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BEAM-BEAM PHENOMENOLOGY

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Introduction

In colliding beam storage rings the beam collision regions are generally so short that the beam-beam interaction can be considered as a series of evenly spaced non-linear kicks superimposed on otherwise stable linear oscillations. Most of the numerical studies on computers were carried out in just this manner. But for some reason this model has not been extensively employed in analytical studies. This is perhaps because all analytical work has so far been done by mathematicians pursuing general transcendental features of non-linear mechanics for whom this specific model of the specific system of colliding beams is too parochial and too repugnantly physical. Be that as it may, this model is of direct interest to accelerator physicists and is amenable to (1) further simplification, (2) physical approximation, and (3) solution by analogy to known phenomena.

We define the simplified system as follows:

(A) head-on collisions of 2 beam bunches at regular intervals, say, once per revolution.

(B) the weak/strong case in which the strong beam is not affected by collisions with the weak beam. Thus, we have in effect, a single particle colliding with a beam bunch.*

(C) The strong beam bunch is short compared to the betatron

*Transition to the strong/strong case is similar to the transition from single particle dynamics in an accelerator to the dynamics of a high intensity beam.

wave length of the colliding particle so that it can be approximated by a δ-function in the longitudinal coordinate s.

(D) Close encounters between particles are negligible, hence the beam-beam force is given by a potential. Moreover, since the strong beam is not affected by the colliding particle, the potential is static. The potential depends on the transverse distribution of the beam bunch and can also be approximated by a δ-function in s.

Nature of the Beam-Beam Forces

(A) Extremely non-linear

To get a rough idea of the degree of non-linearity consider a simple round beam with current I. "Outside" the beam at radial location r the magnetic field is

$$B = \frac{2I}{r} . \tag{1}$$

The conventional non-linear field coefficients are

$$b_n \equiv \frac{1}{n!} \frac{1}{B_o} \frac{d^nB}{dr^n} = (-1)^n \frac{2I}{B_o r^{n+1}} = (-1)^n \frac{B}{B_o} \frac{1}{r^n} \tag{2}$$

where B_o is the external dipole bending field. For colliding beams the electric and the magnetic forces add, and the non-linear force coefficients are, therefore, approximately $2b_n$. Taking normal values:

I ~ amperes

r ~ millimeters

B_o ~ teslas

one gets

$$|b_n| \sim 10^{-4} r^{-n}. \tag{3}$$

This shows that when expressed in units of $[r]^{-n}$ the numerical values of b_n are independent of n, but in bigger units, say cm^{-n}, the numerical values of b_n increase rapidly with n. This should be compared to the non-linearities arising from errors in the external guide field. Even for the rather poor superconducting dipoles the error non-linear field coefficients fall off rather sharply with increasing n when expressed in units of cm^{-n}.

(B) Non-linear forces are localized to "surface" of beam.

The external error non-linear fields are largest at the coil aperture boundary and decrease rapidly toward the center where the beam resides. The non-linear beam-beam forces behave, however, just in the opposite way. They are largest at the "surface" of the beam and decrease sharply toward the aperture boundary. Hence the beam-beam forces affect the beams much more strongly.

(C) The force potential is periodic in s but very rich in harmonics.

Indeed, if the potential is truly a δ-function of s it will have a "white" harmonic spectrum, i.e. equal harmonic content all the way up to infinite order.

