveals zero net result.

6.2.1 Partial Integration

We now start to evaluate the one loop source term (6–6) for a dynamical graviton,

$$\begin{aligned}&\int d^4x' \left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right](x;x') h^{(0)}_{\rho\sigma}(x') \\ &= i\int d^4x' \sqrt{-g(x)}\,\mathcal{P}^{\mu\nu}(x)\sqrt{-g(x')}\,\mathcal{P}^{\rho\sigma}(x')\Big\{\mathcal{F}_0\Big\} h^{(0)}_{\rho\sigma}(x') \\ &+2i\int d^4x' \sqrt{-g(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\sqrt{-g(x')}\,\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')\Big\{\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\mathcal{F}_2\Big\} h^{(0)}_{\rho\sigma}(x') \; . \end{aligned} \tag{6–10}$$

In this expression and henceforth we simply write "$\mathcal{F}_0$" and "$\mathcal{F}_2$" to stand for the full Schwinger-Keldysh expressions,

$$\mathcal{F}_0 \equiv \mathcal{F}_0(y_{++}) - \mathcal{F}_0(y_{+-}) \qquad , \qquad \mathcal{F}_2 \equiv \mathcal{F}_2(y_{++}) - \mathcal{F}_2(y_{+-}) \; . \tag{6–11}$$

The integral (6–10) can be simplified in two steps. First, the projectors $\mathcal{P}^{\mu\nu}(x)$ and $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)$, which act on a function of x^μ, can be pulled outside the integration over x'^μ. Second, the projectors $\mathcal{P}^{\rho\sigma}(x')$ and $\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')$, which act on x'^μ, can be partially integrated to act on the graviton wave function $h^{(0)}_{\rho\sigma}(x')$. After these two steps, the integral (6–10) becomes,

$$\begin{aligned}&\int d^4x' \left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right](x;x') h^{(0)}_{\rho\sigma}(x') \\ &= i\sqrt{-g(x)}\,\mathcal{P}^{\mu\nu}(x)\int d^4x' \sqrt{-g(x')}\,\mathcal{F}_0\Big\{\mathcal{P}^{\rho\sigma}(x') h^{(0)}_{\rho\sigma}(x')\Big\} \\ &+2i\sqrt{-g(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\int d^4x' \sqrt{-g(x')}\,\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\mathcal{F}_2\Big\{\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x') h^{(0)}_{\rho\sigma}(x')\Big\} . \end{aligned} \tag{6–12}$$

Note that the spin zero term drops out due to the tranversality and tracelessness of the dynamical graviton, $h^{(0)}_{\rho\sigma}$:

$$\mathcal{P}^{\rho\sigma} h^{(0)}_{\rho\sigma} = \Big\{ D^\rho D^\sigma - \Big[D^2 + (D-1) H^2 \Big] g^{\rho\sigma} \Big\} h^{(0)}_{\rho\sigma} = 0 \; . \tag{6–13}$$

Thus we only have the spin two term, which gives the linearized Weyl tensor,

$$\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x') h^{(0)}_{\rho\sigma}(x') = \delta C_{\kappa\lambda\theta\phi}(x') \; . \tag{6–14}$$

The one loop source term then reduces to the integral,

$$\int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma} \Big](x;x') h^{(0)}_{\rho\sigma}(x') \\ = 2i \sqrt{-g(x)}\, \mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x) \int d^4x' \sqrt{-g(x')}\, \mathcal{T}^{\alpha\kappa} \mathcal{T}^{\beta\lambda} \mathcal{T}^{\gamma\theta} \mathcal{T}^{\delta\phi} \mathcal{F}_2 \delta C_{\kappa\lambda\theta\phi}(x') \; . \tag{6–15}$$

6.2.2 Extracting Another d'Alembertian

A challenge to evaluating expression (6–15) is the complicated tensor structure of the external projector $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)$ acting on the internal factors of $\mathcal{T}^{\alpha\kappa} \cdots \mathcal{F}_2$. Recall from the flat space limit that all of this was converted to derivatives with respect to x'^μ and then partiall integrated onto the graviton wave function to give zero. To follow this on de Sitter we must make the structure function more convergent by extracting a factor of $\square'$ and then partially integrating it onto the graviton wave function. After this the external projector can be acted, which eliminates four indices, and a final further partial integration can be performed.

