Inflation Basics

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1. Preliminaries

The last few years have yielded remarkable discoveries in physics. In particle physics it appears that a fundamental scalar field exists [1], [2]. The Higgs boson is measured to have a mass of about 126 GeV and to have spin zero and positive parity. The Higgs field is the first fundamental scalar to be discovered in physics.

The Cosmic Microwave Background, CMB, is known to have a uniform temperature to parts per 10^5 but has [3], [4] well measured fluctuations which are thought to evolve gravitationally to provide the seeds of the current structure of the Universe. In addition, the Universe appears to contain, at present, and unknown "dark energy" [5] which is presently the majority energy density of the Universe, larger than either matter or radiation. This may, indeed, be a fundamental scalar field like the Higgs.

"Big Bang" (BB) cosmology is a very successful "standard model" in cosmology [6]. However, it cannot explain the uniformity of the CMB because the CMB consists of many regions not causally connected in the context of the BB model. In addition, the Universe appears to be spatially flat [7]. However in BB cosmology the present spatial curvature is not stable so that the initial conditions for BB cosmology would need to be fantastically fine-tuned in order to successfully predict the presently small value of the observed curvature.

These issues for BB cosmology have led to the hypothesis of "inflation" which postulates an unknown scalar field, not presumably the Higgs field or the dark energy, which causes an exponential expansion of the Universe at very early times [8], [9]. This attractive hypothesis can account for the problems in BB cosmology of flatness and causal CMB connectivity. In addition, the quantum fluctuations of this postulated field provide a natural explanation of the CMB fluctuations which are the seeds of the structure of galaxies.

Researchers are now searching for gravitational waves imprinted on the CMB [10], [11]. These would be a "smoking gun" for inflation since metrical fluctuations, both scalar and tensor, are also produced in inflationary models. Thus, the time appears to be appropriate for a very basic and simple exposition of the inflationary model written from a particle physics perspective. Only the simplest scalar model will be explored because it is easy to understand and contains all the basic elements of the inflationary model [12].

2. Units and Constants

The units used in this note are GeV, m, and sec. There is only one coupling constant which has dimensions which is the gravitational constant of Newton, G_N. That constant is subsumed by using the Planck mass to set mass scales. Since inflation deals with very early times, the Planck mass, M_p, is a natural scale. Related scales are the Planck length, L_{pl} and the Planck time t_{pl}.

$$\begin{aligned} M_p &= \sqrt{\hbar c / G_N} \sim 1.2x10^{19} GeV \\ L_{pl} &= \sqrt{\hbar G_N / c^3} \sim 1.6x10^{-35} m \\ t_{pl} &= L_p / c \sim 5x10^{-44} \sec \end{aligned} \tag{1}$$

Natural units, $\hbar = c = 1, G_N = 1/M_p^2$, are used so that, in principle, all quantities could be expressed in energy units.

Several numerical quantities used in this note appear in Table 1. Present quantities are indicated by an o subscript. The present CMB has cooled off during the expansion and now has a temperature of 2.7 degrees absolute and a number density of about 410 photons/cm^3. The age of the Universe is about 13.7 billion years. The present critical density, the density such that the current spatial curvature is zero or “flat” is presently 5.3 GeV/m^3. The present Universe is 73% dark energy [5], 24% dark matter [13] and with a small fraction of the total by weight due to radiation. The present Universe is “flat” in that the sum of the ratios of energy density to critical density, $\Omega=\rho/\rho_c$, fractional sum is one to good accuracy which means the Universe has a total energy density equal to the critical density.

Table 1: Present Values of Selected Cosmological Quantities [14]

Quantity	Present Value
To – CMB Temperature (^{o}K)	2.725
Ho = 1/to – Hubble Time (1/sec)	1/ (4.1 x 10^{17}), (1/13.7 Gyr)
c/Ho – Hubble distance (m)	1.3 x 10^{26}
ρ_c – critical density (GeV/m^3)	5.3
Ωm – matter fraction (dark matter dominant)	0.24
Ωr – radiation fraction	4.6 x 10^{-5}
Ωv – dark energy or vacuum fraction	0.73

3. Big Bang Cosmology

Assume that the Universe is homogeneous and isotropic [6] which is as simple an assumption as is possible. The space-time metric is then of the Robertson-Walker form. Assume that the Universe is spatially flat from the beginning, in agreement with observation so that the curvature parameter in the metric, k is zero. Spatial coordinates are commoving, participating in the expansion of the Universe and at fixed locations. The physical scale is set by the parameter a(t) which modifies the spatial part of the interval of special relativity. The metric with comoving spatial coordinates r,θ,ϕ is;

$$ds^2 = (cdt)^2 - a^2(t)(dr^2 + r^2 d\Omega^2) \qquad . \qquad (2)$$

The dynamics of the scale factor a(t) is set by the energy content of the Universe as derived from the Einstein field equations. The energy content defines the metric;

$$H^2 = (\dot{a}/a)^2 = (8\pi / 3M_p^2)\rho \qquad . \qquad (3)$$

H is the Hubble "constant" or parameter and the dot over a(t) denotes a time derivative with respect to t. The density ρ refers to all energy densities; matter, radiation and dark energy or vacuum energy.