Measure of Beam-Beam Effects

Although many parameters are required to specify the density distribution of the beam bunch and the dynamics of the particle, for simple beam bunch distributions the effects of the beam-beam forces on the colliding particle can be specified by only a few combinations of these parameters. Let us take a bi-Gaussian beam distribution.

$$\rho = \frac{N}{2\pi\sigma_x\sigma_y} \delta(s) \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right) \tag{4}$$

where s is periodic with the periodicity of the ring circumference. The force potential is, then[1]

$$V(x,y) = \frac{r_o N}{\gamma} \int_o^\infty dt \, \frac{1-\exp\left[-\frac{x^2}{2(\sigma_x^2+t)} - \frac{y^2}{2(\sigma_y^2+t)}\right]}{\sqrt{(\sigma_x^2+t)(\sigma_y^2+t)}}$$

$$= \frac{r_o N}{\gamma} \int_o^\infty d\left(\frac{t}{\sigma_x\sigma_y}\right) G\left(\frac{x^2}{\sigma_x(\sigma_x+\sigma_y)}, \frac{y^2}{\sigma_y(\sigma_x+\sigma_y)}; \frac{t}{\sigma_x\sigma_y} \middle| \frac{\sigma_y}{\sigma_x}\right)$$

$$= \frac{r_o N}{\gamma} F\left(\frac{x^2}{\sigma_x(\sigma_x+\sigma_y)}, \frac{y^2}{\sigma_y(\sigma_x+\sigma_y)} \middle| \frac{\sigma_y}{\sigma_x}\right) \tag{5}$$

where in the last expressions the parametric dependence on σ_y/σ_x is explictly indicated. The Hamiltonian for the motion of the particle is

$$H = \tfrac{1}{2}(p_x^2+K_x x^2)+\tfrac{1}{2}(p_y^2+K_y y^2)+V(x,y)\delta(s). \tag{6}$$

The usual canonical transformation to action-angle variables, namely

$$\begin{cases} x = \sqrt{2\beta_x J_x} \cos\phi_x \\ p_x = -\sqrt{\frac{2J_x}{\beta_x}} \left(\sin\phi_x - \frac{\beta_x'}{2} \cos\phi_x\right) \end{cases} \qquad \text{(similar for y)}$$

and $\theta = \frac{s}{R}$ with $2\pi R$ = circumference, gives the transformed Hamiltonian

$$K = \nu_x J_x+\nu_y J_y+\frac{r_o N}{\gamma} F\left(\frac{\beta_x J_x \cos^2\phi_x}{\sigma_x(\sigma_x+\sigma_y)}, \frac{\beta_y J_y \cos^2\phi_y}{\sigma_y(\sigma_x+\sigma_y)} \middle| \frac{\sigma_y}{\sigma_x}\right) \delta(\theta). \tag{7}$$

Defining the scaled action variables

$$\mathcal{J}_x \equiv \frac{\beta_x J_x}{\sigma_x(\sigma_x+\sigma_y)} \quad , \quad \mathcal{J}_y \equiv \frac{\beta_y J_y}{\sigma_y(\sigma_x+\sigma_y)}$$

we can write the canonical equations for K as

$$\begin{cases} \dfrac{d\phi_x}{d\theta} = \dfrac{\partial K}{\partial J_x} = \nu_x - \dfrac{r_o N\beta_x}{\gamma\sigma_x(\sigma_x+\sigma_y)} \dfrac{\partial F}{\partial J_x}\,\delta(\theta) = \nu_x - 2\pi\xi_x \dfrac{\partial F}{\partial J_x}\,\delta(\theta) \\ \\ \dfrac{dJ_x}{d\theta} = -\dfrac{\beta_x}{\sigma_x(\sigma_x+\sigma_y)} \dfrac{\partial K}{\partial \phi_x} = \dfrac{r_o N\beta_x}{\gamma\sigma_x(\sigma_x+\sigma_y)} \dfrac{\partial F}{\partial \phi_x}\,\delta(\theta) = 2\pi\xi_x \dfrac{\partial F}{\partial \phi_x}\,\delta(\theta). \end{cases} \quad (8)$$

(similar for y).