The first step is extracting the extra d'Alembertian,

$$\mathcal{F}_2 = \frac{\square'}{H^2} \widehat{\mathcal{F}}_2 \; . \tag{6–16}$$

We next commute the $\square'$ through the factor of $\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}$:

$$\begin{aligned}
\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\frac{\square'}{H^2}\widehat{\mathcal{F}}_2 &= \Big(\frac{\square'}{H^2}+4\Big)\Big[\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\widehat{\mathcal{F}}_2\Big] \\
&-\frac{1}{H^2}\widehat{\mathcal{F}'_2}\Big\{\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}+\cdots+\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\frac{\partial y}{\partial x_\delta}\frac{\partial y}{\partial x'_\phi}\Big\} \\
&-\frac{1}{2H^2}\widehat{\mathcal{F}_2}\Big\{g^{\alpha\beta}\frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}+g^{\alpha\gamma}\frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\delta\phi} \\
&\qquad +g^{\alpha\delta}\frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}+g^{\beta\gamma}\frac{\partial y}{\partial x'_\lambda}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\delta\phi} \\
&\qquad\qquad +g^{\beta\delta}\frac{\partial y}{\partial x'_\lambda}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\gamma\theta}+g^{\gamma\delta}\frac{\partial y}{\partial x'_\theta}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\Big\}\ . \qquad (6\text{–}17)
\end{aligned}$$

Exploiting the tracelessness of the Weyl tensor on any two indices, and its antisymmetry on the first two and last two indices, gives,

$$\begin{aligned}
P^{\mu\nu}_{\alpha\beta\gamma\delta}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\frac{\square'}{H^2}\widehat{\mathcal{F}}_2\delta C_{\kappa\lambda\theta\phi} &= P^{\mu\nu}_{\alpha\beta\gamma\delta}\frac{\square'}{H^2}\Big[\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big]\delta C_{\kappa\lambda\theta\phi} \\
= P^{\mu\nu}_{\alpha\beta\gamma\delta}\Big\{4\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}&-\frac{4}{H^2}\widehat{\mathcal{F}'}_2\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big\}\delta C_{\kappa\lambda\theta\phi}\ . \qquad (6\text{–}18)
\end{aligned}$$

For the first term of (6–18) we can partially integrate the $\square'$ onto the linearized Weyl tensor. Then the one loop source term becomes

$$\begin{aligned}
&\int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')h^{(0)}_{\rho\sigma}(x') \\
&= 2i\sqrt{-g(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\int d^4x'\sqrt{-g(x')}\Big\{\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\widehat{\mathcal{F}}_2\frac{\square'}{H^2}\delta C_{\kappa\lambda\theta\phi}(x') \\
&\quad+\Big[4\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}-\frac{4}{H^2}\widehat{\mathcal{F}'}_2\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big]\delta C_{\kappa\lambda\theta\phi}(x')\Big\}\ . \qquad (6\text{–}19)
\end{aligned}$$

This sets the stage for acting the outer projector.

6.2.3 Derivatives of the Weyl Tensor

At this point it is useful to make a short digression on the covariant derivatives of the Weyl tensor. In this sub-section we use $g_{\mu\nu}$ for the full metric, not the de Sitter background. All curvatures are similarly for the full metric.

The Bianchi identity tells us,

$$D_\epsilon R_{\alpha\beta\gamma\delta} + D_\gamma R_{\alpha\beta\delta\epsilon} + D_\delta R_{\alpha\beta\epsilon\gamma} = 0 \; . \tag{6–20}$$

If the stress-energy vanishes, all solutions to the Einstein equation obey,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -3H^2 g_{\mu\nu} \qquad \Longrightarrow \qquad R_{\mu\nu} = 3H^2 g_{\mu\nu} \; . \tag{6–21}$$

In $D = 3+1$ the Weyl tensor can be expressed in terms of the other curvatures as,

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2}\Bigl(g_{\alpha\gamma} R_{\beta\delta} - g_{\gamma\beta} R_{\delta\alpha} + g_{\beta\delta} R_{\alpha\gamma} - g_{\delta\alpha} R_{\gamma\beta}\Bigr) + \frac{1}{6}\Bigl(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\Bigr) R \; . \tag{6–22}$$

Now note that the covariant derivative of the metric vanishes. Substituting (6–21) in (6–22) implies,

$$D_\epsilon C_{\alpha\beta\gamma\delta} = D_\epsilon R_{\alpha\beta\gamma\delta} \; . \tag{6–23}$$

