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ABSTRACT

The qualitative behavior of large-angle scattering by a rotationally invariant optical potential is investigated using the WKB approximation. It is shown that the strong s dependence exhibited by elastic proton-proton scattering requires an energy-dependent potential. The source of the inaccuracy of the Eikonal approximation, at large angles, is found by comparing Eikonal phase shifts with those given by the WKB approximation.

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I. INTRODUCTION

At high energy, near forward elastic proton-proton scattering shows the characteristics of diffraction by a smooth absorber: Re $f(0)/\mathrm{Im}\, f(0)$ and $\sigma_{elastic}/\sigma_{total}$ are both much less than one, and $d\sigma/d\Omega\ (s,t)$ is a strongly decreasing function of $|t|$. The optical model proposed by Serber[1] encorporates these properties in a natural way and provides a qualitative explanation of the t dependence of large-angle proton-proton scattering. The s dependence of $d\sigma/d\Omega\ (s,t)$ remains somewhat mysterious.

If the observed large-angle cross section is reproduced with a symmetric version of the amplitude:

$$A\, s^{\nu/2} (-t)^{w/2} = A\,(2k)^{\nu + w} (\sin\, \theta/\nu)^{w}, \tag{I.1}$$

we find (see Section VII):

$$\nu = -5.75 \qquad w = -2.25.$$

In other words, at large angles, the s dependence is actually more significant than the t dependence. But if Serber's model is solved by the Eikonal approximation (using an energy-independent potential), ν automatically becomes 0.5, which seems to imply that to reproduce (I.1), Serber's potential must be made a function of s, so that, in effect, the s dependence of large-angle scattering is assigned to some unknown dynamic mechanism.

Recently, however, it has been shown that the Eikonal approximation is extremely inaccurate at large angles. For a particular potential studied by Avison,[2] the Eikonal differential cross section is small by as much as a factor of 1000. It is conceivable, therefore, that some energy independent potential, if solved accurately, might produce a cross section which does exhibit the energy dependence required.

Serber's optical model interprets the t dependence of the differential cross section as the result of a spatial distribution of absorber; perhaps the s dependence of the differential cross section may also be interpreted in this way, rather than as a result of the energy dependence of the absorber itself.

In order to explore this possibility, we shall reconsider Serber's optical model using a number of techniques based on the WKB approximation. Unfortunately, it is quite difficult to produce a rigorous bond on the accuracy of these procedures and therefore they will be tested by comparison with Avison's numbers and by comparison with each other. In the process we will find the source of inaccuracy of the Eikonal approximation at large angle.

The final result of all this will be that Serber's equation cannot account for (I.1) using any "reasonably smooth" potential of the form

$$\left[i\,\eta\,\frac{P(\alpha r)}{r},\ 0,\ 0,\ 0\right]$$

unless η and α themselves depend on s. In a subsequent paper, however, we will show that optical potentials of a different sort

("Lorentz contracted") do provide an explanation of (I.1). In addition, once the range and amplitude of the contracted potentials are chosen to reproduce the line $d\sigma/d\Omega$ (k, 90°), a pair of ripples automatically appears in agreement with the observations of Allaby et al.,[6,7] ($k_\perp \sim 1.1$ GeV, 1.9 GeV).

II. THE SERBER MODEL

Serber's optical model is governed by the center-of-mass equation:

$$\left\{P^2 + m^2 - [E + i\eta V(r)^2]\right\} \Psi = 0 \tag{II.1}$$

which describes the scattering of a single Klein-Gordon particle by a spherically symmetric absorber, introduced as the time component of a four vector to insure:

$$\lim_{E \to \infty} \sigma_t(E) > 0.$$

Since large-angle proton-proton scattering occurs at a much greater rate than similar two-body reactions not involving protons, or involving only one proton, it seems reasonable to suppose the proton has a hard core of some sort. Serber therefore chooses V singular at the origin:

$$V(r) = \frac{e^{-\alpha r}}{r}.$$

We will restrict ourselves to Eq. (II.1) for spherically symmetric potentials which go as $1/r$ near the origin and are of finite range. The coupling constant η is assumed real, and $V(r)$ is assumed real when r is on the real axis. If we write:

$$V(r) = \frac{P(r)}{r}, \tag{II.2}$$

$P(0)$ can be nominally set equal to one. We will also assume $P(r)$ has neither cuts nor poles which influence the scattering.

III. WKB APPROXIMATION

From the partial wave equation:

$$\left[\frac{\partial^2}{\partial r^2} + (E+i\eta V)^2 - m^2 - \frac{\ell(\ell+1)}{r^2}\right] \phi_\ell(r) = 0, \tag{III.1}$$

we extract the WKBJ phase shift[3]

$$\delta_\ell = \lim_{R\to\infty}\left[\int_{\hat{r}_0}^{R}\sqrt{(E+i\eta V)^2 - m^2 - \frac{(\ell+\frac{1}{2})^2}{r^2}}\,dr - \int_{r_0}^{R}\sqrt{k^2 - \frac{(\ell+\frac{1}{2})^2}{r^2}}\,dr\right]$$

$$[E + i\eta V(\hat{r}_0)]^2 - m^2 - \frac{(\ell+\frac{1}{2})^2}{\hat{r}_0^2} = 0 \tag{III.2}$$

$$k^2 - \frac{(\ell+\frac{1}{2})^2}{r_0^2} = 0.$$

The function $\hat{r}_0(\ell)$ is considered in Appendix A. Its analytic properties are studied in detail for the case $P(r) = e^{-r}$. If $\eta V(r_0)$ is small we expect $\hat{r}_0 \sim r_0$. The analytic character of δ_ℓ is carried largely by $\hat{r}_0(\ell)$, since δ_ℓ will be analytic at any point ℓ' at which $V[\hat{r}_0(\ell)]$ is finite and $\hat{r}_0(\ell)$ is analytic.

Four different techniques will be considered:

1. The WKB approximation without any simplifying assumptions

2. An asymptotic form of the WKB approximation, correct when k is large (fixed θ)

3. A crude form of (1) correct when η is small

4. An asymptotic form of (3).

Approximations (1) and (2) will be developed for all potentials of form (II.2) and (3) and (4) will only be considered for the choices $P(r) = e^{-r}$, e^{-r^2}.