The energy density has a time dependence determined by the equation of state obeyed by the particular type of energy. The basic relationship has to do with the change of energy within a physical volume V which is a comoving volume times a^3. That change, $d(\rho a^3)$ is due to the pressure, p, doing work, -pdV. The result is dρ+3(da/a)(ρ+p). The time dependence of the energy density in an expanding space is therefore;

$$\dot{\rho} + 3H(\rho + p) = 0 \qquad . \qquad (4)$$

The two terms define the behavior of the fluid in a dynamic Universe. The H term provides the friction, where the density term tracks the reduction in density due to the volume increase during expansion while the pressure term tracks the reduction in energy due to the work done by pressure during expansion. For matter domination, p is henceforth assumed to be zero, while for radiation p = ρ/3.

In the case of temperature, or energy, the scaling as the inverse of a(t) is obvious since the fluid cools upon expansion. Similarly, the physical wavelength scales as a(t) since waves are red shifted by the expansion.

The equation for da/dt can be solved easily in the case where matter or radiation dominates. For matter domination, the density scales with inverse volume or as the inverse cube of the scale a, while for radiation it scales as temperature to the fourth power (Stefan-Boltzmann law) or as the inverse fourth power of a. Note the scaling for matter holds only for pressure-less matter since

$d(\rho V) = -pdV$ where p is the pressure and the volume goes as a^3. Pressure-less matter is at rest with respect to the comoving coordinates. Solving Eq.3 for a in the case of matter domination, the scale a goes as the 2/3 power of t while the energy density goes as the inverse square of t. In the case of radiation dominance a(t) goes as the square root of t while the density again goes as the inverse square of the time t.

These two equations, Eq.3 and Eq.4, can be combined by first differentiating the Hubble expression and then substituting for the time derivative of ρ. The result is the acceleration equation;

$$\ddot{a}/a = -4\pi/(3M_p^2)(\rho + 3p) \quad . \tag{5}$$

Clearly, both for matter and for radiation, the acceleration is negative and an expanding Universe dominated by the energy density of matter or radiation will decelerate. The acceleration in a matter dominated phase goes as $-1/a^2$ while in a radiation dominated phase it goes as $-1/a^3$. In either case the deceleration is large when a(t) is small and then slows as the scale grows.

In the case of dark matter or "vacuum energy" the energy density, ρ_v, of this "cosmological term" is constant with respect to a(t) since the density is proportional to the space-time metric itself and tracks the changes in scale. Specifically the pressure is negative and equal in magnitude to the density so that the time dependence of the density is zero, Eq.4. Therefore, H is a constant, Eq.3, and the scale grows exponentially in time. The acceleration in this case is positive in distinction to the situation with both matter and radiation.

$$\begin{aligned} da/a &= dt\sqrt{8\pi\rho_v/3M_p^2} = Hdt \\ a &\sim e^{Ht} \\ \ddot{a} &= H^2 a \end{aligned} \tag{6}$$

Because the scale a(t) grows exponentially in the case of vacuum energy domination the coordinate distance, dr , travelled by light decreases during this period of acceleration. The physical distances, however, grows exponentially.

In a static Universe, at time t_o a distance ct_o is visible. The Hubble law is that the recession velocity, v, is proportional to physical distance L, v = HL. The Hubble distance, $L_H = c/H$ occurs when the recession velocity is c – a horizon since objects at larger L are unobservable. This distance expands faster than the galaxies so that more of the Universe is included inside the horizon and is therefore visible. This behavior is true for a Universe composed of matter and radiation. In a decelerating Universe, with a scaling of a(t) as t^n, the Hubble distance at present where the recession velocity is c at the time of light emission is $c/H_o = ct_o/n$.

Since the Universe decelerates c/H increases and the Hubble distance expands faster than the Universe, overtaking the receding galaxies. In general the Hubble horizon distance is at $L_H = c/H$

which has a velocity cd(1/H)/dt = v_H = c/n. For radiation dominated behavior, v_H = 2c. In an inflationary period H is approximately constant, so that all objects were observable at some time in the past.