Combining this relation into (6–20) gives,

$$D_\epsilon C_{\alpha\beta\gamma\delta} + D_\gamma C_{\alpha\beta\delta\epsilon} + D_\delta C_{\alpha\beta\epsilon\gamma} = 0 \; . \tag{6–24}$$

Our first key identity derives from contracting α into ϵ, and exploiting the tracelessness of the Weyl tensor,

$$D^\alpha C_{\alpha\beta\gamma\delta} = 0 \; . \tag{6–25}$$

Our second identity derives from contracting D^ϵ into relation (6–24), commuting derivatives and then using relation (6–25),

$$\square C_{\alpha\beta\gamma\delta} = -D_\rho D_\gamma C_{\alpha\beta\delta}{}^{\rho} + D_\rho D_\delta C_{\alpha\beta\gamma}{}^{\rho} \; , \tag{6–26}$$

$$= 6H^2 C_{\alpha\beta\gamma\delta} - R^{\rho}{}_{\alpha\gamma}{}^{\sigma} C_{\rho\beta\delta\sigma} + R^{\rho}{}_{\gamma\beta}{}^{\sigma} C_{\rho\delta\alpha\sigma}$$
$$- R^{\rho}{}_{\beta\delta}{}^{\sigma} C_{\rho\alpha\gamma\sigma} + R^{\rho}{}_{\delta\alpha}{}^{\sigma} C_{\rho\gamma\beta\sigma} - R^{\rho\sigma}{}_{\gamma\delta} C_{\alpha\beta\rho\sigma} \; . \tag{6–27}$$

Relations (6–25) and (6–27) hold, to all orders in the graviton field, for any solution to the source-free Einstein equations. Taking the first order in the graviton field amounts to just replacing the full Weyl tensor by the linearized Weyl $\delta C_{\alpha\beta\gamma\delta}$ we have been using, replacing the full covariant derivative operators by the covariant derivatives in de Sitter background and replacing the full Riemann tensor by its de Sitter limit. When these things are done the two identities become,

$$D^{\alpha}\delta C_{\alpha\beta\gamma\delta} = 0 + O(h^2) \; , \tag{6–28}$$

$$\Box\delta C_{\alpha\beta\gamma\delta} = 6H^2\delta C_{\alpha\beta\gamma\delta} + O(h^2) \; . \tag{6–29}$$

Note also that if the stress-energy had been nonzero the right hand sides of relations (6–28) and (6–29) would have contained simple combinations of derivatives of the stress tensor.

6.2.4 The Final Reduction

We are now ready to act the outer projector on the remaining terms,

$$\int d^4x' \left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right](x;x') h^{(0)}_{\rho\sigma}(x') = 2i\sqrt{-g(x)} \int d^4x' \sqrt{-g(x')}\, \delta C_{\kappa\lambda\theta\phi}(x') \\ \left\{ \mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x) \left[10\widehat{\mathcal{F}}_2 \mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} - \frac{4}{H^2}\widehat{\mathcal{F}'}_2 \frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} \right] \right\} . \tag{6–30}$$

The second line of this expression is quite complicated by itself, but it is greatly simplified when contracted into the linearized Weyl tensor,

$$\delta C_{\kappa\lambda\theta\phi}(x')\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x) \left[10\widehat{\mathcal{F}}_2 \mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} - \frac{4}{H^2}\widehat{\mathcal{F}'}_2 \frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} \right] \\ = \delta C_{\kappa\lambda\theta\phi}(x') \left\{ \frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi} f_1(y) + \frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\theta} f_2(y) \right. \\ \left. + \frac{\partial y}{\partial x'_\lambda}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\kappa(\mu}\mathcal{T}^{\nu)\phi} f_3(y) + \frac{\partial y}{\partial x'_\lambda}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\kappa(\mu}\mathcal{T}^{\nu)\theta} f_4(y) \right\} . \tag{6–31}$$