IV. ASYMPTOTIC FORM

By use of the Watson-Sommerfeld transform, we will be able to show that when k is large, we need consider only a simplified version of (III.2). The scattering amplitude may be written:

$$f(\theta) = \frac{1}{2k} \int_C \left(1 - e^{2i\delta_{\Lambda - \frac{1}{2}}}\right) \frac{P_{\Lambda - \frac{1}{2}}(-\cos\theta)}{\cos\pi\Lambda} \Lambda d\Lambda .$$

The contour C circles each positive real pole of $1/\cos\pi\Lambda$ once in the clockwise direction and does not include singularities carried by the phase shift. If we assume $\delta_{\Lambda - \frac{1}{2}}$ is an analytic function of Λ in a strip which includes the real Λ axis and that $P(r)$ is bounded in a strip which includes the real r axis, we can then deform C as shown in

Fig. 1. But as $k \to \infty$, $\hat{r}_0(\ell) \to r_0(\ell) = \Lambda/k$; thus, if we allow α to grow as $\alpha_0\sqrt{k}$, for large k the horizontal contours will still be in the region in which P(r) is bounded. It is reasonable to assume that $\delta_{\Lambda-\frac{1}{2}}$ is also bounded in this region, and therefore using the relation:

$$\left| \frac{P_{\Lambda-\frac{1}{2}}(-\cos\theta)}{\cos\pi\Lambda} \right| < Ae^{-\theta\,\mathrm{Im}\,\Lambda},$$

we see that the contribution of the horizontal contours will go to zero as $e^{-\theta\alpha_0\sqrt{k}}$. If the vertical contour contributes a power law in k, the horizontal contours can be ignored. As k becomes large, however, $\Lambda/k \to 0$ uniformly on the vertical contour; therefore, we need only determine the behavior of $\delta_{\Lambda-\frac{1}{2}}$ in the limit $\Lambda/k \to 0$.

We define the variable $\Lambda = \ell + \frac{1}{2}$ and decompose δ into three parts:

$$\delta = \delta_1 + \delta_2 + \delta_3$$

$$\delta_1 = \frac{1}{2}\int_D \left\{ \sqrt{\left[E + i\eta\,\frac{P(r)}{r}\right]^2 - m^2 - \Lambda^2/r^2} - \sqrt{k^2 - \Lambda^2/r^2} - \frac{i\eta E/k\,P(r)}{\sqrt{r^2 - \Lambda^2/k^2}} \right\} dr, \qquad \text{(IV.1)}$$

$$\delta_2 = i\eta E/k \int_{r_0}^{\infty} \frac{P(r) - e^{-r}}{\sqrt{r^2 - \Lambda^2/k^2}}\, dr,$$

$$\delta_3 = i\eta E/k \int_{r_0}^{\infty} \frac{e^{-r}}{\sqrt{r^2 - \Lambda^2/k^2}}\, dr = i\eta E/k\, K_0(\Lambda/k),$$

and then decomposes δ_1 again:

$$\delta_1 = \delta_4 + \delta_5$$

$$\delta_4 = \frac{1}{2} \int_D \left\{ \sqrt{\left(E + \frac{i\eta}{r}\right)^2 - m^2 - \Lambda^2/r^2} - \sqrt{k^2 - \Lambda^2/k^2} - \frac{i\eta E/k}{\sqrt{r^2 - \Lambda^2/k^2}} \right\} dr$$

$$\delta_5 = \frac{1}{2} \int_D \left\{ \sqrt{\left[E + \frac{i\eta P(r)}{r}\right]^2 - m^2 - \Lambda^2/r^2} - \sqrt{\left(E + \frac{i\eta}{r}\right)^2 - m^2 - \Lambda^2/r^2} - \frac{i\eta E/k[P(r) - 1]}{\sqrt{r^2 - \Lambda^2/k^2}} \right\} dr.$$

D is the contour shown in Fig. 2.

The square roots are given positive real part above the cuts, which extend from the two branch points to $+\infty$.

The limit of δ_2 as $\Lambda/k \to 0$ is finite. The limit of the first derivative of δ_2 with respect to Λ/k is also finite:

$$\delta_2(0) = -i\eta \frac{E}{k} \left[\int_0^\infty P'(r) \ln r dr - \gamma \right]$$

$$\delta_2'(0) = 0.$$

Therefore by Taylor's theorem, for every $\epsilon > 0$, there exists a neighborhood N of 0, such that:

$$|\delta_2(\Lambda) - \delta_2(0)| < \epsilon \frac{|\Lambda|}{|k|} \text{ if } \frac{\Lambda}{k} \epsilon N.$$

Using the expansion of $K_0(x)$ near $x = 0$, we have:

$$\delta_2 + \delta_3 \sim -i\eta \frac{E}{k} \left(\ell n \frac{\Lambda}{2k} + u \right)$$

$$u = \int_0^\infty P'(r) \ell n \, r dr.$$

δ_5, on the other hand, can be neglected entirely for large values of E. The substitution $r = \gamma/E$ gives:

$$\delta_5 = \int_D \left\{ \sqrt{\left[1 + \frac{i\eta}{\gamma} P\left(\frac{\gamma}{E}\right)\right]^2 - \left(\frac{m}{E}\right)^2 - \left(\frac{\Lambda}{\gamma}\right)^2} - \sqrt{\left(1 + \frac{i\eta}{\gamma}\right)^2 - \left(\frac{m}{E}\right)^2 - \left(\frac{\Lambda}{\gamma}\right)^2} \right.$$

$$\left. - \frac{i\eta \frac{E}{k}\left[P\left(\frac{\gamma}{E}\right) - 1\right]}{\sqrt{\gamma^2 - \Lambda^2}} \right\} d\gamma$$

which must approach 0 as $E \to \infty$ since $P(0) = 1$.

δ_4 is given exactly by the formula:

$$\delta_4 = \frac{\pi}{2} \left(\Lambda - \sqrt{\Lambda^2 + \eta^2} \right) + i\eta \frac{E}{k} (1 + \ell n \ \Lambda)$$

$$- \frac{i}{2} \left\{ \left[\sqrt{\Lambda^2 + \eta^2} + \eta \frac{E}{k} \right] \ell n \left[\sqrt{\Lambda^2 + \eta^2} + \eta \frac{E}{k} \right] \right.$$

$$\left. - \left[\sqrt{\Lambda^2 + \eta^2} - \eta \frac{E}{k} \right] \ell n \left[\sqrt{\Lambda^2 + \eta^2} - \eta \frac{E}{k} \right] \right\}.$$

The preceding results combined yield the following expression for the WKBJ phase shift:

$$\delta = i\eta \frac{E}{k} (1 - u + \ell n \ 2k) + \frac{\pi}{2} \left(\Lambda - \sqrt{\Lambda^2 + \eta^2} \right) - \frac{i\eta E}{2k} \ell n \left(\Lambda^2 - \frac{m^2}{k^2} \eta^2 \right)$$

$$- \frac{i}{2} \sqrt{\Lambda^2 + \eta^2} \ \ell n \left(\frac{\sqrt{\Lambda^2 + \eta^2} + \eta \ E/k}{\sqrt{\Lambda^2 + \eta^2} - \eta E/k} \right). \qquad \text{(IV.3)}$$

This formula includes all terms which do not approach zero as k approaches infinity and Λ/k approaches 0.

The phase shift has branch points at $\Lambda = \pm i\eta$ and $\Lambda = \pm nm/k$ so that the contour of Fig. 1 should actually be the path shown in Fig. 3. This modification does not make expression (IV.3) any the less useful, however, since as $k \to \infty$, $nm/k \to 0$, and the entire contour enters the region in which (IV.3) applies. In fact, as $k \to \infty$ the contribution of the cut approaches zero faster (by one power of k) than does the total amplitude.