Thus, we see that the motion is uniquely characterized by the five parameters

$$\begin{matrix} \nu_x , & \xi_x = \dfrac{1}{2\pi} \dfrac{r_o N\beta_x}{\gamma\sigma_x(\sigma_x+\sigma_y)} , & \\ & & \text{and } \dfrac{\sigma_y}{\sigma_x} . \\ \nu_y , & \xi_y = \dfrac{1}{2\pi} \dfrac{r_o N\beta_y}{\gamma\sigma_y(\sigma_x+\sigma_y)} , & \end{matrix} \quad (9)$$

Furthermore, we can make the following observations

(a) To the lowest order in x and y or J_x and J_y we have

$$F = 2J_x \cos^2\phi_x + 2J_y \cos^2\phi_y \quad (10)$$

and hence the first equation of Eqs. (8) becomes

$$\frac{d\phi_x}{d\theta} = \nu_x - 2\pi\xi_x(2\cos^2\phi_x)\,\delta(\theta). \quad (11)$$

Since the average value of $2\cos^2\phi_x$ is unity we see that to this order ξ_x is just the tune shift.

(b) The betatron wave numbers (tunes) ν_x and ν_y enter only to relate the phases of the kicks given by $V(x,y)\delta(s)$ in the Hamiltonian (6). If the kicks are random (We shall discuss later what random means here.) ν_x and ν_y become irrelevant in so far as the overall characteristics of the motion is concerned.

(c) If there are more than one collision points around the ring and the perturbing kicks at these collision points are random the tune advances between collisions are again irrelevant and the beam-beam effects can be measured by $<\xi_x>$ and $<\xi_y>$ averaged over all the collision points.

(d) The maximum tolerable beam-beam effects are generally reached when one of the two tune-shifts ξ_x and ξ_y reaches its limiting value. Hence if one is only interested in the beam-beam limits the parameter σ_y/σ_x is irrelevant and only one of the two values ξ_x and ξ_y is crucial.

Semi-Quantitative Features of the Beam-Beam Effect

We consider only the equation for one degree-of-freedom x,

$$\frac{d^2x}{ds^2} + K(s)x = -\frac{dV(x)}{dx}\delta(s) \qquad (12)$$

where the independent variable s is periodic with a period equal to the ring circumference. The following observations are important.

(A) Unperturbed $(\frac{dV}{dx} = 0)$ oscillation is linear and long-time stable. Hence accelerators are built to be "linear". Non-linearity can arise from imperfections in design and construction, and from beam-beam interactions. As was seen above, the latter is much larger and is unavoidable in principle. The beam-beam forces impart "kicks" on the colliding particle equal to

$$\Delta x_i' = -\frac{dV(x_i)}{dx_i} \qquad (13)$$

on the i^{th} revolution.

(B) If the kicks $\Delta x_i'$ are random the oscillation amplitude will grow. The increment of the Courant-Synder invariant[2] $W \equiv \gamma x^2 + 2\alpha x x' + \beta x'^2$ caused by all the $\Delta x_i'$ is

$$\Delta W = \sum_i \left[2(\alpha x_i + \beta x_i') \Delta x_i' + \beta (\Delta x_i')^2 \right] = n\beta (\Delta x')^2_{rms} . \qquad (14)$$

Where the terms linear in $\Delta x_i'$ sum to zero for random $\Delta x_i'$ and where n is the total number of kicks received. The corresponding increment in amplitude A is given by

$$\Delta (A^2) = \beta \Delta W = n\beta^2 (\Delta x')^2_{rms} . \qquad (15)$$

The values assumed for Eq. (3) gives a magnetic field on the "surface" of the beam of ~1 gauss. With a beam bunch length of, say, 10^{-1} m and a particle rigidity of 10^{-6} gauss-meter (~30 GeV proton) we get

$$(\Delta x')_{rms} \sim \frac{(1 \text{ gauss})(10^{-1} \text{m})}{10^6 \text{ gauss-m}} = 10^{-7} . \qquad (16)$$

Taking a typical value of $\beta = 10$ m $= 10^4$ mm we get

$$\Delta (A^2) = 10^{-6} n \text{ mm}^2 . \qquad (17)$$

Thus it takes only 5×10^6 kicks to increase A from 2 mm to 3 mm which is very rapid indeed. This is why a beam transport line with a length equivalent to more than 10^7 kicks of this magnitude (not very long compared to the distance travelled by a particle in a storage ring) can not possibly work.