Here the functions $f_i(y)$ are,

$$f_1 = -125\widehat{\mathcal{F}}_2 + 115(2-y)\widehat{\mathcal{F}'}_2 - (68 - 116y + 29y^2)\widehat{\mathcal{F}''}_2 - 2(2-y)(4y-y^2)\widehat{\mathcal{F}'''}_2$$
$$f_2 = -\frac{75}{2}\widehat{\mathcal{F}}_2 + \frac{69}{2}(2-y)\widehat{\mathcal{F}'}_2 - (28 - 44y + 11y^2)\widehat{\mathcal{F}''}_2 - (2-y)(4y-y^2)\widehat{\mathcal{F}'''}_2$$
$$f_3 = -\frac{85}{2}\widehat{\mathcal{F}}_2 + \frac{15}{2}(2-y)\widehat{\mathcal{F}'}_2$$
$$f_4 = -5\widehat{\mathcal{F}}_2 - 13(2-y)\widehat{\mathcal{F}'}_2 - \frac{5}{2}(4y-y^2)\widehat{\mathcal{F}''}_2 \qquad (6–32)$$

Changing the dummy indices in (6–31) gives,

$$\delta C_{\kappa\lambda\theta\phi}(x')\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\Big[10\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} - \frac{4}{H^2}\widehat{\mathcal{F}'}_2\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big]$$
$$= \frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}f(y)\delta C_{\kappa\lambda\theta\phi}(x') \; . \qquad (6–33)$$

Here the function $f(y)$ is,

$$f(y) = -50\widehat{\mathcal{F}}_2 + 60(2-y)\widehat{\mathcal{F}'}_2 - (40 - 62y + \frac{31}{2}y^2)\widehat{\mathcal{F}''}_2 - (2-y)(4y-y^2)\widehat{\mathcal{F}'''}_2 \; . \qquad (6–34)$$

The final reduction is accomplished by one more partial integration. Let us define the integral $I[f]$ of a function $f(y)$ by the relations,

$$\frac{\partial y}{\partial x'_\kappa}f(y) \equiv \frac{\partial}{\partial x'_\kappa}I[f](y) \quad \text{such that} \quad \frac{\partial I[f]}{\partial y} = f(y) \; . \qquad (6–35)$$

Then the one loop source becomes,

$$\int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')h^{(0)}_{\rho\sigma}(x')$$
$$= 2i\sqrt{-g(x)}\int d^4x'\sqrt{-g(x')}\frac{\partial y}{\partial x'_\kappa}f(y)\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}\delta C_{\kappa\lambda\theta\phi}(x') \qquad (6–36)$$
$$= -2i\sqrt{-g(x)}\int d^4x'\sqrt{-g(x')}I[f]\Big\{\frac{D^2 y}{Dx'_\kappa Dx'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}\delta C_{\kappa\lambda\theta\phi}(x')$$
$$+ \frac{D\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}}{Dx'_\kappa}\frac{\partial y}{\partial x'_\theta}\delta C_{\kappa\lambda\theta\phi}(x') + \frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}D^\kappa\delta C_{\kappa\lambda\theta\phi}(x')\Big\} \; . \qquad (6–37)$$

The first and second terms include the metric,

$$\frac{D^2 y}{Dx'_\kappa Dx'_\theta} = H^2(2-y)g^{\kappa\theta}(x') , \qquad \frac{D\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}}{Dx'_\kappa} = \frac{1}{2}\frac{\partial y}{\partial x_{(\mu}}\mathcal{T}^{\nu)(\phi}g^{\lambda)\kappa}(x') \; , \tag{6–38}$$

so they give zero when contracted into the linearized Weyl tensor. The third term vanishes by the transversality of the linearized Weyl tensor (for dynamical gravitons only) which we showed in (6–25). Hence the one loop source term for a dynamical graviton is zero:

$$\int d^4x' \left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right](x;x')h^{(0)}_{\rho\sigma}(x') = 0 \; . \tag{6–39}$$

Before concluding we should comment on the validity of our result (6–39), in view of the enormous difference between de Sitter and the actual expansion history of the universe. Of course equation (6–1) is correct for any geometry, but we only know the graviton self-energy for de Sitter background. This does not make any difference for cosmologically observable tensor perturbations for two reasons:

- As explained section 1.1, de Sitter is an excellent approximation to primordial inflation up until cosmologically observable perturbations experience first horizon crossing. After this time the de Sitter approximation breaks down, but those perturbations are almost constant.
- Our result (6–15) is valid for any geometry, and the linearized Weyl tensor vanishes for constant perturbations. So there is no contribtuion from the portion of the integration which derives from times after the end of inflation.