The validity of (IV.3) depends only on the smoothness of the function P(r). The value of k at which the scattering amplitude given by (IV.3) reproduces the WKB to within a few percent will be controlled by quantities of the form:

$$\frac{\alpha^N P^{(N)}(0)}{k^N} .$$

The scattering amplitude given by (IV.3) depends on only two free parameters, η and α. It will therefore be possible to determine all the information we need by evaluating the Watson-Sommerfeld integral numerically. It remains worthwhile, however, to consider a less accurate form of the WKB approximation, which leads to closed expressions for the phase shift and scattering amplitude. Along the way we will obtain a clearer picture of why the Eikonal approximation fails at large angles.

V. POWER SERIES

Expression (III.2) can be expanded to produce a power series in η:

$$\delta_\ell(\eta) = \eta\,\delta^{(1)}(\ell+\tfrac{1}{2}) + \frac{\eta^2}{2}\,\delta^{(2)}(\ell+\tfrac{1}{2}) + \ldots$$

The first two coefficients are:

$$\delta^{(1)}(\Lambda) = \frac{1}{2}\, i\, \frac{E}{k}\int_E \left[rV(r)\Big/\sqrt{r^2 - \Lambda^2/k^2}\right] dr \qquad \text{(V.1)}$$

$$\delta^{(2)}(\Lambda) = -\frac{1}{2}\,\frac{1}{k}\int_E \frac{rV^2}{\left(r^2 - \dfrac{\Lambda^2}{k^2}\right)^{1/2}}\,dr + \frac{E^2}{2k^3}\int_E \frac{r^3V^2}{\left(r^2 - \dfrac{\Lambda^2}{k^2}\right)^{3/2}}\,dr.$$

Contour E is shown in Fig. 4. If V is a Yukawa potential (energy unit $= \alpha$) we find:

$$\delta^{(1)} = i\,\frac{E}{k}\,K_0\!\left(\frac{\Lambda}{\eta}\right)$$

$$\delta^{(2)} = -\frac{1}{\Lambda}\int_{\frac{2\Lambda}{k}}^{\infty} K_0(u)\,du - \frac{2E^2}{k^3}\,K_0\!\left(\frac{2\Lambda}{k}\right). \qquad \text{(V.2)}$$

The linear term $\eta\delta^{(1)}$ is just the phase shift given by the Eikonal approximation. The quadratic term, however, is not included in the Eikonal phase shift, and for a potential singular at the origin, this term contributes a singularity at $\Lambda = 0$ which has a drastic effect on the large-angle scattering amplitude.

When Λ is close to 0, (V.2) leads to the formula:

$$e^{2i\delta_\Lambda} = g\left(\frac{\Lambda}{k}\right)^{x+iy} e^{-iz/\Lambda}$$

$$g = \left\{ \exp\left[2\eta \frac{E}{k} \gamma + 2i \frac{\eta^2}{k} \left(1 + \frac{m^2}{k^2} \gamma \right) \right] \right\} \times \left(\frac{1}{2^{2\eta E/k}} \right)$$

$$x = 2\eta E/k \qquad z = \frac{\pi\eta^2}{2} \tag{V.3}$$

$$y = 2\eta^2 \frac{m^2}{k^3} \qquad \gamma = \text{Euler's constant.}$$

It is worth noting that the pole which appears in the phase shift given by (V.2) and (V.3) does not appear in the exact WKB phase shift, nor in (IV.3). Its source may be found, however, by expanding (IV.3) as a power series. The term:

$$\frac{\pi}{2} \quad \Lambda - \sqrt{\Lambda^2 + \eta^2}$$

becomes:

$$-\frac{\pi}{4} \frac{\eta^2}{\Lambda} + \ldots$$

The pole at $\Lambda = 0$ in (V.2) represents the weak coupling limit of the effect of the branchpoints at $\Lambda = \pm i\eta$. These branchpoints must also be present in the exact solution since the radial equation corresponding to (II.1):

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1) + \eta^2 e^{-2r}}{r^2} + 2iE\eta \frac{e^{-r}}{r} \right\} U_\ell = 0$$

if solved by a power series:

$$U_\ell = r^\rho \sum_{\nu=0}^{\infty} c_\nu r^\nu$$

leads to the indical equations:

$$\rho(\rho - 1) = \ell(\ell + 1) + \eta^2$$

$$\rho = \tfrac{1}{2} \pm \sqrt{(\ell + \tfrac{1}{2})^2 + \eta^2}$$

with branchpoints at $\Lambda \equiv \ell + \frac{1}{2} = \pm i\eta$.

We now make the approximate substitution:

$$\sqrt{\frac{2}{\pi q \Lambda}} \cos\left(q\Lambda - \frac{\pi}{4}\right) \rightarrow P_\ell(\cos\theta) \quad q \equiv 2 \sin\theta/2 \tag{V.4}$$

(see Appendix B), replace the angular momentum sum by an integral, and rotate contours across the complex plane.[4] The scattering amplitude becomes:

$$f \sim \left(\frac{-ig}{k^{x+iy+1}\sqrt{2\pi q}}\right)\left(\frac{z}{q}\right)^{\frac{x+iy}{2}+\frac{3}{4}} \Big[2e^{(x+iy+1)\pi i/2} K_{x+iy+3/2}(2\sqrt{zq}) - \pi i\, e^{-(x+iy+1)\pi i/2} H^{(2)}_{x+iy+3/2}(2\sqrt{zq})\Big] \tag{V.5}$$

But since:

$$2\sqrt{zq} = 2\eta\sqrt{\pi \sin\frac{\theta}{2}}$$

if either η or $\sin\theta/2$ is small, the Bessel functions can be conveniently expanded as power series. The leading term of the series for $H^{(2)}$, added to the first term given by K, duplicates Serber's asymptotic form for the Eikonal [actually there is a small difference caused by (V.4)]. For weak coupling or small angles, we expect the Eikonal to be fairly accurate, as in fact is the case. At large angles for strong coupling, however, higher terms in the power series of the Bessel functions contribute and the Eikonal becomes inaccurate. These terms are the result of the η^2 coefficient of (V.1) which produces the essential singularity in the partial wave amplitude of (V.3).

We note that if $\eta \gtrsim 1$ the term $H^{(2)}$ dominates the large-angle amplitude. If θ is fixed:

$$|f| \propto \frac{1}{k^{2n+1}} . \tag{V.6}$$

But at fixed k, if $\eta \gtrsim 1$ $\partial \ln |H^{(2)}|/\partial \ln q$ is small and therefore:

$$|f| \propto \frac{1}{q^{n+5/4}} . \tag{V.7}$$

We have the approximate relationship:

$$\left(\frac{\partial \ln |f|}{\partial \ln k}\right)_q \simeq 2 \cdot \left(\frac{\partial \ln |f|}{\partial \ln q}\right)_k + \frac{3}{2} . \tag{V.8}$$

These equations will turn out to be rather useful.

