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EQUATIONS OF MOTION OF AN ACCELERATED PARTICLE

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Summary

The purpose of this paper is to derive the equations of motion of a particle accelerated in a circular machine when it is performing betatron and phase oscillations at the same time. The derivation is done formally by introducing proper pairs of canonical variables and an Hamiltonian function. The main ingredient of the analysis is a reference closed curve which is supposed to be the orbit of a "fictitious" particle which moves along it with the necessary momentum function p_o. The motion of the actual particle is described in proximity of that reference curve and the reference momentum function p_o.

The Canonical Equations of Motion

1. Let us make use of the system of coordinates defined in Appendix B of Courant-Snyder paper.[1] The symbols we are going to introduce have the same meaning as in the referenced paper, except when mentioned.

The three pairs of canonical variables are

$$(x,p_x) \quad (z,p_z) \quad \text{and} \quad (s,p_s).$$

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The Hamiltonian equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} \qquad \dot{p}_z = -\frac{\partial H}{\partial z}$$

$$\dot{s} = \frac{\partial H}{\partial p_s} \qquad \dot{p}_s = -\frac{\partial H}{\partial s}$$

where

$$H = eV + c\left\{m^2c^2 + \frac{(p_s-eA_s)^2}{(1+\Omega x)^2} + (p_x-eA_x)^2 + (p_z-eA_z)^2\right\}^{1/2}. \quad (1)$$

Here, we are assuming the reference curve is contained by a plane so that torsion is everywhere zero.

2. Using the last canonical equations of motion (s and p_s), we can change to s as an independent variable instead of time t. The equations transform to

$$x' = -\frac{\partial p_s}{\partial p_x} \qquad p_x' = \frac{\partial p_s}{\partial x}$$

$$z' = -\frac{\partial p_s}{\partial p_z} \qquad p_z' = \frac{\partial p_s}{\partial z}$$

$$t' = \frac{\partial p_s}{\partial H} \qquad H' = -\frac{\partial p_s}{\partial t}$$

and

$$p_s = eA_s + (1+\Omega x)\left\{\left(\frac{H-eV}{c}\right)^2 - m^2c^2 - (p_x-eA_x)^2 - (p_z-eA_z)^2\right\}^{1/2}.$$

The canonical pair (s,p_s) has been replaced by the new canonical pair $(t,-H)$.

3. Let p_o be the momentum of the particle which has the reference curve as trajectory, and t_o be the time of this particle. In particular, t_o can be also identified with any local clock time. From the Hamiltonian (1) we can derive the quantity

$$H_o = c\sqrt{p_o^2 + m^2c^2}$$

to associate to p_o with the assumption that A_x, A_z and A_s are zero on the reference orbit, and V is also identically zero for $t = t_o$.

Another definition of p_o is also the following. At any location of the magnetic ring, the following relationship must be satisfied

$$eB_o = cp_o\Omega$$

where B_o is the guide magnetic field and a function of the time t_o. If we assume Ω is a constant, p_o must change with B_o.

From p_o, we derive the velocity

$$v_o(t_o) = \frac{p_o c}{\sqrt{p_o^2+m^2c^2}} \equiv \frac{ds}{dt_o} .$$

This equation can be integrated to give s as a function of t_o

$$s = \int_0^{t_o} \frac{p_o c}{\sqrt{p_o^2+m^2c^2}}\, dt_o = s(t_o)$$

which we assume we are able to reverse to obtain t_o as a function of s

$$t_o = t_o(s)$$

which inserted in p_o gives p_o as a function of s

$$p_o = p_o(s).$$

4. Introduce the following new variables

$$\tau = t - t_0 \qquad \text{and} \qquad w = H_0 - H.$$

The transformation to the new variables is done by taking the generating function

$$S = (w - H_0)(t - t_0)$$

which relates the new variables to the old variable

$$-H = \frac{\partial S}{\partial t} \,. \qquad \tau = \frac{\partial S}{\partial w} \,.$$

The new Hamiltonian is

$$G = p_s - \frac{\partial S}{\partial s} = eA_s + (1+\Omega x)\left\{p_0^2 + \frac{(w+eV)^2}{c^2} - 2\frac{H_0(w+eV)}{c^2} + \right.$$

$$\left. - (p_x - eA_x)^2 - (p_z - eA_z)^2\right\}^{1/2} + wt_0' + \tau H_0'.$$

(In the Hamiltonian, we ignore constants and functions of only s.)

The equations of motion are now

$$x' = -\frac{\partial G}{\partial p_x} \qquad p_x' = \frac{\partial G}{\partial x}$$

$$z' = -\frac{\partial G}{\partial p_z} \qquad p_z' = \frac{\partial G}{\partial z}$$

$$\tau' = -\frac{\partial G}{\partial W} \qquad w' = \frac{\partial G}{\partial \tau}$$

Observe that, in the Hamiltonian G, the quantities A_s, A_x, A_z and V are not only functions of x and z (and not of p_x, p_z and w) but also of s and t, the latter being now replaced by $t_0(s) + \tau$. Besides, Ω is only a periodic function of s with periodicity C, which is just the length of the reference closed curve.