A particle horizon is different in that some events cannot ever be observed if they are outside the causal light cone. Photons travel on null geodesics, $ds^2 = 0$ in the Robertson-Walker metric. In that case, cdt = a(t)dr and a photon emitted at time t = 0 and absorbed at a time t has a "conformal time" τ which is defined to be;

$$\tau = c\int_0^t dt / a(t) \quad . \tag{7}$$

This definition of conformal time accounts for the expansion of the Universe during the travel time of the photon and restores the 'light cones" familiar from the Minkowski flat space of special relativity. Light starting from χ = 0 at τ = 0 arrives at χ = τ. Any more distant point is outside the light cone and is never visible.

$$ds^2 = 0 = a^2(t)[d^2\tau - dr^2], r = \chi \tag{8}$$

Clearly, conformal times slow down as the Universe expands. The maximum comoving distance, dr, that light can go in time dt is just dτ. This is the commoving particle horizon while the physical horizon is a(t)dτ. The particle horizon occurs at τ = 0, $\chi = \tau_o$, = L_P/a_o. The particle horizon has a physical length $L_P = nL_H/(1-n)$ in a power law Universe. The relationships of conformal and coordinate time for the three types of energy density are shown in Table 2.

The expected power law behaviors of the three types of energy density are shown in Table 2 for energy density, scale factor, conformal time, Hubble parameter and comoving Hubble parameter which scales with t as does τ.

Table 2: Time Dependence of Selected Cosmological Quantities.

	ρ(a)	a(t)	τ	a(τ)	H	1/Ha
Matter	$1/a^3$	$t^{2/3}$	$t^{1/3}$	τ^2	(2/3)/t	$t^{1/3}$
Radiation	$1/a^4$	$t^{1/2}$	$t^{1/2}$	τ	(1/2)/t	$t^{1/2}$
Vacuum Energy	constant	e^{Ht}	$-e^{-Ht}$	-1/Hτ	constant	e^{-Ht}

4. Radiation and Matter Dominated Times

Assuming the dominance of matter over radiation at the present time (Table 1), the scaling of a(t) with time t can be used to project backward in time. The radiation term will become more important and will, at some past time, become dominant.

Numerical results are obtained by assuming the present time, $t_{o,}$ is 13.43 billion years since the Big Bang. The present energy densities are as defined in Table 1. Assuming matter domination, the inverse matter density goes as t squared while the radiation density rises as t to the 8/3 power. The two densities then become equal at an earlier time. At that time, teq, the energy density is 1.907 x $10^{\mathbf{11}}$ GeV/$m^3$and the temperature of the radiation, scaling inversely as a(t) is 14200 degrees Kelvin. That temperature corresponds to an energy of 1.18 eV. The time teq is 1.1 x 10^{12} sec or 35600 years. It is assumed that the "dark matter" which now predominates over ordinary matter has the same time dependence for a(t) as ordinary matter.

Before teq radiation is the dominant form of energy in the Universe. One can distinguish between teq and the decoupling or recombination time when the plasma of photons and electrons, which is opaque, becomes a system of hydrogen atoms and photons, transparent to light. The light from this epoch is the CMB or the surface of last photon scattering.

The time can be scaled to earlier periods reliably since the atomic and nuclear physics which is in play is well understood. Going to 0.01 sec, the energy density increases to 2.42 x 10^{39} GeV/$m^3$and the radiation temperature is 1.51 x 10^{11} degrees Kelvin or 12 MeV. This energy scale remains in the domain of nuclear physics and should be reliable.

The energy density of matter and radiation from 0.01 sec to the present is shown in Fig. 1. All Figures in this note were made by MATLAB scripts which are available upon request to the author. The two densities cross at teq assuming a simple scaling in the matter and radiation dominated regimes. Also shown are representative densities of water, the sun and a white dwarf. The dark energy density given in Table 1 is assumed to be a cosmological term and therefore constant in time.

The energy of the radiation component of the Universe is shown in Fig. 2. It goes as the inverse of the scale a(t), but the time dependence of a(t) depends on whether the Universe is radiation or matter dominated. From Table 2, the energy goes as the inverse square root of t. Two representative scales are also shown; atomic scale is the ionization energy of hydrogen while the nuclear scale is set by the binding energy of the deuteron. The Universe is invisible to electromagnetic probes for times less than about 10^{10} sec because the plasma of electrons and photons is opaque. This makes the CMB the earliest object of study barring the possible future ability to detect gravitational waves or primordial neutrinos.

5. The Flatness and Causality Issues for BB Cosmology

There are two major issues for Big Bang (BB) cosmology [15]. One is called the flatness problem. If the Universe is not spatially flat, there is a modified radial term $dr^2/(1-kr^2)$ in the Robertson-Walker metric, Eq.2. The current value of the flatness parameter [14]r is that the Universe is within a few percent of the critical energy density which divides positive and negative curvature. The problem is that if curvature is now near to flatness, then in the past it must have been incredibly fine-tuned to be almost exactly one. Eq.3 defines the critical value of ρ because k equal to zero was assumed there. The value is defined in terms of the present value of the Hubble parameter H and the Planck mass, Eq.9.