(C) If the kicks are periodic all evils are concentrated into resonances. On resonance, $\Delta x_i'$ add coherently and A grows proportional to n. Off resonance, $\Delta x_i'$ cancell systematically to give zero amplitude growth.

(D) For perturbations arising from external field errors only low order non-linearities are sizeable. Therefore only low order resonances are excited in appreciable strength. As long as these resonances are avoided the amplitude growth should be negligible. The drop-off of high order non-linearity is a general characteristic of all fields generated by charges and currents outside the aperture and is a consequence of the vacuum Maxwell equations. This discussion shows also that the resonance expansion is useful only when the resonances excited are limited to low orders.

(E) When the perturbations arise from the field generated by a beam bunch through which the colliding particle travels, the non-linearity and the harmonics of the forces extend to extremely high orders. The tune-space is covered dense (density of rational numbers) by resonances and the unperturbed tune ν_o sits in a continuum of high order resonances even when all strong low order resonances are avoided. This means that the part of $\Delta x_i'$ which contributes to the continuum of resonances in the neighborhood (within the "line width") of ν_o appears to be random, the corresponding part of the motion is ergodic, and the oscillation amplitude grows.* This is similar to the statement that a signal which is random in the time domain has a continuous "white" spectrum in the frequency domain. The "natural line width" is rather small, but since ν_o is always wobbled by some random noises in the external field, with this ν_o-wobble included the "total line width" could be substantial.

*It may be objected that this is contrary to the KAM theorem which states that for 1 degree-of-freedom when the non-linear perturbation is sufficiently small well behaved KAM surfaces exist and prevent the growth of the oscillation amplitude. There is indication, however, that KAM theorem holds only for extremely small perturbations, much smaller than any physically realistic values. In any case we can always consider the motion in 1 degree-of-freedom as the projection of a motion in 2 degrees-of-freedom for which Arnol'd diffusion does occur and cause unrestricted growth in oscillation amplitude.

(F) Following the reasonings given above and using the bi-Gaussian potential, Eq. (5), we can derive a semi-quantitative formula for the amplitude growth. Putting $\sigma_x = \sigma_y \equiv \sigma$ (round beam) and y = 0 in Eq. (5) we get

$$V(x) = \frac{r_o N}{\gamma} \int_0^\infty dt \; \frac{1-\exp\left[\frac{-x^2}{2(t+\sigma^2)}\right]}{t+\sigma^2}$$

$$= \frac{r_o N}{\gamma} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n (n+1)!} \frac{1}{2(n+1)} \left(\frac{x^2}{\sigma^2}\right)^{n+1} \qquad (18)$$

and

$$\Delta x' = -\frac{dV}{dx} = \frac{r_o N}{\gamma\sigma^2} \left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n (n+1)!} \left(\frac{x^2}{\sigma^2}\right)^n\right] x. \qquad (19)$$

If only resonances of order m (a large integer) and above can fall inside the ν_o line-width, the random part of $\Delta x'$ contains only terms with n>m. Thus, in the expression for $(\Delta x')_{rms}$ the summation should only be from m to ∞. The amplitude growth is, then, given by Eq. (14) to be

$$\frac{dW}{dt} = f\beta(\Delta x')^2_{rms} = 8\pi^2 f\xi^2 \left[\sum_{n=m}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} \left(\frac{W}{\varepsilon/\pi}\right)^n\right]^2 W \qquad (20)$$

where we have used the relations

$$\xi \equiv \frac{1}{4\pi} \frac{r_o N\beta}{\gamma\sigma^2}$$

$$\left(\frac{x^2}{\beta}\right)_{rms} \sim \frac{W}{2}$$

$$\frac{4\pi\sigma^2}{\beta} = \varepsilon = \text{emittance of beam}$$

and

$f = \frac{dn}{dt}$ = rate of collision between particle and beam bunch.

