

EDDY CURRENTS IN SUPERCONDUCTING DIPOLE BANDING

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PURPOSE

The banding used to contain the coils in a superconducting dipole might be simplified if stainless steel were used rather than the fabricated plastic bands now envisaged. The use of metal bands depends on the magnitude of the fields and the associated power loss arising from eddy currents. An estimate of these effects is made using excitation currents and eddy currents confined to cylindrical sheets. Also, for simplicity no outer magnetic shield is included. For reasonable bands and a frequency of 1 Hz the field distortion is less than $1 \div 10^6$ and the power dissipated is less than 1 W/magnet.

VECTOR POTENTIAL OF EXCITING CURRENTS

All geometrical quantities are depicted in Fig. 1. The exciting current is assumed to be

$$i_z = i_0 \cos \phi \text{ (abA/cm)} \quad (1)$$

and confined to the cylindrical sheet at $\rho = a$. Then for the gauge in which $\nabla \cdot \vec{A} = 0$,

$$A_\rho = 0 \quad A_\phi = 0 \quad A_z = \begin{cases} A_1 \rho \\ \frac{B_1}{\rho} \end{cases} \cos \phi \quad \begin{matrix} 0 < \rho < a \\ a < \rho < \infty \end{matrix} \quad (2)$$

The corresponding fields are

$$H_\rho = \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} = \begin{Bmatrix} -A_1 \\ -\frac{B_1}{\rho^2} \end{Bmatrix} \sin \phi, \quad H_\phi = -\frac{\partial A_z}{\partial \rho} = \begin{Bmatrix} -A_1 \\ \frac{B_1}{\rho^2} \end{Bmatrix} \cos \phi. \quad (3)$$

At $\rho = a$: $H_\phi^+ - H_\phi^- = 4\pi i_z$, $H_\rho^+ = H_\rho^-$ or

$$A_1 = 2\pi i_0 \quad B_1 = 2\pi i_0 a^2. \quad (4)$$

FIELDS AND POTENTIALS FROM EDDY CURRENT SHEET AT $\rho = b$

In order to satisfy $\nabla \cdot \vec{i} = 0$ for the surface eddy currents at $\rho = b$ choose the following harmonic expansion.

$$i_z = \cos \phi \sum_n i_n \sin \frac{n\pi z}{d}, \quad i_\phi = -\sin \phi \sum_n \frac{n\pi b}{d} i_n \cos \frac{n\pi z}{d}. \quad (5)$$

To be consistent with the gauge previously chosen, one again seeks a vector potential that satisfies $\nabla \cdot \vec{A} = 0$. However, the most general vector satisfying this condition may be written as

$$\vec{A} = \vec{L} W_1 + \nabla \times \vec{L} W_2 \quad (6)$$

where, for the problem at hand the \vec{L} - operator is most conveniently expressed as

$$\vec{L} = \vec{k} \times \nabla \quad (\vec{k} = z - \text{unit vector}). \quad (7)$$

The properties of the \vec{L} - operator¹ suggest that

$$\nabla^2 W_1 = 0, \quad \nabla^2 W_2 = 0 \quad (8)$$

will insure that \vec{A} satisfies $\nabla \times \nabla \times \vec{A} = 0$. In this case, however, Eq. (6) may be replaced by

$$\vec{A} = \vec{L} W + \nabla \phi \quad (9)$$

where

$$W = W_1 \quad \nabla^2 \phi = 0. \quad (10)$$

Thus, to permit a connection with the dipole excitation potentials, choose

$$W = \begin{cases} \sum_n \frac{d}{n\pi} C_n I_1\left(\frac{n\pi\rho}{d}\right) \cos \frac{n\pi z}{d} \\ \sum_n \frac{d}{n\pi} D_n K_1\left(\frac{n\pi\rho}{d}\right) \cos \frac{n\pi z}{d} \end{cases} \sin\phi \quad \begin{matrix} 0 < \rho < b \\ b < \rho < \infty \end{matrix} \quad (11)$$

$$\phi = \begin{cases} \sum_n \frac{d}{n\pi} E_n I_1\left(\frac{n\pi\rho}{d}\right) \cos \frac{n\pi z}{d} \\ \sum_n \frac{d}{n\pi} F_n K_1\left(\frac{n\pi\rho}{d}\right) \cos \frac{n\pi z}{d} \end{cases} \cos\phi \quad \begin{matrix} 0 < \rho < b \\ b < \rho < \infty \end{matrix} \quad (12)$$