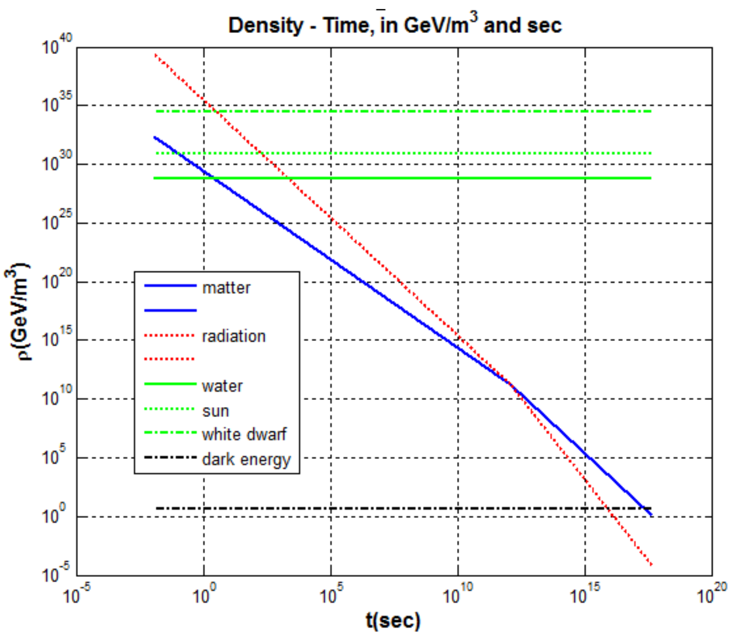

Figure 1: Time dependence of the energy density for matter, radiation and dark energy. The radiation and matter densities are equal at a time teq = 1.2 x 10^{12} sec. Horizontal densities of dark energy, water, the sun and a white dwarf are shown to set the scales.

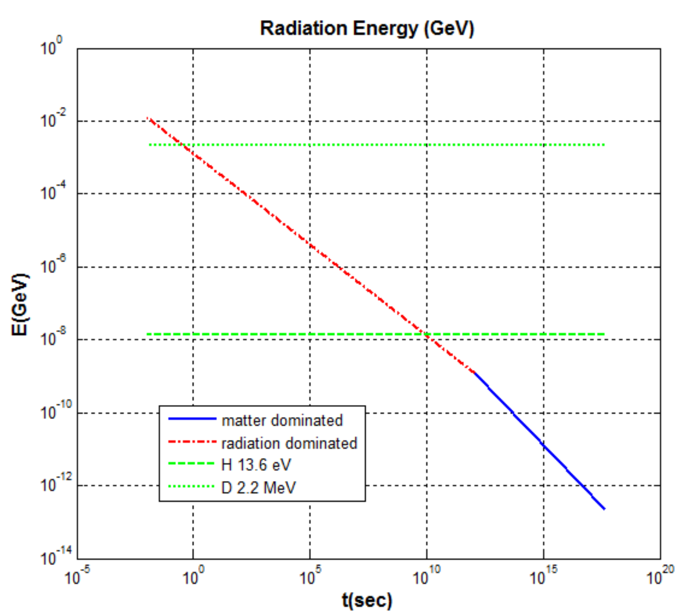

Figure 2: Time dependence of the energy (Temperature) of the radiation. The horizontal scales for atomic and nuclear physics are shown to set the scales.

$$(\dot{a}/a)^2 = (8\pi/3M_p^2)\rho - k/a^2 = H^2$$
$$\rho_c = H_o^2/(3M_p^2/8\pi) \quad (9)$$
$$\Omega = \rho/\rho_c, |\Omega - 1| = k/(Ha)^2$$

As seen from Table 2, Ha decreases as t increases so that the curvature parameter, |Ω-1|, is driven away from one at late times. Indeed, Ha for matter domination goes as the inverse 1/3 power of t while for radiation domination it goes as the inverse square root of t. Therefore a flat space is unstable and fine tuning seems difficult to avoid as an initial condition on the Universe. Note that, from Table2, vacuum energy behaves oppositely, driving the flatness parameter to zero.

The curvature issue can be explored by making an exact numerical integration of the differential equation for a(t). Using the present energy densities for matter, radiation and vacuum energy as given in Table 1, the equation to solve follows from Eq.3 and the behavior of a(t) shown in Table 2, and is;

$$\alpha = a(t)/a_o$$
$$\dot{\alpha} = d\sqrt{1/\alpha + b/\alpha^2 + f\alpha^2} \quad . \quad (10)$$

The constant is, $d = \sqrt{(8\pi/3)M_p^2\Omega_m\rho_c}$ = 1.18 x 10^{-18} /sec. The other terms are b equal the ratio or radiation to matter density at present and f the ratio of vacuum to matter density at present. The result of this numerical integration appears in Fig. 3.