To see the second point, note that general coordinate invariance requires matter contributions to the graviton self-energy to take the form (4–51), provided one uses expressions (4–12)-(4–13) to define the projectors for a general metric, and provided the general form of expression (4–52) is related to the geodetic length function through (2–14). That form is all we required to derive equation (6–15).

CHAPTER 7
CONCLUSION

We have computed the one loop contribution to the graviton self-energy from a massless, minimally coupled scalar on a locally de Sitter background. We used it to solve the one loop-corrected, linearized Einstein field equations to study the effect of inflationary scalars on dynamical gravitons. The computation was done using dimensional regularization and renormalized by absorbing the divergences with BPHZ counterterms.

The graviton self-energy has been given in two forms. The first form (3–11) is fully dimensionally regulated, with the ultraviolate divergences neither localized nor subtracted off with counterterms. This version of the result agrees with the stress tensor correlator recently computed by Perez-Nadal, Roura and Verdaguer [99]. Our second form is fully renormalized, with the unregulated limit taken,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x') = \sqrt{-\overline{g}(x)}\,\mathcal{P}^{\mu\nu}(x)\sqrt{-\overline{g}(x')}\,\mathcal{P}^{\rho\sigma}(x')\Big[\mathcal{F}_{1R}(y)\Big] \\ +2\sqrt{-\overline{g}(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\sqrt{-\overline{g}(x')}\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')\Big[\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\mathcal{F}_{2R}(y)\Big]\,. \qquad (7\text{–}1)$$

In this expression the spin zero operator $\mathcal{P}^{\mu\nu}$ was defined in (4–14), the spin two operator $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}$ was defined in (4–15), and the bitensor $\mathcal{T}^{\alpha\kappa}$ was given in (4–52). Our results for the renormalized spin zero and spin two structure functions are expressions (4–139) and (4–163), respectively.

An interesting application of this work is the transverse-traceless projector (4–15), which played a crucial role in the recent solution for the graviton propagator in de Donder gauge [103, 104]. It should be noted that equations (4–139) and (4–163) are the first (and so far only) fully renormalized results for the graviton structure functions on de Sitter background. All previous results [86, 99] have been specialized to non-covincident points, and so cannot be used in the effective field equations.

Our second form (7–1) is manifestly transverse, as required by gauge invariance. It is also de Sitter invariant, despite the fact that the massless, minimally coupled scalar propagator breaks de Sitter invariance [62], because the de Sitter breaking term drops out of mixed second derivatives (2–32). Our result agrees with the flat space limit [61]. And the divergent parts of the counterterms we used to subtract off the divergences agree with those found long ago by ‘t Hooft and Veltman [9]. We actually included finite renormalizations of Newton's constant and of the cosmological constant. Such renormalizations are presumably necessary when considering the effective field equations of quantum gravity if the parameters Λ and G are to have their correct physical meanings.

The point of this exercise is to discover whether or not the inflationary production of scalars has a significant effect on gravitational radiation and the force of gravity. In order to check this we have employed the quantum-corrected linearized Einstein equation,

$$\sqrt{-g}\,\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) - \int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x') h_{\rho\sigma}(x') = \frac{1}{2}\kappa\sqrt{-g}\; T^{\mu\nu}_{\rm lin}(x) \; , \qquad (7\text{–}2)$$

where $\mathcal{D}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator (3–35) specialized to de Sitter background. Because we only know the self-energy at order κ^2, all we can do is to solve (6–1) perturbatively by expanding the graviton field and the self-energy in powers of κ^2,

$$h_{\mu\nu}(x) \;=\; h^{(0)}_{\mu\nu}(x) + \kappa^2 h^{(1)}_{\mu\nu}(x) + O(\kappa^4) \; , \qquad (7\text{–}3)$$

$$\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm ren}\Big](x;x') \;=\; \kappa^2 \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{1}\Big](x;x') + O(\kappa^4) \; . \qquad (7\text{–}4)$$

Of course $h^{(0)}_{\mu\nu}(x)$ obeys the classical, linearized Einstein equation. Given this solution, the corresponding one loop correction is defined by the equation,

$$\sqrt{-g(x)}\,\mathcal{D}^{\mu\nu\rho\sigma} h^{(1)}_{\rho\sigma}(x) = \int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{1}\Big](x;x') h^{(0)}_{\rho\sigma}(x') \; . \qquad (7\text{–}5)$$