The vector potential due to eddy currents becomes

$$A_\rho = -\frac{1}{\rho} \frac{\partial W}{\partial \phi} + \frac{\partial \phi}{\partial \rho} = \begin{cases} \sum_n \left[-\frac{d}{n\pi\rho} C_n I_1\left(\frac{n\pi\rho}{d}\right) + E_n I_1'\left(\frac{n\pi\rho}{d}\right) \right] \cos \frac{n\pi z}{d} \\ \sum_n \left[-\frac{d}{n\pi\rho} D_n K_1\left(\frac{n\pi\rho}{d}\right) + F_n K_1'\left(\frac{n\pi\rho}{d}\right) \right] \cos \frac{n\pi z}{d} \end{cases} \cos\phi \quad (13)$$

$$A_\phi = \frac{\partial W}{\partial \rho} + \frac{1}{\rho} \frac{\partial \phi}{\partial \phi} = \begin{cases} \sum_n \left[C_n I_1'\left(\frac{n\pi\rho}{d}\right) - \frac{d}{n\pi\rho} E_n I_1\left(\frac{n\pi\rho}{d}\right) \right] \cos \frac{n\pi z}{d} \\ \sum_n \left[D_n K_1'\left(\frac{n\pi\rho}{d}\right) - \frac{d}{n\pi\rho} F_n K_1\left(\frac{n\pi\rho}{d}\right) \right] \cos \frac{n\pi z}{d} \end{cases} \sin\phi \quad (14)$$

$$A_z = \frac{\partial \phi}{\partial z} = \begin{cases} -\sum_n E_n I_1\left(\frac{n\pi\rho}{d}\right) \sin \frac{n\pi z}{d} \\ -\sum_n F_n K_1\left(\frac{n\pi\rho}{d}\right) \sin \frac{n\pi z}{d} \end{cases} \cos\phi \quad (15)$$

The corresponding fields become

$$H_\rho = -\frac{\partial^2 W}{\partial \rho \partial z} = \begin{cases} \sum_n \frac{n\pi}{d} C_n I_1'\left(\frac{n\pi\rho}{d}\right) \sin \frac{n\pi z}{d} \\ \sum_n \frac{n\pi}{d} D_n K_1'\left(\frac{n\pi\rho}{d}\right) \sin \frac{n\pi z}{d} \end{cases} \sin\phi \quad (16)$$

$$H_{\phi} = - \frac{1}{\rho} \frac{\partial^2 W}{\partial \phi \partial z} = \left\{ \begin{array}{l} \sum_n \frac{1}{\rho} C_n I_1\left(\frac{n\pi\rho}{d}\right) \sin \frac{n\pi z}{d} \\ \sum_n \frac{1}{\rho} D_n K_1\left(\frac{n\pi\rho}{d}\right) \sin \frac{n\pi z}{d} \end{array} \right\} \cos \phi \quad (17)$$

$$H_z = - \frac{\partial^2 W}{\partial z^2} = \left\{ \begin{array}{l} \sum_n \frac{n\pi}{d} C_n I_1\left(\frac{n\pi\rho}{d}\right) \cos \frac{n\pi z}{d} \\ \sum_n \frac{n\pi}{d} D_n K_1\left(\frac{n\pi\rho}{d}\right) \cos \frac{n\pi z}{d} \end{array} \right\} \sin \phi \quad (18)$$

$$E_{\rho} = -j\omega A_{\rho}, \quad E_{\phi} = -j\omega A_{\phi}, \quad E_z = -j\omega(A_z + A_z(\text{excit.})) \quad (19)$$