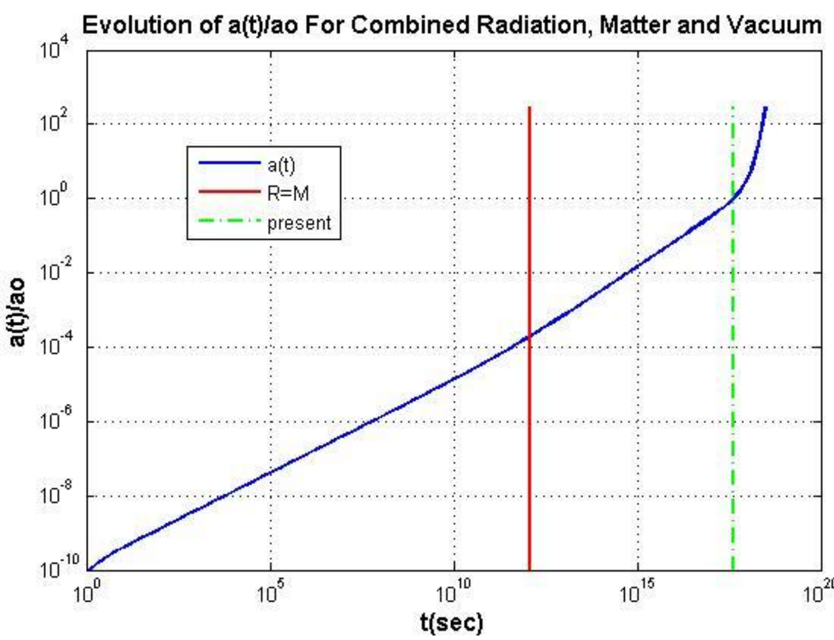

Figure 3: Time dependence of the scale factor a(t) from 1 sec to the present and a bit later. All sources of energy density are treated simultaneously.

The plot shows a(t) in the general case where all three sources of energy density are treated simultaneously. The present time and teq are shown explicitly as vertical lines. Note that at future times the vacuum energy begins to drive an exponential expansion. This is illustrative, but is assumed to be unrelated to inflation at very early times.

The flatness parameter is defined by the behavior of $(Ha)^2$ as seen in Eq. 9. Since vacuum energy has H^2 proportional to the vacuum energy density, H is a constant and, solving for a, the scale factor is exponentially increasing, as seen in Table 2. Therefore a vacuum energy density would drive the $1/(Ha)^2$ rapidly to zero. This behavior is seen in Fig. 4 at future times. In the past, the curvature, $|\Omega-1|$, increased by a factor roughly 10^{15} from a time of one second to the present. The expected linear dependence of the flatness parameter on t, Table 2, is seen in Fig.4 for early times, dominated by radiation with a softer $t^{2/3}$ behavior for later, matter dominated times.

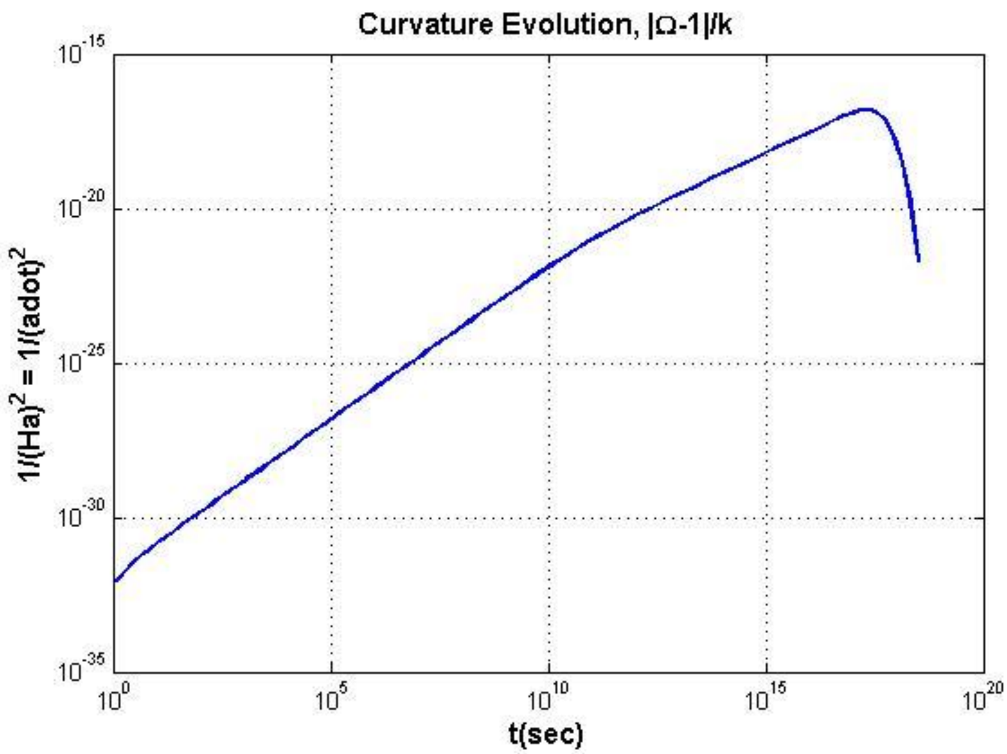

Figure 4: Evolution of the curvature parameter from a time of 1 sec to the present and near future. Note that vacuum energy drives down the curvature which grows rapidly in the matter or radiation dominated epochs.