VI. COMPARISONS

As mentioned earlier, Serber's proton-proton potential:

$$V = \frac{e^{-\alpha r}}{r} \qquad \eta = 1.0 \qquad \alpha = 0.2646 \text{ GeV}$$

has been solved exactly by Avison whose results disagree with the Eikonal differential cross section, at large angle, by as much as a factor of 1000.

Using the Watson-Sommerfeld contours shown in Fig. 5, both the WKB (III.2)--and its asymptotic form (IV.3)--reproduce Avison's scattering amplitude quite reliably. In obtaining the large-angle scattering amplitude corresponding to (III.2) we use the Watson-Sommerfeld transform rather than a partial wave summation, since at large angles a partial wave sum involves a great deal of cancellation and is therefore quite sensitive to small errors in the phase shift integral.

The scattering amplitudes given by the WKB approximation, and its asymptotic form, are compared with Avison's result in Table I (k_{lab} = 11 GeV) and Table II (k_{lab} = 30 GeV). The disagreement between the WKB amplitude and Avison's is less than a factor of four, which, for a qualitative study of the sort intended here, is more than sufficient. Even this error occurs only at backward angles and is therefore masked by the forward amplitude when the symmetrized cross section is determined. It is worth noting that the energy dependence at fixed angle of both the WKB and its asymptotic form, reproduces Avison's result to within a small factor:

$$\left.\frac{\frac{d\sigma}{d\Omega}\,(11\text{ GeV},\ 90^\circ)}{\frac{d\sigma}{d\Omega}\,(30\text{ GeV},\ 90^\circ)}\right]_{\text{Avison}} = 13.2$$

$$\left.\frac{\frac{d\sigma}{d\Omega}\,(11\text{ GeV},\ 90^\circ)}{\frac{d\sigma}{d\Omega}\,(30\text{ GeV},\ 90^\circ)}\right]_{\text{WKB}} = 18.6$$

$$\left.\frac{\frac{d\sigma}{d\Omega}\,(11\text{ GeV},\ 90^\circ)}{\frac{d\sigma}{d\Omega}\,(30\text{ GeV},\ 90^\circ)}\right]_{\text{ASYM. WKB}} = 16.6$$

Figure 6 compares the three differential cross sections at 11 GeV. Figure 7, which corresponds to 30 GeV, shows only two lines since at this energy the complete WKB is indistinguishable from its asymptotic form. It is clear that the WKB approximation reproduces the angular dependence of the differential cross section quite reliably.

If η and α are raised we might expect the asymptotic form to become less accurate. However, even for the choice $\eta = 1.5$ and $\alpha = 0.467$ GeV the scattering amplitude given by (IV.3) is within 15% of that given by (III.2) at both $k_{lab} = 11$ GeV and $k_{lab} = 30$ GeV, for scattering angles greater than 50°.

Expression (V.2) reproduces Avison's results just as dependably as does the WKB. If η is much larger than 1, however, (V.2) becomes inaccurate since powers of η greater than two become significant. For the choice η = 3.687 and u = -0.3853 + ℓn GeV (a Yukawa potential with α = 0.382 GeV assigns this value to u). (V.2) gives differential cross sections larger than those of (IV.3) by a factor of 10. A partial wave summation is convenient for calculating the scattering amplitude corresponding to (V.2) since the necessary Bessel functions can be rapidly evaluated to great precision.

The asymptotic form of the scattering given by (V.2), expression (V.5), agrees with (V.2) fairly well. Table III compares the two amplitudes for various values of η and α at k_{lab} = 30 GeV, $\theta = 180^\circ$. For $\theta = 90^\circ$, the agreement is only slightly worse.

VII. CONCLUSION

We will now apply these methods to the problem of proton-proton scattering.[12] It should certainly be possible to fit the scattering amplitude given by (IV.3) to the energy dependence of 90° scattering: the normalization of the amplitude corresponding to (IV.3) can be controlled by choosing the quantity u, and the energy slope fixed by choice of η (since most of the energy dependence of the phase shift is carried by the term proportional to η ℓn k). In fact, the 90° scattering[5] is reproduced quite accurately on the internal 8 GeV $\leq k_{lab} \leq$ 31 GeV by the constants:

$$\eta = 3.687 \qquad\qquad u = -0.3853 + \ell n\ \mathrm{GeV}. \qquad \text{(VII.1)}$$

The angular slope given by these parameters, however, is much greater than observed. At 60°, for example, the cross section predicted by (VII.1) is greater than observation by a factor of 10. And if either η or u is altered, to reproduce the correct angular dependence, the energy dependence becomes incorrect, since $\partial \ell n\, [d\sigma(k, 90^\circ)/d\Omega]/\partial \ell n\, k$ is a monotonic function of η, and $d\sigma(k, 90^\circ)/d\Omega$ is a monotonic function of u.

We must conclude no "reasonably smooth" potential of the form:

$$i\eta \, \frac{P(\alpha r)}{r}$$

can account for the observed scattering unless η and α are themselves functions of s. By "reasonably smooth" we mean a potential for which the quantities

$$\frac{\alpha^N P^{(N)}(0)}{k^N}$$

are small enough that the asymptotic form is reliable in the region $8 \text{ GeV} \leq k_{lab} \leq 30 \text{ GeV}$.

By comparing (V.6) and (V.7) with the experimental data we can get some understanding of why energy-independent potentials fail. If the observed data are fit to a symmetrized[12] differential cross section calculated from the amplitude:

$$A(2k)^{\nu + w}(\sin\frac{\theta}{2})^w$$

we find:

$$\nu = -5.75 \qquad\qquad w = -2.25 \tag{VII.2}$$

The choice $\nu + w = -8$ requires $\eta = 3.5$ (V.6), but $w = -2.25$ implies $\eta = 1.0$ (V.7). In other words, (VII.2) violates (V.8).

If the scattering given by expression (V.1) is compared with experiment, our conclusion remains the same. The cross section corresponding to (V.1) for either of the potentials:

$$V_1 = \frac{e^{-\alpha r}}{r} \qquad\qquad V_2 = \frac{e^{-\alpha^2 r^2}}{r} \tag{VII.3}$$

invariably shows less energy slope than experimentally observed whenever the observed angular slope is correctly reproduced.

APPENDIX A

The function $r(\Lambda)$ defined by the equation:

$$\left[E + i\eta \frac{P(r)}{r}\right]^2 - m^2 = \frac{\Lambda^2}{r^2} \tag{A.1}$$

can be evaluated by performing the iteration:

$$r^N = -i\eta \frac{E}{k^2} P(r^{N-1}) + \sqrt{\frac{\Lambda^2}{k^2} - \frac{\eta^2 m^2}{k^4}\left[P(r^{N-1})\right]^2} \tag{A.2}$$

$$r^o = \frac{\Lambda}{k}$$

if η is sufficiently small and $P(r)$ is well behaved (we assume also that P is an entire function of r), since according to the contraction mapping fixed-point theorem,[10] the equation:

$$x = F(x) \tag{A.3}$$

has a unique solution in any region Q such that

$$\begin{aligned} &x, y \in Q \rightarrow |F(x) - F(y)| \leq \alpha |x - y| \\ &0 \leq \alpha < 1 \end{aligned} \tag{A.4}$$

and this solution is given by x^{∞} where:

$$x^N = F(x^{N-1}) \quad x^o = z \tag{A.5}$$

for any point $z \in Q$.