Generally, the first term in the summation is the largest and we have approximately

$$\frac{dW}{dt} = kf\xi^2 \left(\frac{W}{\varepsilon/\pi}\right)^{2m} W, \qquad k \equiv 2\left[\frac{2\pi}{(m+1)!}\right]^2 \tag{21}$$

Two comments are useful.

(1) The line-width cannot be derived from this crude model. Thus, m must be considered an adjustable parameter. Furthermore, depending on how much reliance one puts on the measured beam emittance ε and on the validity of the approximations, it may be well to consider k also as an adjustable parameter.

(2) Larger line-width corresponds to lower m, hence larger k and larger dW/dt. Thus, the effect of external noise in increasing dW/dt is magnified by the non-linear beam-beam forces through a widening of the line-width.

<u>Comparison of Different Systems</u>

(A) According to the beam and collision geometry

(1) Continuous beams

(a) Crossing at an angle - Kicks are one dimensional (only in direction perpendicular to the crossing plane), hence the motion should be relatively stable.

(b) Colliding head-on - Kicks are two dimensional, hence the motion is expected to be more unstable.

(2) Long bunched beams - The force potential is identical to that of the corresponding case of continuous beams except at the ends of the beam bunches which constitute only a negligible part of the long bunches. The synchrotron motion of particles in the beam bunch will, however, enhance the instability. This can be understood

simply by noting that the number of resonances is increased by the synchro-betatron side-bands and the continuum of resonances is therefore much denser than without the synchro-betatron resonances.

(3) Short bunched beams - If the length of the beam bunches is comparable to their widths the kicks from the beam-beam forces are three dimensional whether the beams are crossing at an angle or colliding head-on. This plus the synchrotron oscillation will make this the most unstable geometry.

(B) According to the particle type

(1) Electrons (positrons)

At the present storage ring energies the synchrotron radiation from these particles is sizeable. The synchrotron radiation produces two major effects on the particle oscillations: (i) damping and (ii) quantum fluctuation which acts as random kicks to blow up the oscillation. In terms of the Courant-Snyder invariant W defined in Eq. (14) we can write

$$\frac{dW}{dt} = Q - \frac{W}{\tau} \tag{22}$$

where $Q(>0)$ is the blowup due to quantum fluctuation and τ is the damping time due to synchrotron radiation. With some modification and reinterpretation the beam-beam effect can be obtained from Eq. (19). The electron beams are not round but flat ribbons with $\sigma_x >> \sigma_y$, hence the vertical (y) effect is larger and gives the limitations. We first rewrite Eq. (19) as

$$\Delta y' = \frac{r_o N}{\gamma \sigma_y (\sigma_x + \sigma_y)} (\sigma_x + \sigma_y) \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n (n+1)!} \left(\frac{y^2}{\sigma_y^2} \right)^{n+\frac{1}{2}}$$

$$\cong \frac{2\pi \xi_y}{\beta_y} \sigma_x \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sqrt{2}}{(n+1)!} \left(\frac{y^2}{2\sigma_y^2} \right)^{n+\frac{1}{2}} . \tag{23}$$

Eq. (20) then becomes

$$\frac{dW}{dt} = f\beta(\Delta y')^2_{rms} = 8\pi^2 f\xi^2 \left(\frac{\sigma_x^2}{\beta}\right)\left[\sum_{n=m}^{\infty} \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{W}{\varepsilon/\pi}\right)^{n+\frac{1}{2}}\right]^2 \qquad (24)$$

where the subscript y is omitted. Again, taking only the largest term n = m in the summation we get

$$\frac{dW}{dt} = kf\xi^2\left(\frac{\sigma_x^2}{\beta}\right)\left(\frac{W}{\varepsilon/\pi}\right)^{2m+1}, \qquad k = 2\left[\frac{2\pi}{(m+1)!}\right]^2 . \qquad (25)$$

