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Practical Beam-Beam Tune Shift Formulae for Simulation Cross-Check*

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Abstract

In this note, practical analytical formulae for head-on beam-beam tune shift are derived for the case of arbitrary crossing angle and non-negligible hourglass effect. The formulae are evaluated using numerical integration, and the results used to cross-check with the particle tracking code Lifetrac.

Introduction

The general approach to derive the betatron tune shift for a particle with small amplitude passing through the electromagnetic field of a counter-propagating beam bunch follows that of [1] and [2]. The former case treats the hourglass effect, and the latter provides formulae for bunches colliding at an angle. Our derivation combines the two approaches.

Tune Shift Formulae

The expressions for electromagnetic field of a 3D Gaussian ellipsoid are [3]:

$$
\begin{aligned}
E_x &= \frac{eN\gamma}{2\varepsilon_0\sqrt{\pi}^3}x\int_0^\infty dw\frac{exp\left(-\frac{x^2}{2\sigma_x^2+w}-\frac{y^2}{2\sigma_y^2+w}-\frac{\gamma^2(z-ct)^2}{2\gamma^2\sigma_z^2+w}\right)}{(2\sigma_x^2+w)^{3/2}\left(2\sigma_y^2+w\right)^{1/2}(2\gamma^2\sigma_z^2+w)^{1/2}}\\
E_y &= \frac{eN\gamma}{2\varepsilon_0\sqrt{\pi}^3}y\int_0^\infty dw\frac{exp\left(-\frac{x^2}{2\sigma_x^2+w}-\frac{y^2}{2\sigma_y^2+w}-\frac{\gamma^2(z-ct)^2}{2\gamma^2\sigma_z^2+w}\right)}{(2\sigma_x^2+w)^{1/2}\left(2\sigma_y^2+w\right)^{3/2}(2\gamma^2\sigma_z^2+w)^{1/2}}\\
B_x &= -\frac{E_y}{c}, B_y = \frac{E_x}{c}
\end{aligned}
\tag{1}
$$

Here N is the number of particles in the bunch, x,y,z are the Cartesian horizontal, vertical and longitudinal coordinates in the system of strong bunch, σ_x, σ_y, σ_z are the horizontal, vertical and longitudinal beam sizes. We denote the coordinates and fields for a particle in the weak bunch in the system of that bunch using the $'$ sign. If the weak bunch is moving at a horizontal angle θ with respect to the strong bunch axis, the coordinate transformations are:

$$
\begin{cases}
x = x'\cos(\theta) + z'\sin(\theta)\\
y = y'\\
z = z'\cos(\theta) - x'\sin(\theta)
\end{cases}
\tag{2}
$$

The equations of motion for a particle in the weak beam are:

$$
\begin{cases}
x(t) = -ct\sin(\theta) + x_0\\
y(t) = y_0\\
z(t) = -ct\cos(\theta) + z_0
\end{cases}
\tag{3}
$$

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The horizontal and vertical Lorentz force for a particle in the weak beam is:

$$\begin{cases} F_x' = F_x = e\left(E_x - v_z B_y\right) = eE_x(1+\cos(\theta)) \\ F_y' = F_y = e\left(E_y + v_z B_x\right) = eE_y(1+\cos(\theta)) \end{cases} \tag{4}$$

Tune shift for a particle in the weak beam due to a thin transverse focusing element is:

$$\begin{cases} \Delta Q_x = \dfrac{\beta_x}{4\pi} G_x = \dfrac{\beta_x}{4\pi}\dfrac{1}{p_0}\dfrac{\partial \Delta p_x'}{\partial x'} = \dfrac{\beta_x}{4\pi}\dfrac{dt}{p_0}\dfrac{\partial \Delta F_x'}{\partial x'} \\ \Delta Q_y = \dfrac{\beta_y}{4\pi} G_y = \dfrac{\beta_y}{4\pi}\dfrac{1}{p_0}\dfrac{\partial \Delta p_y'}{\partial x'} = \dfrac{\beta_y}{4\pi}\dfrac{dt}{p_0}\dfrac{\partial \Delta F_y'}{\partial y'} \end{cases} \tag{5}$$

Assuming no external fields in the interaction region, and the minimum beta-function coinciding with the z=0 position, one can express the horizontal and vertical beam sizes as functions of z:

$$\begin{cases} \sigma_x(z) = \sqrt{\varepsilon_x \beta_x^*(1 + z^2/\beta_x^*)} \\ \sigma_y(z) = \sqrt{\varepsilon_y \beta_y^*\left(1 + z^2/\beta_y^*\right)} \end{cases} \tag{6}$$

Here ε_x, ε_y are the horizontal and vertical emittance of the strong beam, β_x^* and β_y^* are the horizontal and vertical beta-functions at the IP.

