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Calculation of the longitudinal emittance dilution in a RF cavity

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A. Shemyakin, Fermilab, Batavia, IL 60510, USA
shemyakin@fnal.gov

Abstract

Dilution of the longitudinal rms emittance of a bunch passing a RF cavity is estimated for the sinusoidal dependence of the energy gain on phase and Gaussian beam distribution.

1. Model

Dilution of the longitudinal rms emittance of a bunch passing a RF cavity can be estimated caused in a simplest model:

a) Energy ΔE gained by a particle depends only on its phase φ and the energy gain on crest U as

$$\Delta E = U \cdot \cos\varphi. \tag{1}$$

b) The energy gain of all particles is small in comparison with the total kinetic energy

$$U \ll (\gamma - 1)Mc^2 , \tag{2}$$

where γ is the relativistic factor, M is particle mass, and c is the speed of light.

c) The longitudinal distribution is Gaussian

$$\rho(z, z') \equiv \frac{d^2N}{dz\, dz'} = \frac{N_0}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon_0}}, J = \frac{1}{2}(\gamma_T z^2 + 2\alpha_T z \cdot z' + \beta_T z'^2), \tag{3}$$

where N_0 is the total number of particles in the bunch, ε_0 is the rms longitudinal emittance and $\alpha_T, \beta_T, \gamma_T$ are the longitudinal Twiss functions. The distribution is expressed in terms of the distance to the synchronous particle z and the rate of changing this distance along the longitudinal coordinate s,

$$z' \equiv \frac{dz}{ds} = -\frac{\Delta\beta}{\beta} , \tag{4}$$

where βc is the velocity of the synchronous particle and $\Delta\beta c$ is the velocity deviation.

d) Effects of space charge, beam loading, etc. are ignored. The only considered effect is the different energy gain for particles with different z.

2. Calculation

After passing the cavity, the particle velocity with respect to the synchronous particle changes by $\delta\beta c$, so that

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$$z_1' = z' - \frac{\delta\beta}{\beta} = z' - \frac{1}{\beta^2\gamma^3}\frac{\Delta E - U\cdot\cos\varphi_s}{Mc^2} \equiv z' + \delta z',$$
$$\delta z'(z) = -\frac{1}{\beta^2\gamma^3}\frac{U}{Mc^2}[\cos(\varphi_s - \Delta\varphi) - \cos\varphi_s],$$
$$\Delta\varphi = \frac{2\pi f_c z}{\beta c}, \tag{5}$$

where f_c is the cavity frequency, and φ_s is the synchronous phase.
The diluted rms emittance ε_1 is calculated by usual averaging over all particles:

$$\varepsilon_1^2 = \overline{z^2}\,\overline{z_1'^2} - \overline{zz_1'}^2 = \overline{z^2}\left(\overline{z'^2} + 2\overline{z'\delta z'} + \overline{\delta z'^2}\right) - \left(\overline{zz'}^2 + 2\overline{zz'}\cdot\overline{z\delta z'} + \overline{z\delta z'}^2\right) =$$
$$= \varepsilon_0^2 + \sigma_z^2\cdot 2\overline{z'\delta z'} + \sigma_z^2\cdot\overline{\delta z'^2} - 2\overline{zz'}\cdot\overline{z\delta z'} - \overline{z\delta z'}^2,$$
$$\varepsilon_0^2 \equiv \overline{z^2}\,\overline{z'^2} - \overline{zz'}^2,\ \ \sigma_z^2 \equiv \overline{z^2} = \beta_T\varepsilon_0, \qquad \overline{zz'} = -\alpha_T\varepsilon_0. \tag{6}$$

Averaging of each component of Eq.(6) can be made by explicit integration using Eq.(1), Eq.(3), and Eq.(5). The first component is integrated as follows.

$$\overline{z'\delta z'} = \frac{1}{N_0}\int_{-\infty}^{\infty} dz\int_{-\infty}^{\infty} z'\delta z'\rho(z,z')dz'\,. \tag{7}$$

To simplify the expression, the z, z' variables are expressed through the variables u, v:

$$z = u\sigma_z \equiv u\sqrt{\beta_T\varepsilon_0}\,,\ \ z' = \frac{v - \alpha_T u}{\sqrt{\frac{\beta_T}{\varepsilon_0}}}, \tag{8}$$

Eq.(7) can be re-written as

$$\overline{z'\delta z'} = \frac{1}{2\pi}\sqrt{\frac{\varepsilon_0}{\beta_T}}\int_{-\infty}^{\infty}\delta z'(u)du\int_{-\infty}^{\infty}(v - \alpha_T u)e^{-\frac{u^2+v^2}{2}}dv = -\frac{\alpha_T}{\sqrt{2\pi}}\sqrt{\frac{\varepsilon_0}{\beta_T}}\int_{-\infty}^{\infty}\delta z'(u)ue^{-\frac{u^2}{2}}du\,. \tag{9}$$

Substituting Eq.(8) to Eq.(5),

$$\delta z'(u) = -\frac{1}{\beta^2\gamma^3}\frac{U}{Mc^2}[\cos(\varphi_s - ku) - \cos\varphi_s] = A(\cos\varphi_s\cos ku + \sin\varphi_s\sin ku - \cos\varphi_s),$$
$$A \equiv -\frac{1}{\beta^2\gamma^3}\frac{U}{Mc^2}, \qquad k \equiv \frac{2\pi f_c\sigma_z}{\beta c}. \tag{10}$$