The scattering problem studied by Serber is

$$\begin{aligned} &P(r) = e^{-\alpha r} \qquad \alpha = 0.2646 \text{ GeV} \\ &\eta = 1.0 . \end{aligned} \tag{A.6}$$

If energy is measured in units of α at $k_{lab} = 11$ GeV then:

$$\begin{aligned} k &= 8.33 \\ E &= 9.05 \\ m &= 3.55 \\ P(r) &= e^{-r} . \end{aligned} \tag{A.7}$$

Using this collection of information, it can be shown that a region Q_Λ, $\Lambda/k \in Q_\Lambda$, exists fulfilling condition (A.4) and so forcing the convergence of iteration (A.2) if Λ is contained in the territory:

$$\{\Lambda \in \mathbb{C} \mid |\Lambda| \geq 0.74, \ \mathrm{Re}\Lambda \geq 0\} \ \cup \ \{\Lambda \in \mathbb{C} \mid \Lambda = iu, u \in \mathbb{R}\} \tag{A.8}$$

The square root must be evaluated according to the convention:

$$\sqrt{\Lambda^2/k^2 + x} = \Lambda/k\sqrt{1 + k^2\, x/\Lambda^2} \quad \mathrm{Re}\sqrt{1 + k^2\, x/\Lambda^2} \geq 0 \tag{A.9}$$

unless Λ is very close to the imaginary axis, in which case either choice of the sign of the square root, adhered to consistently, will yield a convergent series.

The function $r(\Lambda)$ is differentiable and hence analytic if:

$$\frac{\partial}{\partial r}\{[Er + i\eta P(r)]^2 - m^2 r^2\} \neq 0 \text{ at } r = r(\Lambda), \tag{A.10}$$

which is true at all points in region (A.8). We can also prove that $r(\Lambda)$ can be continued to almost every point of the positive half plane by considering the function:

$$F(r,\Lambda) = \Lambda^2 - [E + i\eta P(r)]^2 + m^2 r^2 \tag{A.11}$$

$r(\Lambda)$ is defined by:

$$F[r(\Lambda),\Lambda] = 0. \tag{A.12}$$

If (A.10) is fulfilled we have:

$$\frac{\partial}{\partial r} F(r,\Lambda) \neq 0 \text{ at } r = r(\Lambda). \tag{A.13}$$

And so, by a well-known theorem,[11] we have the result that if $r(\Lambda)$ exists at Λ_0 and if (A.10) is fulfilled at this point then $r(\Lambda)$ exists and is analytic in an open region of Λ_0. Thus all points on the frontier of the maximum region to which $r(\Lambda)$ can be extended violate (A.10).

(A.10) must definitely be violated at least once since if Λ is carried along a closed loop through the shaded region of (A.8) beginning and ending at 0, the function $r(\Lambda)$ does not return to its original value, and so there must be at least one branch point somewhere in the shaded region. By using the fixed-point theorem on (A.10) it can be shown, however, that there is exactly one such point, $\hat{r}$, violating (A.10) and therefore at most one $\hat{\Lambda}$ at which $r(\Lambda)$ cannot be defined. But of course for this value of Λ we already have a $r(\Lambda)$, namely $\hat{r}$, thus $r(\Lambda)$ is defined over the entire region $\mathrm{Re}\,\Lambda > 0$ and is analytic at all points in this region except $\hat{\Lambda}$.

To first order in η we have:

$$\hat{\Lambda} = \eta m/k. \tag{A.14}$$

APPENDIX B

The approximation:

$$\frac{i}{2k} \sum_{\ell=0}^{\infty} A(\ell) P_{\ell}(\cos\theta)(2\ell+1) \sim ik \int_0^{\infty} A(k\rho - \tfrac{1}{2}) \; J_0(\sqrt{-t}\rho)\rho d\rho \tag{B.1}$$

has been studied in some detail.[4] If $A(k\rho - 1/2)$ is analytic in ρ at $\rho = 0$, and the lead term in this power series has exponent less than 5, the error caused by (B.1), for large t, is less than a factor of 2. If the leading exponent is 1, at large t (B.1) will introduce no errors at all.

The additional error caused by

$$J_0(q\Lambda) \sim \sqrt{\frac{2}{\pi q \Lambda}} \cos\left(q\Lambda - \frac{\pi}{4}\right) \tag{B.2}$$

is a factor of $\sqrt{2}$ if the leading exponent is 1, 1.06 if the leading exponent is 3, and corresponding less if this exponent is greater than 3.

Since the partial wave amplitude given by (V.2) has an essential singularity at $\rho = 0$ the significant exponent here, in effect, is negative. At 180° the Watson-Sommerfeld version of the asymptotic amplitude given by (V.3) can be converted to a rapidly convergent series of Bessel functions, the value of which differs from the quantity found combining approximations (B.1) and (B.2) by less than a factor of 2, even when η is as large as 4.

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FOOTNOTES AND REFERENCES

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[12]We construct a spin-averaged differential cross section for elastic p-p scattering by the formula

$$\frac{d\sigma}{d\Omega}(t) = \left|f(\theta)\right|^2 + \left|f(\pi - \theta)\right|^2 - \mathrm{Re}\,\{f(\theta)f^*(\pi - \theta)\}$$

which is symmetric under the exchange $\theta \leftrightarrow \pi - \theta$, as must be the case since the two protons are identical. For a more detailed discussion see Ref. 2.

Table I. Scattering Amplitudes for Serber's Yukawa Potential.
$k_{lab} = 11$ GeV.

θ	Avison	WKB	Asymptotic WKB
30°	$0.62 + 1.85i\ (10^{-2})$	$0.551 + 1.41i\ (10^{-2})$	
50°	$3.00 + 0.66i\ (10^{-3})$	$2.21 + 0.548i\ (10^{-3})$	$2.03 - 1.22i\ (10^{-3})$
70°	$16.0 - 1.6i\ (10^{-4})$	$9.00 - 0.085i\ (10^{-4})$	$7.66 - 3.56i\ (10^{-4})$
90°	$9.7 - 1.8i\ (10^{-4})$	$4.63 - 0.291i\ (10^{-4})$	$3.92 - 1.57i\ (10^{-4})$
110°	$8.6 - 0.70i\ (10^{-4})$	$2.88 - 0.23i\ (10^{-4})$	$2.45 - 0.89i\ (10^{-4})$
130°	$8.00 - 1.2i\ (10^{-4})$	$2.08 - 0.186i\ (10^{-4})$	$1.77 - 0.62i\ (10^{-4})$

Table II. Scattering Amplitude for Serber's Yukawa Potential.
$k_{lab} = 30$ GeV.