Substituting (2), (3) into (1) and combining (4) and (5), and integrating (5) over t, we obtain:

$$\Delta Q_x = \frac{r_0 N}{2\pi^{3/2}}(1+\cos(\theta)) \int_0^\infty dw \int_{-\infty}^{\infty} dz\, \beta_x(z) \left(\cos(\theta) + 2z^2 \sin(\theta)^2 \left[\frac{\gamma^2 z^2(\cos(\theta) - 1)}{2\gamma^2\sigma_z^2 + w} - \frac{1}{2\sigma_x(z)^2 + w} \right] \right) \times \\ \times \frac{exp\left(-\dfrac{z^2 \sin(\theta)^2}{2\sigma_x(z)^2 + w} - \dfrac{\gamma^2 z^2 (1+\cos(\theta))^2}{2\gamma^2\sigma_z^2 + w} \right)}{(2\sigma_x(z)^2 + w)^{3/2} \left(2\sigma_y(z)^2 + w\right)^{1/2} \left(2\gamma^2\sigma_z^2 + w\right)^{1/2}} \tag{7}$$

$$\Delta Q_y = \frac{r_0 N}{2\pi^{3/2}}(1+\cos(\theta)) \int_0^\infty dw \int_{-\infty}^{\infty} dz\, \beta_y(z) \frac{exp\left(-\dfrac{z^2 \sin(\theta)^2}{2\sigma_x(z)^2 + w} - \dfrac{\gamma^2 z^2 (1+\cos(\theta))^2}{2\gamma^2\sigma_z^2 + w} \right)}{(2\sigma_x(z)^2 + w)^{1/2} \left(2\sigma_y(z)^2 + w\right)^{3/2} (2\gamma^2\sigma_z^2 + w)^{1/2}}$$

For negligible hourglass effect ($\sigma_z/\beta_{x,y} \ll 1$) one can neglect the $\sigma_x(z)$ and $\sigma_y(z)$ substituting them with the constant values $\sigma_{x,y}^* = \sqrt{\varepsilon_{x,y}\beta_{x,y}^*}$. Then, the integrals can be taken for θ=0, arriving at the well-known expressions:

$$\Delta Q_x = \frac{r_0 N \beta_x^*}{2\pi\gamma\sigma_x^*\left(\sigma_x^* + \sigma_y^*\right)} = \xi_{x0}$$

$$\Delta Q_y = \frac{r_0 N \beta_y^*}{2\pi\gamma\sigma_y^*\left(\sigma_x^* + \sigma_y^*\right)} = \xi_{y0}$$

For simplicity of the following analysis we introduce the normalized tune shift factors:

$$R_x = \frac{\Delta Q_x}{\xi_{x0}}, R_y = \frac{\Delta Q_y}{\xi_{y0}}$$

Head-on Tune Shift for Round Beams

Tune shift for round beams in the case of full head-on collision (θ=0) can be obtained analytically with a much simpler derivation. Noting that for the ultra-relativistic case the longitudinal field component can be neglected, and using the axial symmetry of the system, the expression for the electrical field of a round beam is derived from the Gauss' law:

$$2\pi r \cdot E_r = \frac{1}{\varepsilon_0}\rho(z)dz\int_0^r \rho_r(r')2\pi r'dr'$$

Substituting Gaussian longitudinal and transverse densities, we obtain:

$$E_r = \frac{eN}{2\pi\varepsilon_0}\frac{e^{-z^2/2\sigma_z^2}}{\sqrt{2\pi}\sigma_z}dz\frac{1}{r}\left[1-e^{-r^2/2\sigma^2}\right]$$

Then, using $F_r = 2eE_r$ and (5), the tune shift is:

$$\Delta Q_x = \Delta Q_y = \frac{r_0 N}{2\pi\gamma}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}\sigma_z}\beta(z)e^{-z^2/2\sigma_z^2}\frac{\partial}{\partial x}\left[\frac{1-e^{-r^2/2\sigma(z)^2}}{r}\right]\Bigg|_{r=0} =$$

$$\frac{r_0 N}{4\pi\gamma}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}\sigma_z}\frac{\beta(z)}{\sigma(z)^2}e^{-z^2/2\sigma_z^2} = \frac{r_0 N}{4\pi\gamma}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}\sigma_z}\frac{\beta(z)}{\varepsilon\beta(z)}e^{-z^2/2\sigma_z^2} = \frac{r_0 N}{4\pi\gamma\varepsilon}$$

Hence, the head-on tune shift is not affected by hourglass effect.