BOUNDARY CONDITIONS

At $\rho = b$ the following conditions produce the corresponding conditions among the coefficients

$$H_{\rho}^{+} = H_{\rho}^{-} : C_n I_1'\left(\frac{n\pi b}{d}\right) = D_n K_1'\left(\frac{n\pi b}{d}\right) \quad (20)$$

$$H_{\phi}^{+} - H_{\phi}^{-} = 4\pi i_z : D_n K_1\left(\frac{n\pi b}{d}\right) - C_n I_1\left(\frac{n\pi b}{d}\right) = 4\pi b i_n \quad (21)$$

$$H_z^{-} - H_z^{+} = 4\pi i_{\phi} : D_n K_1\left(\frac{n\pi b}{d}\right) - C_n I_1\left(\frac{n\pi b}{d}\right) = 4\pi b i_n \quad (22)$$

$$E_{\rho}^{+} - E_{\rho}^{-} = 4\pi\sigma : \text{Will not need the surface charge.} \quad (23)$$

$$E_{\phi}^{+} = E_{\phi}^{-} : E_n I_1\left(\frac{n\pi b}{d}\right) = F_n K_1\left(\frac{n\pi b}{d}\right) \quad (24)$$

The eddy currents \vec{i} in the surface, $\rho = b$, are related to the electric field \vec{E} by Ohm's law: $s \vec{E} = \vec{i}$, where s is the surface conductivity. Hence

$$s E_{\phi} = i_{\phi} : C_n I_1'\left(\frac{n\pi b}{d}\right) - \frac{d}{n\pi b} E_n I_1\left(\frac{n\pi b}{d}\right) = \frac{n\pi b}{j\omega s b} b i_n \quad (25)$$

Finally, the condition, $s E_z = i_z$, gives for $b > a$

$$-j\omega s 2\pi \frac{a^2}{b} i_0 + j\omega s \sum_n E_n I_1\left(\frac{n\pi b}{d}\right) \sin \frac{n\pi z}{d} = \sum_n i_n \sin \frac{n\pi z}{d} \quad (26)$$

BANDING CONDITION

The surface currents are to exist in a width c with a repeat distance of d . Hence the first term in Eq. (26) may be multiplied by

$$\left\{ \begin{array}{ll} 0 & 0 < z < \frac{1}{2}(d-c) \\ 1 & \frac{1}{2}(d-c) < z < \frac{1}{2}(d+c) \\ 0 & \frac{1}{2}(d+c) < z < \infty \end{array} \right\} = \frac{4}{\pi} \sum_{n=1,3,5} \frac{1}{n} \sin \frac{n\pi}{2} \sin \frac{n\pi c}{2d} \sin \frac{n\pi z}{d} . \quad (27)$$

Thus, combining Eq. (27) with Eq. (26) gives

$$- 8j\omega sb \frac{a^2}{b^2} i_0 \frac{\sin \frac{n\pi}{2}}{n} \sin \frac{n\pi c}{2d} + j\omega s E_n I_1 \left(\frac{n\pi b}{d} \right) = i_n . \quad (28)$$

COEFFICIENTS

From Eqs. (20-28) one has

$$C_n = 4\pi \left(\frac{n\pi b}{d} \right) K_1' \left(\frac{n\pi b}{d} \right) b i_n \quad (29)$$

$$D_n = 4\pi \left(\frac{n\pi b}{d} \right) I_1' \left(\frac{n\pi b}{d} \right) b i_n \quad (30)$$

$$E_n = \frac{\left(\frac{n\pi b}{d} \right)^2}{I_1 \left(\frac{n\pi b}{d} \right)} \left[4\pi I_1' \left(\frac{n\pi b}{d} \right) K_1' \left(\frac{n\pi b}{d} \right) - \frac{1}{j\omega sb} \right] b i_n \quad (31)$$

$$F_n = \frac{\left(\frac{n\pi b}{d} \right)^2}{K_1 \left(\frac{n\pi b}{d} \right)} \left[4\pi I_1' \left(\frac{n\pi b}{d} \right) K_1' \left(\frac{n\pi b}{d} \right) - \frac{1}{j\omega sb} \right] b i_n \quad (32)$$

$$i_n = \frac{8j\omega sb \frac{a^2}{b^2} \sin \frac{n\pi}{2} \sin \frac{n\pi c}{2d}}{n \left\{ \left(\frac{n\pi b}{d} \right)^2 \cdot \left[4\pi j\omega sb I_1' \left(\frac{n\pi b}{d} \right) K_1' \left(\frac{n\pi b}{d} \right) - 1 \right] - 1 \right\}} i_0 \quad (33)$$

MEDIAN PLANE FIELD

$$H_y(x, z) = -2\pi i_0 + \frac{1}{x} \sum_n C_n I_1\left(\frac{n\pi x}{d}\right) \sin \frac{n\pi z}{d} \quad (34)$$