θ	Avison	WKB	Asymptotic WKB
30°	$2.3 + 1.8i\ (10^{-3})$	$2.065 + 1.72i\ (10^{-3})$	
50°	$6.2 + 0.6i\ (10^{-4})$	$5.23 + 0.60i\ (10^{-4})$	$4.98 - 1.04i\ (10^{-4})$
70°	$4.4 + 0.0i\ (10^{-4})$	$2.04 + 0.08i\ (10^{-4})$	$1.95 - 0.32i\ (10^{-4})$
90°	$2.6 - 0.8i\ (10^{-4})$	$1.07 + 0.02i\ (10^{-4})$	$1.02 - 0.15i\ (10^{-4})$
110°	$2.2 + 0.08i\ (10^{-4})$	$6.75 + 0.09i\ (10^{-5})$	$6.50 - 0.88i\ (10^{-5})$
130°	$2.1 - 0.0i\ (10^{-4})$	$4.91 + 0.05i\ (10^{-5})$	$4.76 - 0.62i\ (10^{-5})$

Table III. Scattering Amplitudes Given by (V.5) and (V.2) for Various Yukawa Potentials.

$\theta = 180^\circ$

$k_{lab} = 30$ GeV

η	α	f - (V.5) - 10^{-18} cm	f - (V.2) - 10^{-18} cm
1.0	0.10 GeV	0.570 - 0.568i	0.696 - 0.657i
2.0	0.23 GeV	0.168 + 0.237i	0.105 + 0.261i
3.5	0.33 GeV	-0.016 + 0.168i	-0.052 + 0.147i

FIGURE CAPTIONS

Fig. 1. Contour C.

Fig. 2. Contour D.

Fig. 3. Modified contour C.

Fig. 4. Contour E.

Fig. 5. Watson-Sommerfeld contour.

Fig. 6. Differential cross section as a function of -t, $k_{lab} = 11$ GeV. The solid line is Avison's result, the dashed line is the scattering curve given by the WKB approximation, and the dotted line is the asymptotic form of the WKB approximation.

Fig. 7. Differential cross section as a function of -t, $k_{lab} = 30$ GeV. The solid line is Avison's result, and the dashed line is the curve given by the WKB approximation. At 30 GeV the WKB approximation and its asympotic form are indistinguishable.