In addition to the beam-beam effect we can also add an external noise term P. Altogether Eq. (22) is modified to

$$\frac{dW}{dt} = P+Q-\frac{W}{\tau}+kf\xi^2\left(\frac{\sigma_x^2}{\beta}\right)\left(\frac{W}{\varepsilon/\pi}\right)^{2m+1} . \qquad (26)$$

The maximum tune shift ξ_{max} that can be obtained is given by the condition $\frac{dW}{dt} = 0$ at a value of W of the order of and proportional to ε/π, since the two beams are approximately equal in height. This gives

$$kf\xi^2_{max}\left(\frac{\sigma_x^2}{\beta}\right)\left(\frac{W}{\varepsilon/\pi}\right)^{2m+1} = \frac{W}{\tau} - Q - P . \qquad (27)$$

This leads immediately to the energy (E) dependence of ξ_{max} because we have

$W \propto \varepsilon/\pi \propto E^2$, hence $\frac{W}{\varepsilon/\pi} \propto E^0$;

$\frac{1}{\tau} \propto E^3$, hence $\frac{W}{\tau} \propto E^5$,

$Q \propto E^5$, coupled over from horizontal;

$\sigma_x \propto E^0$, because σ_x is likely aperture limited, and

$P \propto E^0$.

The energy dependence of ξ_{max} can, thus, be

$$\xi_{max} = (aE^5 - b)^{\frac{1}{2}} . \tag{28}$$

In actuality the measured data from SPEAR[3] can be fitted quite well with b=0, i.e. no external noise. Fig. 1 shows the fit with

$$\xi_{max} = 0.01\ E^{\frac{5}{2}} \qquad \text{(E in GeV)}. \tag{29}$$

The energy dependence of the maximum luminosity L_{max} is related to that of ξ^2_{max} by[4]

$$L_{max} \propto E^2\ \xi^2_{max} \propto E^7. \tag{30}$$

Figure 2 shows the fit to SPEAR data with

$$L_{max} = 0.03\ E^7 \qquad \text{(E in GeV)}. \tag{31}$$

(2) Protons (antiprotons)

For present storage rings at energies less than tens of TeV the synchrotron radiation for these particles is negligible and the amplitude (or W) growth equation is given by Eq. (21) for round beams to be

$$\frac{dW}{dt} = P + kf\xi^2 \left(\frac{W}{\varepsilon/\pi}\right)^{2m} W. \tag{32}$$

Several conclusions can be drawn from this equation.

(a) With all terms positive on the right-hand-side there cannot be any threshold behavior as in the case of electrons. The beam growth rate will simply increase with increasing ξ.

(b) If the beam growth rate is measured by the beam loss on a collimator aperture, the collimator has to be fitted rather tightly around the beams. As was stated at the beginning, the non-linear beam-beam forces are localized to the "surface" of the beam and fall off rapidly going away from the beam.

(c) Unlike electron beams, proton (antiproton) beams generally do not have Gaussian transverse density distributions. The distribution tends to be more squarish and more truncated. Nevertheless, the qualitative or perhaps even the semi-quantitative features of the development given above should still be valid.

(d) Eq. (32) indicates a beam growth rate proportional to ξ^2. The same quadratic dependence in Eq. (27) led to the fit shown in Eq. (29). Experiments by Keil[5] and Zotter[6] on the CERN-ISR seem, however, to indicate an exponential dependence. This discrepancy must be resolved.