Averaging this over z gives

$$\langle H_y(x) \rangle_{Av} = -2\pi i_0 + \frac{2}{\pi x} \sum_n \frac{1}{n} C_n I_1\left(\frac{n\pi x}{d}\right) \quad (35)$$

which, upon expanding in x gives

$$\langle H_y(x) \rangle_{Av} = -2\pi i_0 + \frac{1}{d} \sum_n C_n \left[1 + \frac{1}{2} \left(\frac{n\pi x}{2d} \right)^2 + \dots \right] \quad (36)$$

At $x = 0$ one has

$$\frac{\Delta H}{H} = \frac{\langle H_y(0) \rangle_{Av} - (-2\pi i_0)}{-2\pi i_0} = - \frac{\sum_n C_n}{2\pi i_0 d} \quad (37)$$

In addition

$$\frac{H''}{H} = \frac{\langle H_y''(0) \rangle_{Av}}{-2\pi i_0} = - \frac{\pi \sum_n n^2 C_n}{8 i_0 d^3} \quad (38)$$

POWER DISSIPATED IN BANDS

$$P = \text{Real} \left\{ \frac{1}{2} \iint \vec{E} \cdot \vec{i}^* b d\phi dz \right\} \quad (39)$$

$$P = \frac{b}{2s} \sum_n \sum_m i_n i_m^* \iint \left\{ \cos^2 \phi \sin \frac{n\pi z}{d} \sin \frac{m\pi z}{d} + \frac{n\pi b}{d} \cdot \frac{m\pi b}{d} \sin^2 \phi \cos \frac{n\pi z}{d} \cos \frac{m\pi z}{d} \right\} d\phi dz$$

$$P = \frac{bd}{8s} \sum_n [1 + (\frac{n\pi b}{d})^2] i_n i_n^* \text{ (ergs/sec - one band)} \quad (40)$$

NUMERICAL RESULTS

In order to apply to the doubler dipole put

$$a = 1.1325'' = 2.8766 \text{ cm} \quad b = 2.0'' - .375'' = 4.1275 \text{ cm}$$

$$c = .375'' = .9525 \text{ cm} \quad d = 1.0'' = 2.54 \text{ cm}$$

$$f = \omega/2\pi = 1 \text{ Hz}$$

$$s = \frac{.100 \times 2.54}{50 \times 10^{-6}} \times 10^{-9} = 5.08 \times 10^{-6} \text{ (emu)}$$

$$H_0 = 45 \text{ kG .}$$

This gives

$$\omega sb = .0001317 \quad a/b = .6969$$

$$\pi b/d = 5.1051$$

$$\sin \frac{\pi c}{2d} = .5556$$

$$I'_1(5.1051) = 24.554$$

$$K'_1(5.1051) =$$

$$- .004018.$$

Thus

$$i_1 = \frac{8j \times .0001317 \times .5556 \times .6969 \times .6969}{-4\pi j \times .001317 \times 24.554 \times .004018 \times (5.1051)^2 - 1 - 5.1051 \times 5.1051} i_0$$

$$= \frac{.0002843j}{-.00426j - 27.0620} = -10.51 \times 10^{-6} j i_0$$

i_n = negligible for $n = 3, 5, 7$ etc. because of

$$K'_1(5.1051 n)$$

$$C_1 = 4\pi \times 5.1051 \times .004018 \times 4.1275 \times 10.51 \times 10^{-6} j i_0 = 11.18 j i_0$$

C_n = negligible for $n = 3, 5, 7$, etc.

$$i_0 = H_0/2\pi = 45000/2\pi = 7162 \text{ abA/cm}$$

$$\Delta H/H = -C_1/(2\pi i_0 d) = -.70 \times 10^{-6} \text{ j}$$

$$H''/H = -\pi C_1/(8i_0 d^3) = -1.73 \times 10^{-6} \text{ j/in/in.}$$

$$P/d = b[1 + (\pi b/d)^2] i_1 i_1^* / 8s = .0040 \text{ W/in.}$$

$$P(240'') = .95 \text{ W/magnet}$$

REFERENCE

1. S.C. Snowdon, Properties of the L-Operator, MURA Technical Note TN-506, Oct. 19, 1964.