For a dynamical graviton of wave vector $\vec{k}$, the classical 0th order solution takes the form [70],

$$h^{(0)}_{\rho\sigma}(x) = \epsilon_{\rho\sigma}(\vec{k}) a^2 u(\eta, k) e^{i\vec{k}\cdot\vec{x}} \; , \tag{7–6}$$

where the tree order mode function is,

$$u(\eta, k) = \frac{H}{\sqrt{2k^3}} \Big[1 - \frac{ik}{Ha}\Big] \exp\Big[\frac{ik}{Ha}\Big] \; , \tag{7–7}$$

and the polarization tensor obeys all the same relations as in flat space,

$$0 = \epsilon_{0\mu} = k_i \epsilon_{ij} = \epsilon_{jj} \quad \text{and} \quad \epsilon_{ij}\epsilon^*_{ij} = 1 \; . \tag{7–8}$$

Our result is that the inflationary production of MMC scalars has no effect on dynamical gravitons at one loop order. There is nothing very surprising about this result. It is exactly what happens in flat space [9] and we have reviewed it in Section 5.2.3. Although the scalar contribution to the graviton self-energy is enormously more complex in de Sitter than in flat space, we showed in section 6.2 that all of this complexity can be absorbed into surface integrations at the initial time. It is plausible that these surface integrations can be regarded as perturbative redefinitions of the initial state which involve two scalars and one graviton. The null effect of flat space certainly has this interpretation, which implies the same for the highest derivative part of the de Sitter result. What has yet to be proved — and so must be labeled a conjecture — is that the lower derivative, intrinsically de Sitter parts have the same interpretation. Checking this requires a computation like that recently completed for the self-interacting scalar [7].

Another way of understanding this result is to consider the number of the MMC scalars. The one loop corrections we seek to compute represent the response (of either dynamical gravitons or the force of gravity) to the vast ensemble of infrared scalars which are produced by inflation. It is simple to show that the occupation number for *each* *mode* with wave number $\vec{k}$ grows like [49] (as reviewed in the introduction section 1.2.3),

$$N(k,\eta) = \left(\frac{Ha(\eta)}{2k}\right)^2 \tag{7–9}$$

This growth is balanced by expansion of the 3-volume so that the number density of infrared particles with $0 < k < Ha$ remains fixed,

$$n(\eta) = \int \frac{d^3k}{(2\pi a)^3}\, \theta(Ha - k) N(k,\eta) = \frac{H^3}{8\pi^2} \; . \tag{7–10}$$

The constant density of virtual scalars has no effect at all on dynamical gravitons (after field strength renormalization) [9].

This is the math behind our result; the physics is that ultraviolet virtual scalars affect gravitons the same as in flat space, and infrared scalars carry too little stress-energy to have much effect. The effect of ultraviolet scalars is limited, as on flat space, to inducing higher derivative counterterms. Although primordial inflation produces many scalars, they are all highly infrared so they interact only weakly with gravtions. This seems to be why inflationary gravitons have no significant effect on MMC scalars [96]. One might worry that a very infrared graviton would still suffer some effect from absorbing a comparably infrared scalar. To understand why this is not so, let us model the process by simply replacing the graviton's co-moving wave number k with a new one k',

$$0 = \ddot{u}(t,k) + 3H\dot{u}(t,k) + \frac{k^2}{a^2(t)} u(t,k) \longrightarrow \ddot{u}(t,k) + 3H\dot{u}(t,k) + \frac{k'^2}{a^2(t)} u(t,k) \; . \tag{7–11}$$

The effect on the mode function is negligible after both $1/a^2$ terms have redshifted into insignificance.

One last project which remained to be completed is to investigate how inflationary scalars affect the force of gravity. That can be done by solving (6–6) to correct for the linearized response to a stationary point mass M [71],

$$h^{(0)}_{00}(x) = a^2 \times \frac{2GM}{a\|\vec{x}\|} \; , \; h^{(0)}_{0i}(x) = 0 \; , \; h^{(0)}_{ij}(x) = a^2 \times \frac{2GM}{a\|\vec{x}\|} \times \delta_{ij} \; . \tag{7–12}$$

The same reduction procedures we laid out in section 6.2 can be applied in this case except that:

- The spin zero projector $\mathcal{P}^{\rho\sigma}(x')$ does not annihilate (7–12); and
- The linearized stress tensor does not vanish.