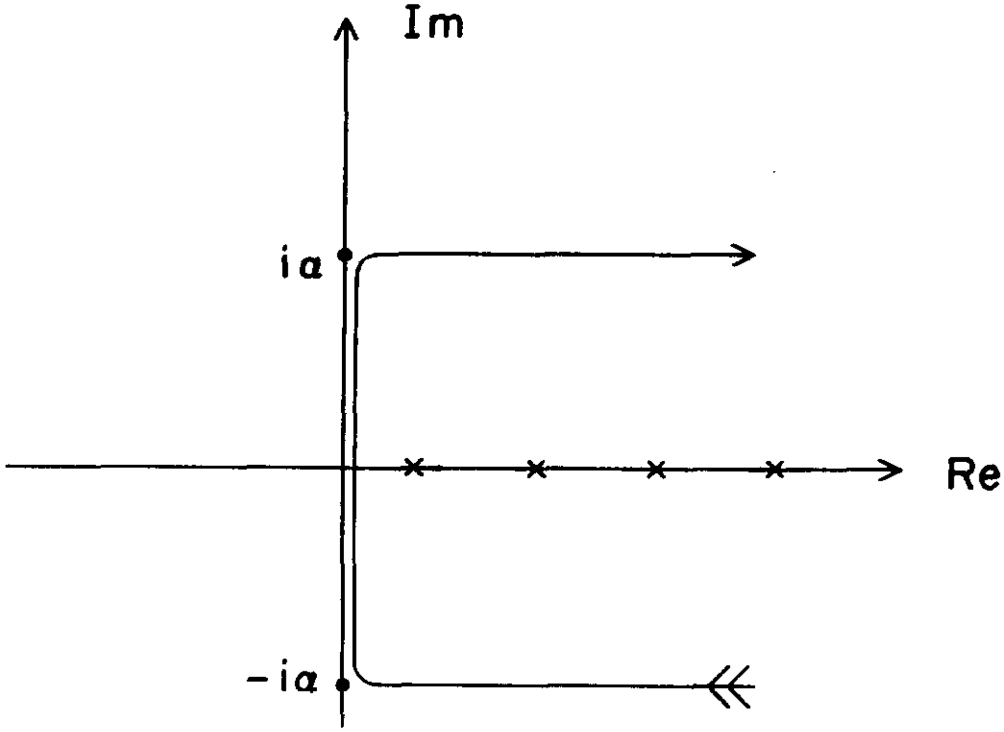


Fig. 1

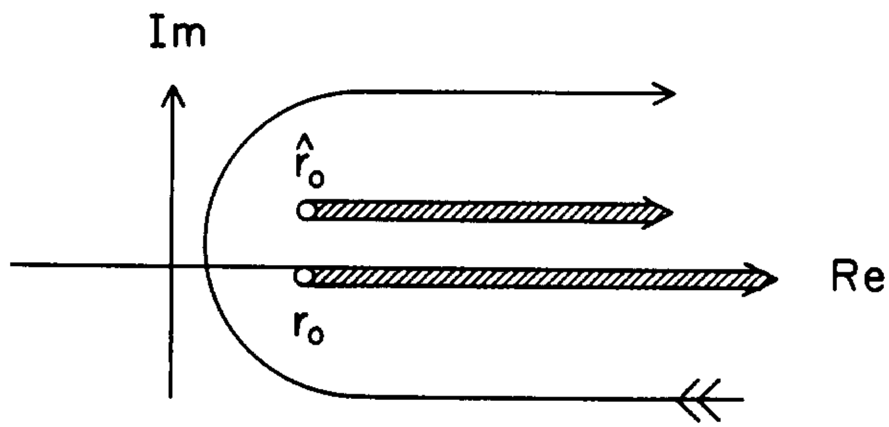


Fig. 2

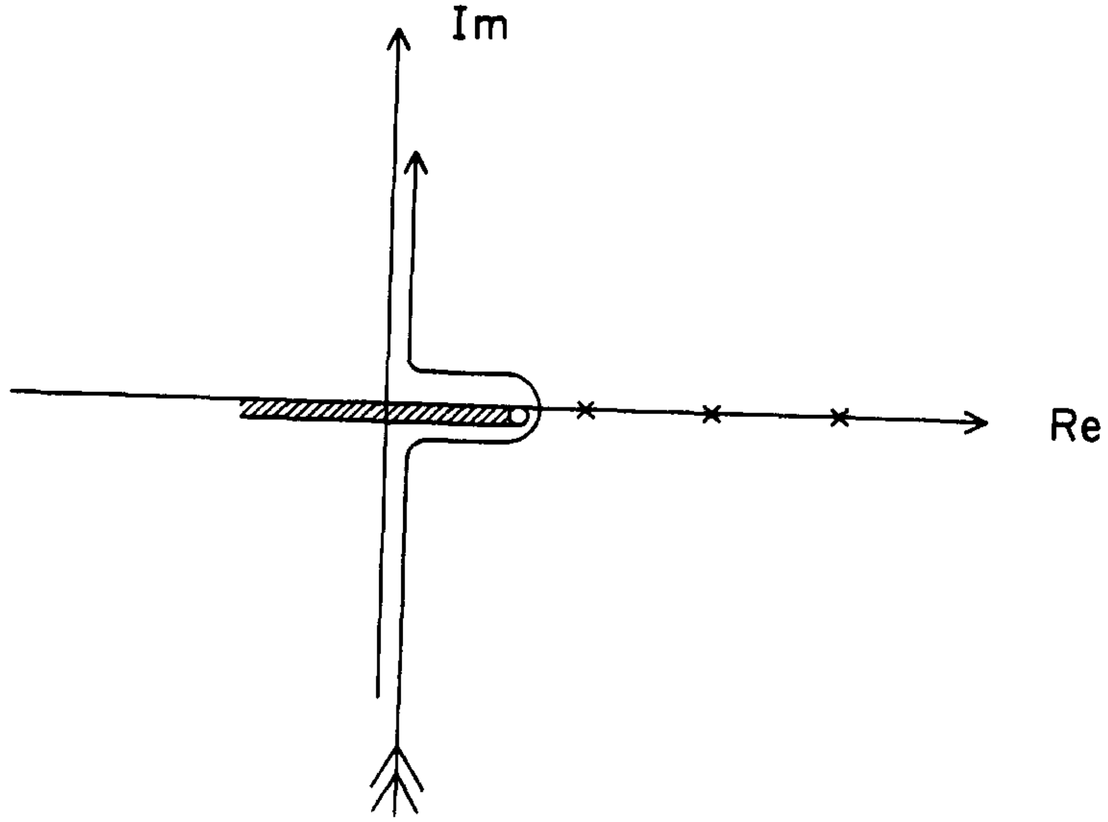


Fig. 3

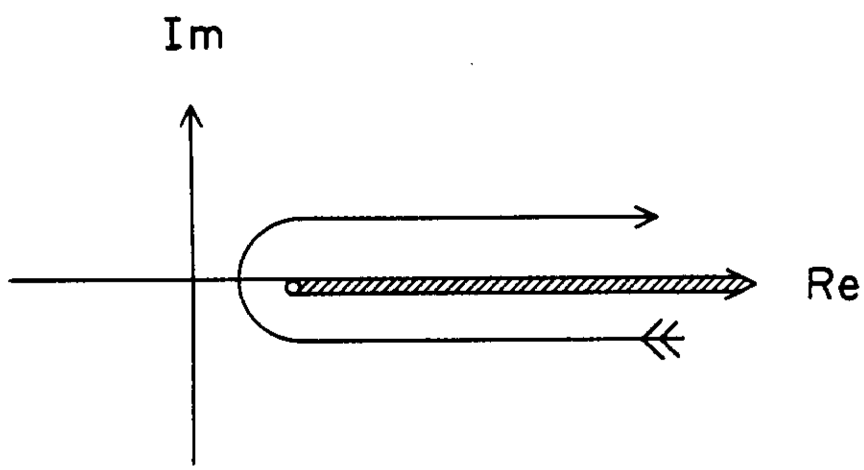
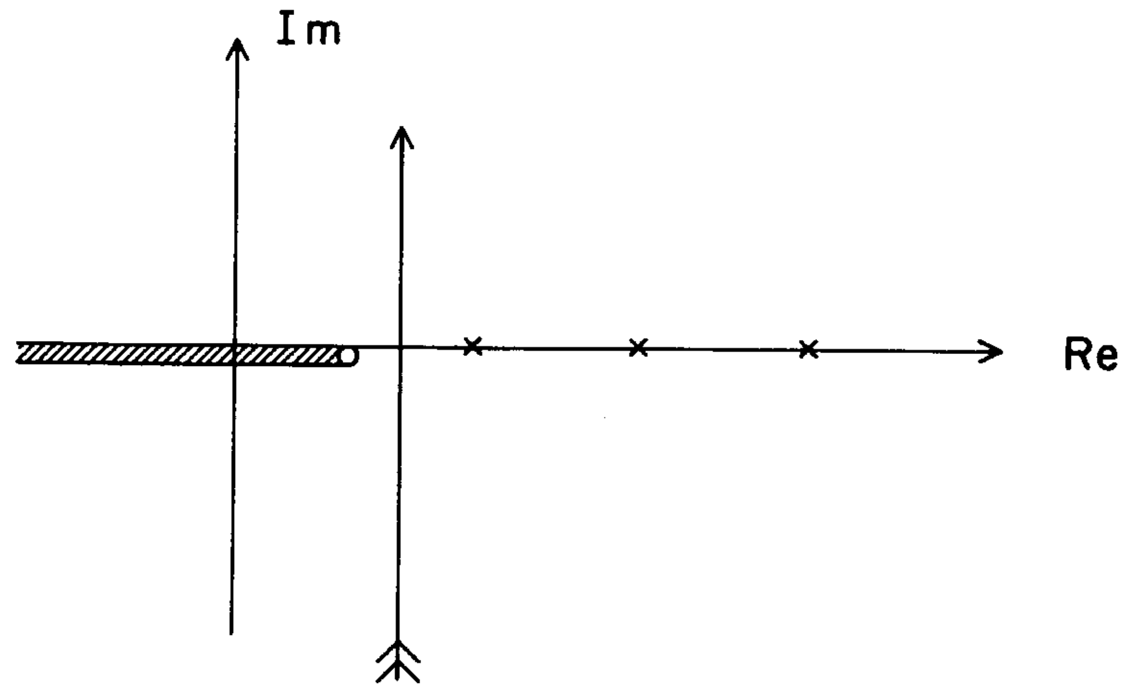