Integral of the first and third terms in Eq. (10) is zero by parity, and Eq. (9) yields

$$\overline{z'\delta z'} = -\frac{\alpha_T}{\sqrt{2\pi}}\sqrt{\frac{\varepsilon_0}{\beta_T}}A\sin\varphi_s\int_{-\infty}^{\infty}\sin ku\,ue^{-\frac{u^2}{2}}du = -\alpha_T\sqrt{\frac{\varepsilon_0}{\beta_T}}A\sin\varphi_s\,ke^{-\frac{k^2}{2}}. \tag{11}$$

Integration of other terms of Eq.(6) is similar:

$$\overline{\delta z'^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta z'^2(u)du\int_{-\infty}^{\infty}e^{-\frac{u^2+v^2}{2}}dv =$$
$$= \frac{A^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[1 + \frac{\cos 2\varphi_s}{2} + \frac{1}{2}\cos 2\varphi_s\cos 2ku - (1 + \cos 2\varphi_s)\cos ku\right]e^{-\frac{u^2}{2}}du =$$

$$= A^2\left[1+\frac{\cos 2\varphi_s}{2}+\frac{\cos 2\varphi_s\, e^{-2k^2}}{2}-(1+\cos 2\varphi_s)e^{-\frac{k^2}{2}}\right], \tag{12}$$

$$\overline{z\delta z'} = \frac{\sigma_z}{2\pi}\int_{-\infty}^{\infty} u\cdot\delta z'(u)du\int_{-\infty}^{\infty} e^{-\frac{u^2+v^2}{2}}dv = A\sigma_z \sin\varphi_s\, ke^{-\frac{k^2}{2}}. \tag{13}$$

Substitution of Eq.(11) – (13) into Eq. (6) gives

$$\varepsilon_1^2-\varepsilon_0^2 = -2\sigma_z^2\alpha_T\sqrt{\frac{\varepsilon_0}{\beta_T}}A\sin\varphi_s\, ke^{-\frac{k^2}{2}}+$$
$$+A^2\sigma_z^2\left[1+\frac{\cos 2\varphi_s}{2}+\frac{\cos 2\varphi_s\, e^{-2k^2}}{2}-(1+\cos 2\varphi_s)e^{-\frac{k^2}{2}}\right]+$$
$$+2\alpha_T\varepsilon_0 A\sigma_z\sin\varphi_s\, ke^{-\frac{k^2}{2}}-\left(A\sigma_z\sin\varphi_s\, ke^{-\frac{k^2}{2}}\right)^2. \tag{14}$$

After combining the terms in Eq. (14), the final expression for the diluted emittance is

$$\varepsilon_1^2-\varepsilon_0^2 = A^2\sigma_z^2\left\{\cos 2\varphi_s\left[\frac{1}{2}+\frac{e^{-2k^2}}{2}-e^{-\frac{k^2}{2}}+\frac{k^2}{2}e^{-k^2}\right]+1-e^{-\frac{k^2}{2}}-\frac{k^2}{2}e^{-k^2}\right\}. \tag{15}$$

For a small dilution, the relative emittance change can be expressed as

$$\frac{\delta\varepsilon}{\varepsilon_0}\equiv\frac{\varepsilon_1-\varepsilon_0}{\varepsilon_0}\approx A^2\frac{\sigma_z^2}{2\varepsilon_0^2}\left\{\cos 2\varphi_s\left[\frac{1}{2}+\frac{e^{-2k^2}}{2}-e^{-\frac{k^2}{2}}+\frac{k^2}{2}e^{-k^2}\right]+1-e^{-\frac{k^2}{2}}-\frac{k^2}{2}e^{-k^2}\right\}. \tag{16}$$

3. Case of the small bunch length

The variable k introduced in Eq. (5) is the rms bunch length expressed in the phase units, $k\equiv\sigma_\varphi$. Typically, this value is small, $\sigma_\varphi \ll 1$. For this case, Eq. (16) can be expanded to the first non-zero terms:

$$\frac{\delta\varepsilon}{\varepsilon_0}\approx A^2\frac{\sigma_z^2}{2\varepsilon_0^2}\left[\cos^2\varphi_s\left(\frac{3}{8}{\sigma_\varphi}^4+\frac{19}{48}{\sigma_\varphi}^6\right)+\frac{{\sigma_\varphi}^6}{6}\right]. \tag{17}$$

For a bunching cavity, $\varphi_s=-\frac{\pi}{2}$, Eq. (17) can be transformed to show explicitly the initial parameters as follows:

$$\frac{\delta\varepsilon}{\varepsilon_0}\approx\frac{1}{12}\left(\frac{1}{\beta^2\gamma^3}\frac{U}{Mc^2}\cdot\frac{\beta c}{2\pi f_c\varepsilon_0}\right)^2{\sigma_\varphi}^8. \tag{18}$$

For convenience, Eq. (18) can be expressed through the RF wavelength $\lambda=\frac{c}{f_c}$ and normalized emittance $\varepsilon_n=\beta\gamma^3\varepsilon_0$:

$$\frac{\delta\varepsilon}{\varepsilon_0}\approx\frac{1}{12}\left(\frac{U}{Mc^2}\cdot\frac{\lambda}{2\pi\varepsilon_n}\right)^2{\sigma_\varphi}^8. \tag{19}$$

For an accelerating cavity, $|\varphi_s| - \frac{\pi}{2} \gg \sigma_\varphi \sqrt{\frac{2}{9}}$, Eq. (17) is approximated

$$\frac{\delta\varepsilon}{\varepsilon_0} \approx \frac{3}{16}\left(\frac{U\cos\varphi_s}{Mc^2} \cdot \frac{\lambda}{2\pi\varepsilon_n}\right)^2 {\sigma_\varphi}^6. \tag{20}$$

Note that in Eq. (20), the synchronous phase enters only as the energy gain by the synchronous particle.

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