References

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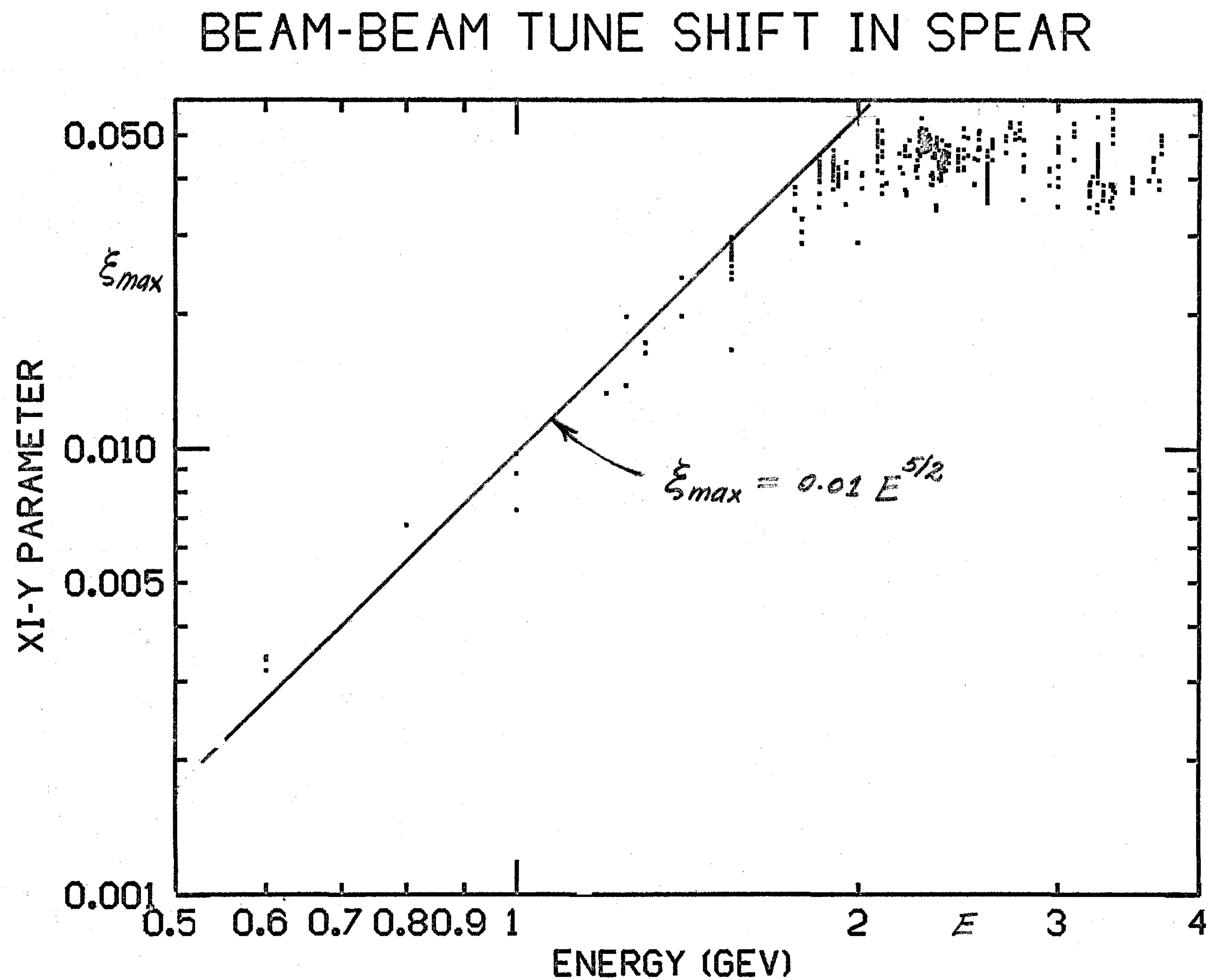

Figure 1. Maximum vertical tune-shift versus energy in SPEAR.