The causality issue in BB cosmology has to do with the observed uniformity of the CMB radiation. A rough order of magnitude estimate is as follows. Consider the CMB and extrapolate back to time teq when the scale was about 10^{-4} times the present scale (Fig. 3). The distance ct_o at present, Table 1, corresponds to 1.3×10^{22} m at CMB time, roughly teq. However light has only gone, since the t = 0 Big Bang, about a distance of cteq or 3.3 x 10^{20} m. Hence, there is no way that the CMB volume can be causally connected which is needed to easily explain its' temperature uniformity. That conclusion, of course, assumes that no new physics intervenes going back from teq to time zero or the "Big Bang".

The conformal time from one second to the present is shown in Fig. 5. Note that, as defined here, conformal time is dimensionless. This time dependence was derived using the numerical results of Fig. 3 and ignoring small contributions. The expected behavior that conformal time goes as the square root of t at early times, Table 2, is observed. Numerically, the conformal time from one sec to teq is about 0.02, while from teq to t_o it is about 3.7.

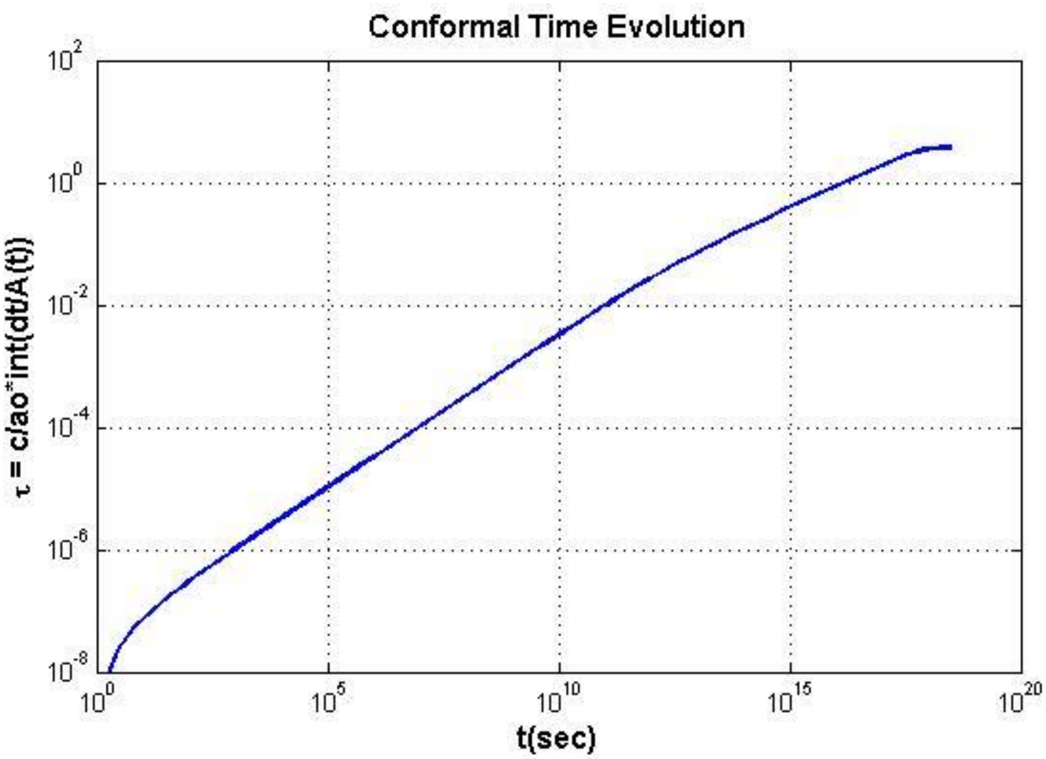

Figure 5: The evolution of conformal time from t = 1 sec to the present time as derived from the numerical results of integrating Eq. 3 with all three sources of energy, matter, radiation and vacuum.

The maximum conformal time occurs now and is about 3.6. The initial conformal time is here defined to be zero. Light travels on straight lines inclined by 45 degrees (the light cones), Eq.8, in the (τ, χ) plane. Causally connected events occur within the past light cones as in the case of special relativity. The cone for the present is shown in Fig. 6. Also indicated is the conformal time at teq, scaled by 10 or a CMB conformal time about 100 times less than the present time (Fig. 5). Clearly, events spanning the CMB range of χ extrapolated to zero conformal time cannot be causally connected since the CMB conformal time light cones do not reach the (0,0) origin.