Cross-Check with Lifetrac

The formulae (7) were integrated numerically for a number of beam parameters. Below the results of these calculations are compared with the tune shifts calculated with the particle tracking code Lifetrac. The tune shift calculation in Lifetrac was performed via the normal mode analysis, in which a one turn linear map of the machine is calculated by tracking a sample particle with varying initial conditions and the subsequent derivation of a linearized map. This procedure has limited precision, which explains the observed minor difference with the analytical calculation. Otherwise, the agreement is very good.

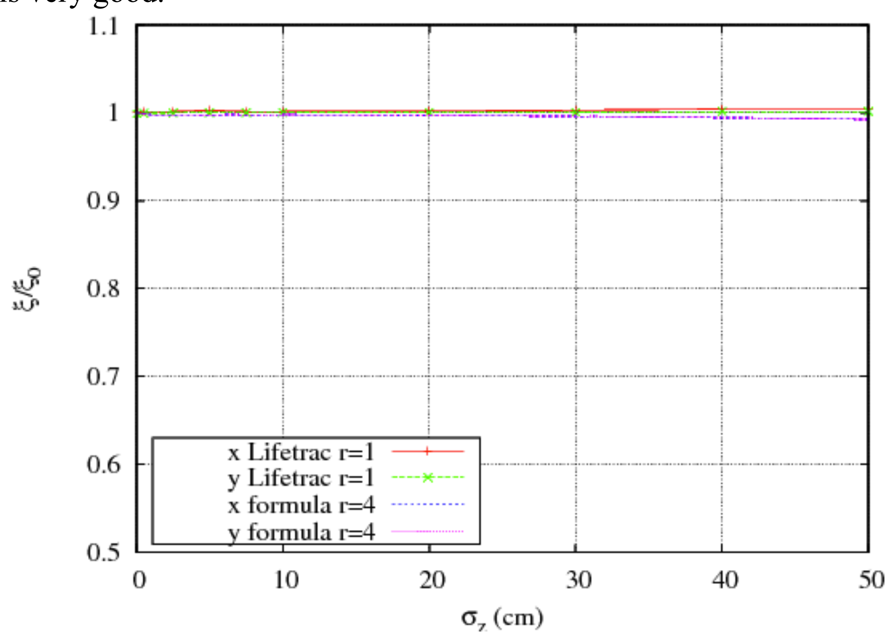

Figure 1: Tune shift factor vs. bunch length for θ=0. Round beam β_x=15 cm, β_y=15 cm, ε_x=ε_y.

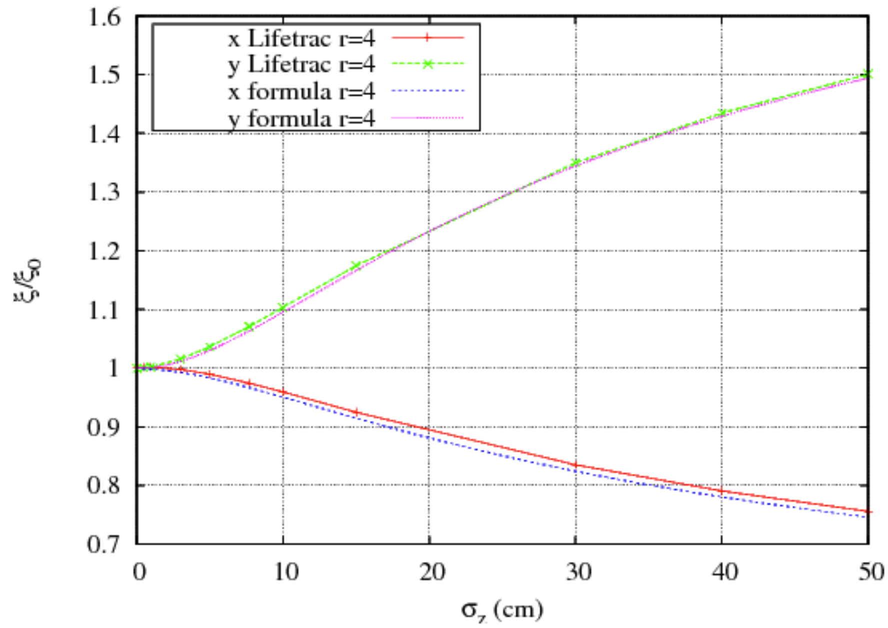

Figure 2: Tune shift factor vs. bunch length for θ=0. Flat beam β_x=30 cm, β_y=7.5 cm, $\varepsilon_x=\varepsilon_y$.