TM-973
2040

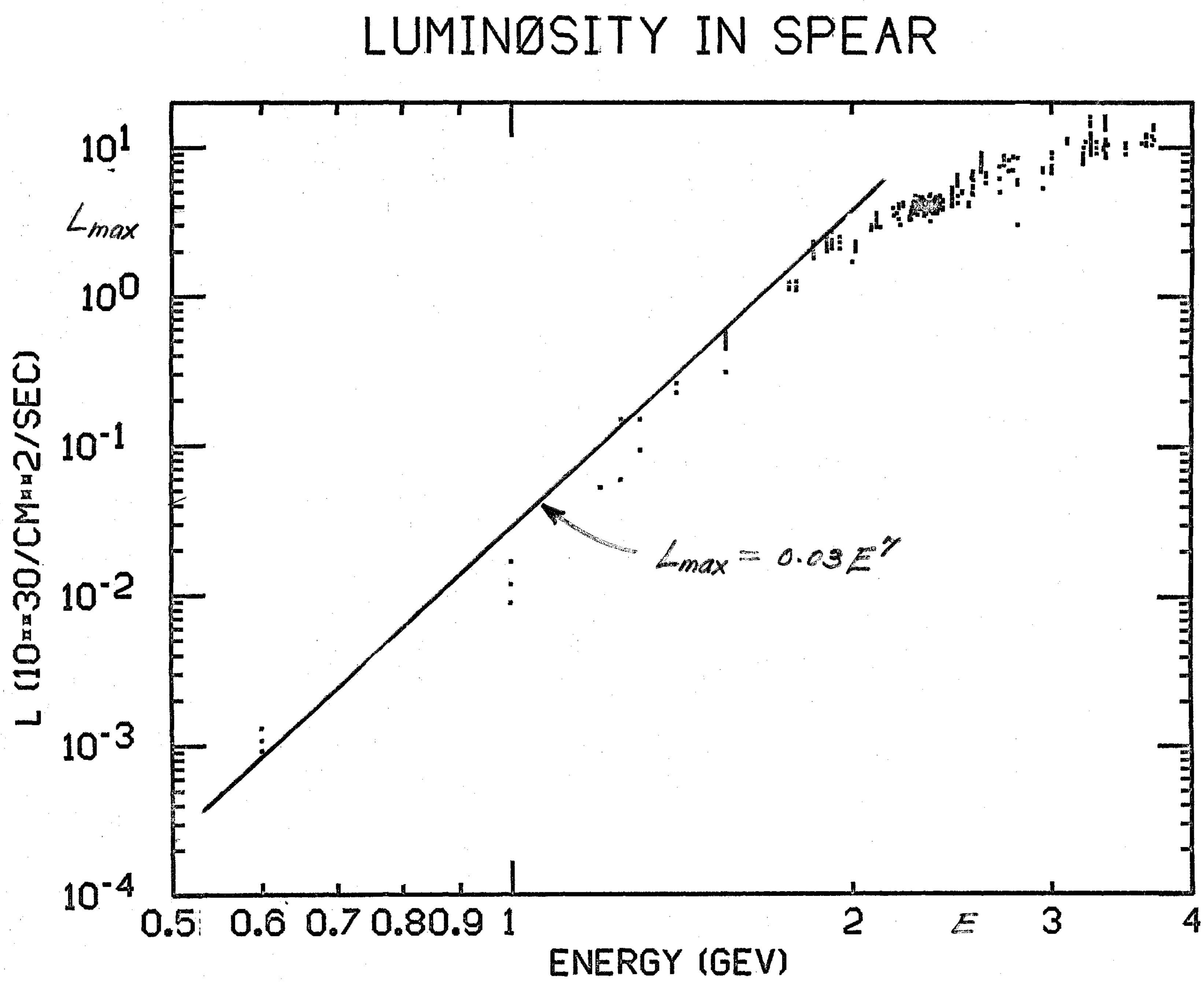

Figure 2. Maximum luminosity versus energy in SPEAR.

TM-973
2040