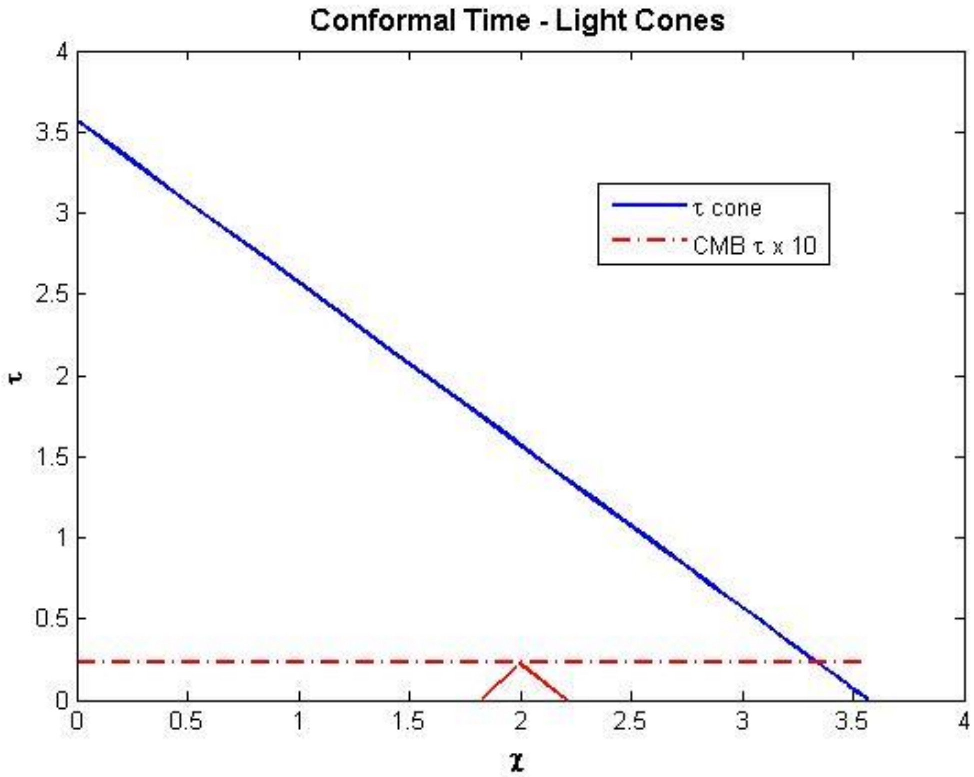

Figure 6: Light ray conformal time from the present time back to the CMB. The CMB conformal time is scaled up by a factor 10 in order to make the CMB past light cones visible. They cannot communicate with the (0,0) origin or with each other except very locally.

6. Inflation and Scalar Fields

Inflation aims to solve the BB cosmology problems by postulating a rapid expansion of the scale factor at very early times, thus invalidating the simple extrapolation to time zero assumed so far in BB cosmology. New physics intervenes at very early times. In the approximation of a cosmological constant, ρ_v, or vacuum energy term Λ, H is a constant, $H = \sqrt{\Lambda / 3}$. The related energy density is a constant since it is the negative of the pressure (Eq. 4). The scale factor, Table 2, is exponential with argument Ht. Therefore, the curvature is driven to extremely low values so that even with the subsequent growth, Fig. 4, the presently observed flatness can easily be accommodated. The causal issues for the CMB are resolved because the early epoch of inflation has conformal time, $\tau = -(c / H)e^{-Ht}$, which is large and negative and allows sufficient time to have all segments of the CMB in causal contact at very early times.

The physical mechanism for the existence of an approximately constant value of H posits a scalar field which has a potential energy which is sensibly constant (called "slow roll") for a sufficient period of time to solve the flatness and causal problems which were mentioned above.

The field, φ, is uniform in space, but decreases with time [16]. The dimension of the field is mass as can be inferred from noting that it appears as a potential term, $m^2\varphi^2/2$, in the Lagrange density from which the space and time integral leads to a dimensionless action. Since this term is an energy density, it has the dimension of mass to the fourth power. The field mass is taken to be m. The field energy density and pressure are; $\rho_\varphi = \dot{\varphi}^2/2 + V(\varphi), p_\varphi = \dot{\varphi}^2/2 - V(\varphi)$ where V(φ) is the potential energy of the scalar field. The density has a kinetic term which is a special case of the Klein-Gordon equation term $\varphi^* \partial_\mu \partial^\mu \varphi$ for a spatially uniform field. Allowing for a pressure source term, Eq. 5, the acceleration of a(t) is positive if the potential term dominates. The Hubble parameter, density time derivative, and scale factor acceleration are special cases of Eqs.3, 4 and 5;

$$\begin{aligned} &H^2 = (8\pi/3M_p^2)(\dot{\varphi}^2/2 + V(\varphi)) \\ &\dot{\rho} + 3H\dot{\varphi}^2 = 0 \\ &\ddot{a}/a = -(8\pi/3M_p^2)(\dot{\varphi}^2/2 - V(\varphi)) \end{aligned} \qquad (11)$$