Fig. 4



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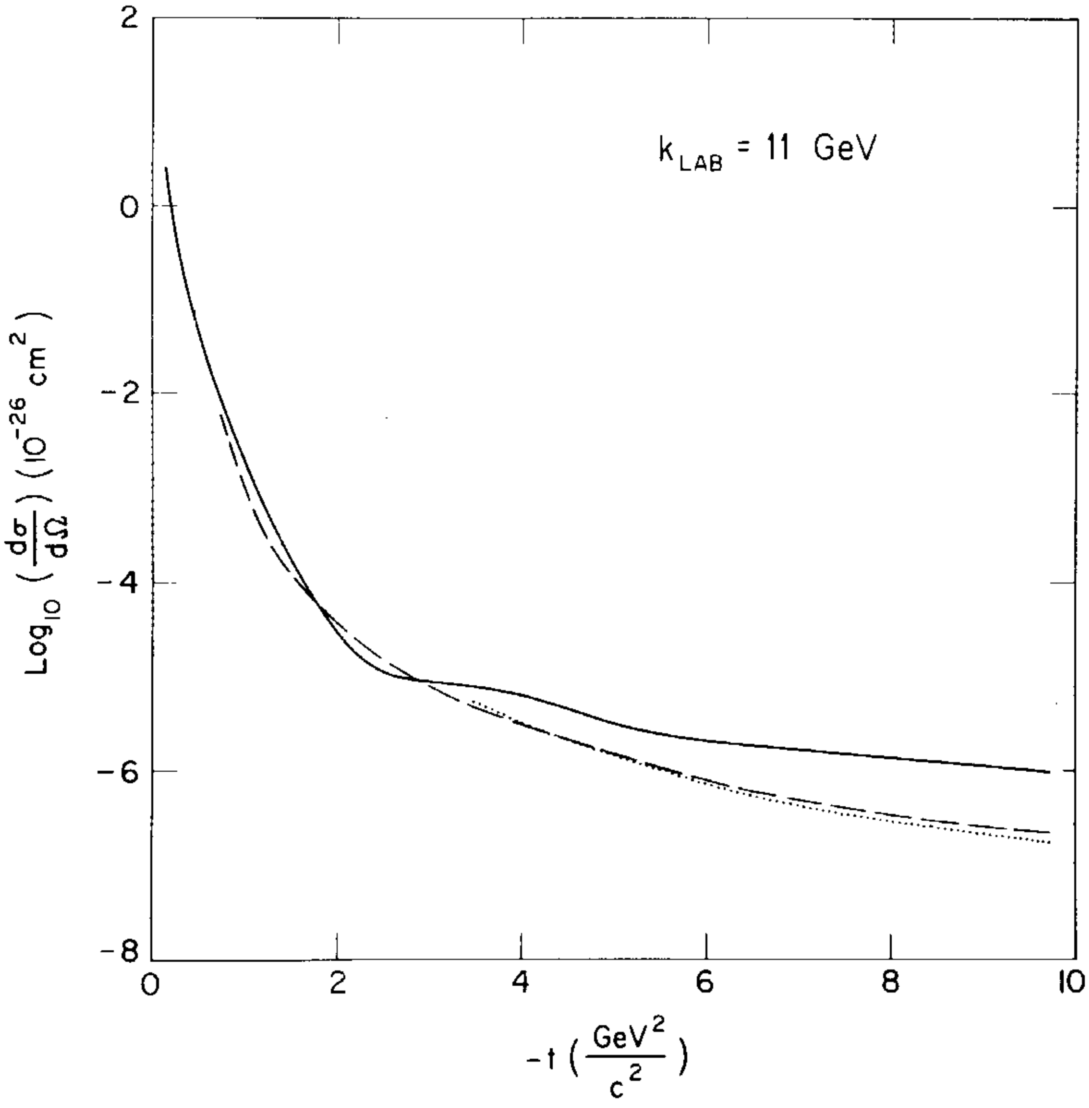
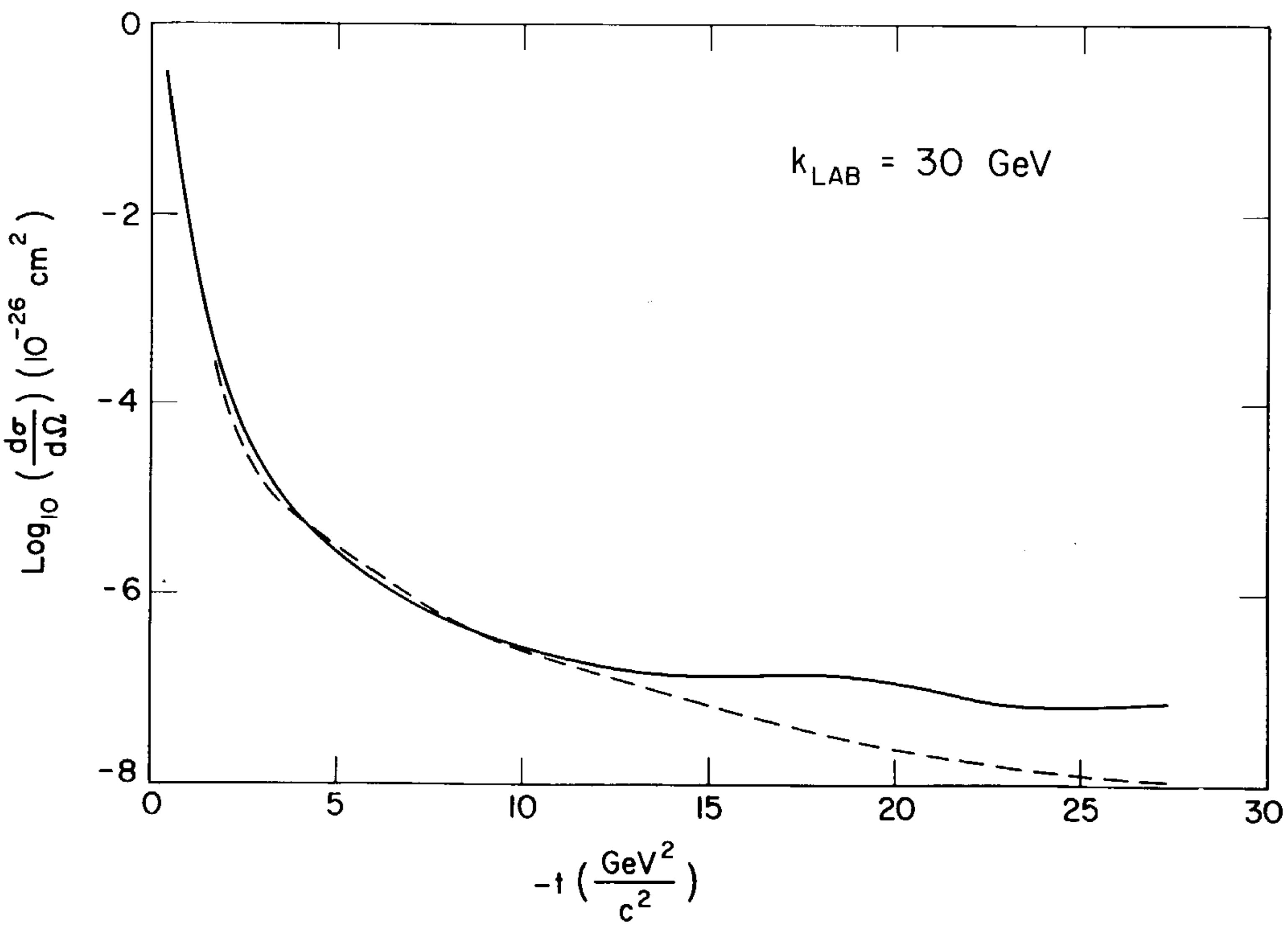


k_{LAB} = 11 GeV
$\text{Log}_{10}\left(\frac{d\sigma}{d\Omega}\right)(10^{-26}\,cm^2)$
$-t\left(\frac{GeV^2}{c^2}\right)$
2
0
-2
-4
-6
-8
0
2
4
6
8
10

Figure 6



k_{LAB} = 30 GeV
0
-2
-4
-6
-8
$\mathrm{Log}_{10}\left(\frac{d\sigma}{d\Omega}\right)(10^{-26}\ \mathrm{cm}^2)$
0
5
10
15
20
25
30
$-t\left(\frac{\mathrm{GeV}^2}{c^2}\right)$

Figure 7