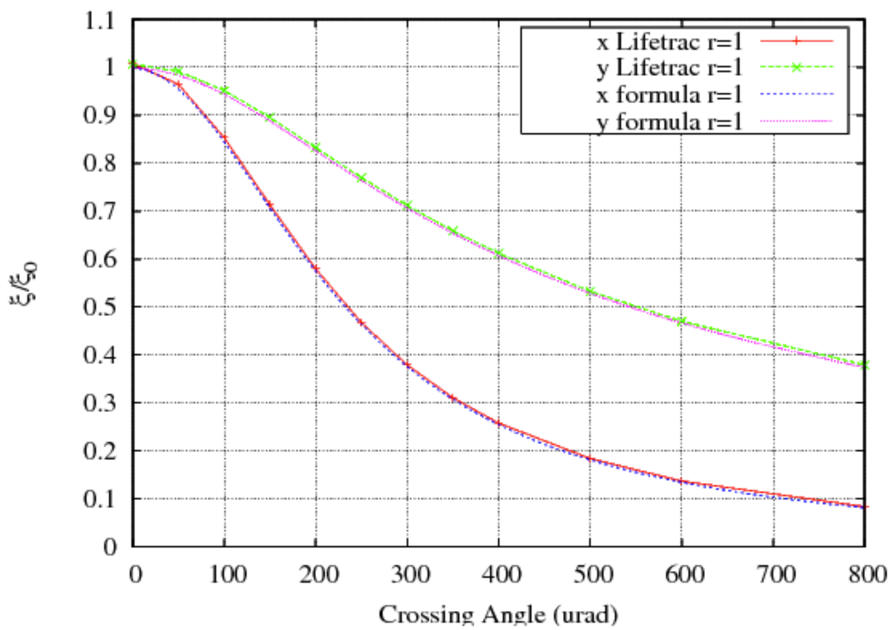

Figure 3: Tune shift factor vs. crossing angle for σ_z=7.5 cm. Round beam β_x=15 cm, β_y=15 cm, $\varepsilon_x=\varepsilon_y$.

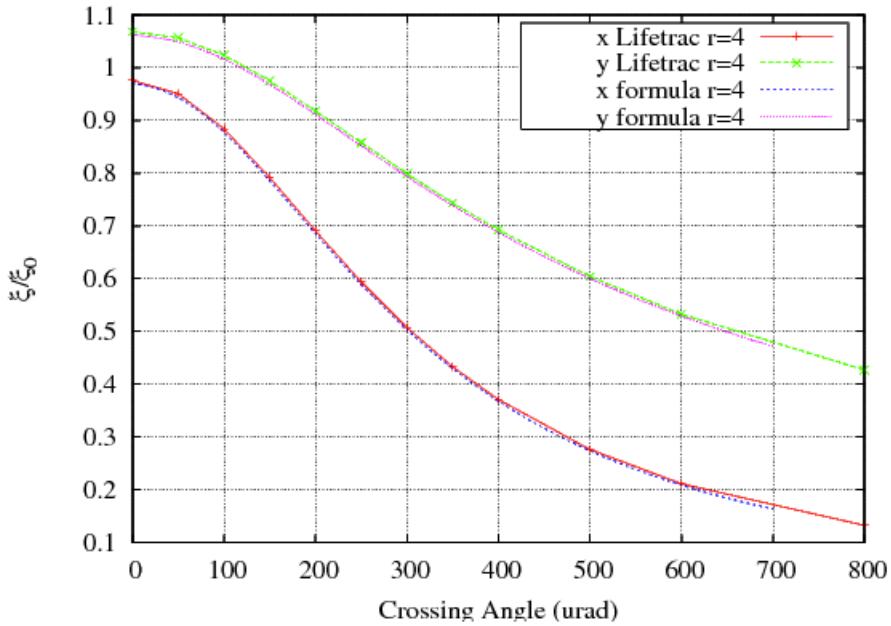

Figure 4: Tune shift factor vs. crossing angle for σ_z=7.5 cm. Flat beam β_x=30 cm, β_y=7.5 cm, ε_x=ε_y.

References

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