The computation has been reduced to a single integration which I will complete soon.

It should be noted that the virtual scalars of flat space do induce a correction to the classical potential [43, 44] which is reviewed in section 5.2.5, and we expect one as well on de Sitter background. On dimensional grounds the flat space result must (and does) take the form,

$$\Phi_{\rm flat} = -\frac{GM}{r}\left\{1 + \text{constant} \times \frac{G}{r^2} + O(G^2)\right\} . \tag{7–13}$$

On de Sitter background there is a dimensionally consistent alternative provided by the Hubble constant H and by the secular growth driven by continuous particle production,

$$\Phi_{\rm dS} = -\frac{GM}{r}\left\{1 + \text{constant} \times GH^2 \ln(a) + O(G^2)\right\} . \tag{7–14}$$

If such a correction were to occur its natural interpretation would be as a time dependent renormalization of the Newton constant. The physical origin of the effect (if it is present) would be that virtual infrared quanta which emerge near the source tend to collapse to it, leading to a progressive increase in the source.

Both math and physics suggest that inflationary gravitons might do something interesting to other gravitons. The graviton contribution to the graviton self-energy has been derived at one loop order [60] so the computation can be made. Of course one can reduce the effect to a temporal surface term, as we did in section 6.2, but it seems likely that this surface term will depend upon the observation time η so that it cannot be absorbed into a perturbative correction to the initial state. The reason for this is that the graviton contribution contains de Sitter-breaking, infrared logarithms [60], unlike the scalar contribution. The physical principle involved would be that gravitons possess spin and even very infrared gravitons continue to interact via the spin-spin coupling which doesn't exist for scalars. This is presumably why inflationary gravitons induce a secular enhancement of the field strength of massless fermions [97].

Final comment concerns the possible comparison with cosmological data. One consequence of our computation would be on the tensor component of the anisotropies in the CMB (Cosmic Microwave Background radiation). This has not been resolved yet but high precision measurements of the so-called B-modes of the CMB polarization, for example from the Planck satellite, may allow us to probe it[106, 107]. There are also three NASA-funded balloon probes set to go up over the course of the next year[108]. None of these measurements would be sensitive to the one loop corrections I am computing, but they are first steps. On the very long — several decades — term, there is enough data to resolve one loop corrections from very early proto-structures[109]. The observable is the tensor power spectrum, which is $k^3/2\pi^2$ times the spatial Fourier transform of $< \Omega|h_{ij}^{TT}(t,0)h_{ij}^{TT}(t,\vec{x})|\Omega >$. My result is that a single scalar loop does not give any correction to the graviton mode functions. However as we have discussed in the previous paragraph, it seems likely that the one loop effects from gravitons might be significant because gravitons can interact through their spins, which do not redshift.

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BIOGRAPHICAL SKETCH

Sohyun Park came from South Korea. She earned her undergraduate degree in mathematics at Pusan National University (PNU). In the last year of her undergraduate study, she had a chance to participate in computational biology research which led her to do her master's in biology with thesis title,"Implementation of Cellular Automata for the Spatio-Temporal Analysis of Population Dynamics and Diffusion Model of Pine Needle Gall Midge." It was a good experience from which she learned mathematical modeling and computer simulation techniques. However, she found herself having more interest in fundamental theory and she decided to study high energy physics. Thus she finally switched to the department of physics at PNU and took another master's degree in physics with a thesis entitled, "A Study on the Snyder-Yang Discrete Space-time" under the direction of Prof. Chang Gil Han. After completing her master's, she had teaching jobs in educational institutes and a technical college while planning to study abroad.

She came to the University of Florida in the Fall of 2007. In her first year, she took the graduate core courses, Particle Physics and Quantum Field Theory. In her second and third year, she took General Relativity, Standard Model and a Special Topics course on Dark Matter. In the Fall of 2009, she started her doctoral research on quantum field theory in curved space under the direction of Prof. Richard Woodard which led to this dissertation. In Fall 2011, she won a Fermi National Accelerator Laboratory (Fermilab) Fellowship to spend her last year of graduate study doing research under the direction of Dr. Scott Dodelson at Fermilab. Her winning proposal was to work out structure formation in a new type of modified gravity theories which have been suggested to explain the current phase of cosmological acceleration. Sohyun received her Ph.D. in the Summer of 2012.