5. The circular accelerator is composed of magnets and cavities, which produce a distributed magnetic field and electric field. The scalar potential function V and the three components A_x, A_z, A_s of the vector potential describe entirely, apart from constants or at most functions of only s, the magnetic and electric field. The electric field in the cavities changes (rather quickly) with time, thus a magnetic field should be also related. Conversely, if we have acceleration, the magnetic field in the magnets changes (rather slowly) with time, thus an electric field should also be related. We believe it is a good approximation to neglect these extra fields. Namely, we shall assume the scalar potential function V contributes only to electric field in the cavities, and that the potential vector of components A_x, A_z, A_s contributes only to the magnetic field in the magnets.

6. We want to retain in the Hamiltonian G only the terms up to the second order in any of the canonical variables, except τ. At this purpose it is sufficient to expand A_s, A_x, A_z in the following way.

$$A_s = p_o(ax + bz + gx^2 + dzx + qz^2)$$

$$A_x = -p_o fz$$

$$A_z = p_o fx.$$

The coefficients a, b, g, d, q, f are only periodic functions of s with periodicity C.

We obtain

$$G = ep_o(ax + bz + gx^2 + dxz + qz^2) + \Omega p_o x +$$

$$-\frac{(w+eV)}{v_o}\Omega x - \frac{w+eV}{v_o} - \frac{(w+eV)^2}{2p_o v_o^2 \gamma_o^2} +$$

$$-\frac{(p_x+ep_o fz)^2}{2p_o} - \frac{(p_z-ep_o fx)^2}{2p_o} + wt_o' + \tau H_o'$$

where

$$\gamma_o^2 = \frac{c^2}{c^2-v_o^2}.$$

7. Since the reference curve is supposed to be the equilibrium orbit of a particle with momentum p_o, then it must be

$$b = 0, \qquad ea = -\Omega$$

and (from the Maxwell equations)

$$2(g+q) = \Omega a.$$

Also, if we assume there is no coupling between the two transverse modes it is

$$f = d = 0.$$

In this case, the motion on the horizontal plane is completely independent of the motion on the vertical plane. The latter has the following equations

$$z' = \frac{p_z}{p_o} \qquad \text{and} \qquad p_z' = 2eqp_o z$$

which can be condensed in the following second order differential equations

$$z'' + \frac{p_o'}{p_o} z - 2eqz = 0.$$

8. The equations of motion on the horizontal plane are obtained ignoring z and p_z in the Hamiltonian, which now is

$$G = ep_o gx^2 - \frac{p_x^2}{2p_o} - \frac{e}{v_o} V + \tau H_o' + - \frac{w+eV}{v_o} \Omega x - \frac{(w+eV)^2}{2p_o v_o^2 \gamma_o^2} .$$

V is generated by one or more cavities which we regard as lumped elements. Then V is a stepwise function, with the jump in correspondence of the cavity position. The height of the jump depends on τ and on the time t_o at which the reference particle crosses the cavity in consideration.

We shall assume the cavities are located in places where $\Omega = 0$. Also, between two cavities V is constant and, since the motion of a particle in an equipotential environment does not depend on this constant, we shall take it zero.

Thus the motion is as well described by the following Hamiltonian

$$G = ep_o gx^2 - \frac{p_x^2}{2p_o} - \frac{e}{v_o} V + \tau H_o' + - \frac{\Omega}{v_o} wx - \frac{w^2}{2p_o v_o^2 \gamma_o^2}.$$

The equation of the radial betatron oscillations then is

$$x'' + \frac{p_o'}{p_o} x' - 2egx = -\frac{w\Omega}{v_o p_o} \qquad (2)$$

and the equations for the phase motion are

$$\tau' = \frac{\Omega}{v_o} x + \frac{w}{p_o v_o^2 \gamma_o^2} \qquad (3)$$

$$w' = H_o' - \frac{e}{v_o} \frac{\partial V}{\partial \tau} . \qquad (4)$$

9. Observe that

$$\frac{\partial V}{\partial \tau} = \frac{\partial .}{\partial t_o}$$

so that the energy change, after crossing one cavity, is

$$\Delta w = -eV_p(t_o+\tau) \sin \int_{t_o}^{t_o+\tau} \omega(t)dt$$

where V_p is the peak voltage across the cavity, and $\omega/2\pi$ is the rf frequency.

Reference

1. E.D. Courant and H.S. Snyder, Annals of Physics: 3, 1-48 (1958).