If the potential dominates and varies slowly with time, H is quasi-constant and the acceleration is positive, in contrast to acceleration for matter and radiation. Using Eq. 11, the equation of motion for the field is;

$$\ddot{\varphi} + 3H\dot{\varphi} + dV/d\varphi = 0 \qquad (12)$$

The field acts like a simple harmonic oscillator (SHO), with H again supplying the damping effect. The length of time for inflation depends on the shape of the potential. Parameters are defined which characterize the first and second derivative of the scalar potential. Neglecting $\dot{\varphi}$ in the expression for H and the second time derivative in the harmonic equation is called the slow roll approximation, $H^2 \sim (8\pi/3M_p^2)V(\varphi), 3H\dot{\varphi} \sim -dV/d\varphi$. The two dimensionless parameters specifying the shape of the potential are;

$$\begin{aligned} &\varepsilon = (M_p^2/16\pi)[(dV/d\varphi)/V]^2 \\ &\eta = (M_p^2/8\pi)[(d^2V/d^2\varphi)/V] \end{aligned} \quad . \qquad (13)$$

In order to solve the BB flatness and causality issues a sufficient expansion of a(t) is needed. It is characterized by the number of "e-folds", N. Using Eq.11 in the slow roll approximation, $\int Hdt \sim \int 3H^2 d\varphi/(dV/d\varphi)$;

$$\begin{aligned} &a \sim e^{Ht}, N \sim \int Hdt \\ &N = (8\pi/M_p^2)\int [V/(dV/d\varphi)]d\varphi \\ &\quad = (2\sqrt{\pi}/M_p)\int d\varphi/\sqrt{\varepsilon} \end{aligned} \quad . \qquad (14)$$

The number of “e-folds” depends on how fast the field is fractionally decreasing.

7. Simplest Scalar Field

The simplest field has a quadratic potential energy similar to a harmonic oscillator, $V(\varphi) = m^2\varphi^2 / 2$. The slow roll parameters for this potential depend on the square of the field, $\varepsilon = \eta = (1/4\pi)(M_p / \varphi)^2$. Small values of the field lead to more rapid changes in the field near the end of the inflationary period when the field has largely fallen off. Using Eq.11 in the slow roll approximation the time dependence of the field can easily be solved for;

$$\begin{aligned} \dot{\varphi} &= -mM_p / \sqrt{12\pi} \\ \varphi &= \varphi_i - (mM_p / \sqrt{12\pi})t \end{aligned} \quad . \qquad (15)$$

The subscripts i and f refer to the start and end of the inflationary period. Neglecting the final field, the number of e folds is, Eq.14 $N = 2\pi(\varphi_i / M_p)^2$**.** The Hubble parameter depends linearly on the field while the scale parameter depends exponentially on the time integral of the field;

$$\begin{aligned} H &= \sqrt{4\pi/3}(m/M_p)\varphi \\ a(t) &= a_i e^{\sqrt{4\pi/3}(m/M_p)\int \varphi dt} \end{aligned} \quad . \qquad (16)$$

There must be a sufficiently large value of N to solve the BB problems of flatness and CMB causality which means the initial value of the field must be large enough.

8. Inflationary Numerology

At present, fundamental particle physics is understood up to about the one TeV scale. Scaling from teq of about 10^{12} sec, with 1.2 eV energy, using Table 2 for the behavior of a(t) in a radiation dominated regime, the scaling in energy to one TeV decreases a(t) by about a factor of 10^{12} or the time decreases by a factor of 10^{24} going down to about 10^{-12} sec.

The postulated physics of inflation operates a at vastly increased scale of energy and a vastly shorter time period, quite beyond what is now understood. The Planck scale for masses, Eq.1, is invoked and time scales are in the range 10^{-36} to 10^{-40} sec. Between this time range and the understood time range much new physics could exist, such as superstrings, supersymmetry, and grand unification over the range 10^{-36} to 10^{-12} sec. In this note a radiation or relativistic dominated regime is assumed to cover the physics beyond current knowledge. In spite of this uncertainty, it is very important to explore the inflationary paradigm and see where it leads.

Numerical estimates for inflation are made in the simplest possible scalar model. Inflation ends when the slow roll approximation breaks down, when ε is one or $\varphi_f / M_p = 1/\sqrt{4\pi}$ = 0.28. A sufficiently large value of N is needed to solve the BB issues, N ~ 60, which sets the scale, Eq.14, for the initial field, $\varphi_i / M_p = \sqrt{N/2\pi}$ = 3.1. The value of the H parameter as a function of time depends on the scalar mass, m, since it defines the time dependence of the field, Eq.15, and thus the scale factor a(t), Eq. 16. A value of m/M_p of 10^{-6} is chosen. That choice leads to a slope of the field of 2.96 x 10^{36} sec^{-1} and a time when inflation is active of about φ_f/slope = 1.05 x 10^{-36} sec. The evolution in the inflationary regime is therefore tracked from 10^{-40} to 10^{-36} sec. Other choices of the range of time can be made for other choices of m.

The resulting Hubble parameter is shown in Fig. 7. It decreases linearly with the rolling field, Eq. 16.

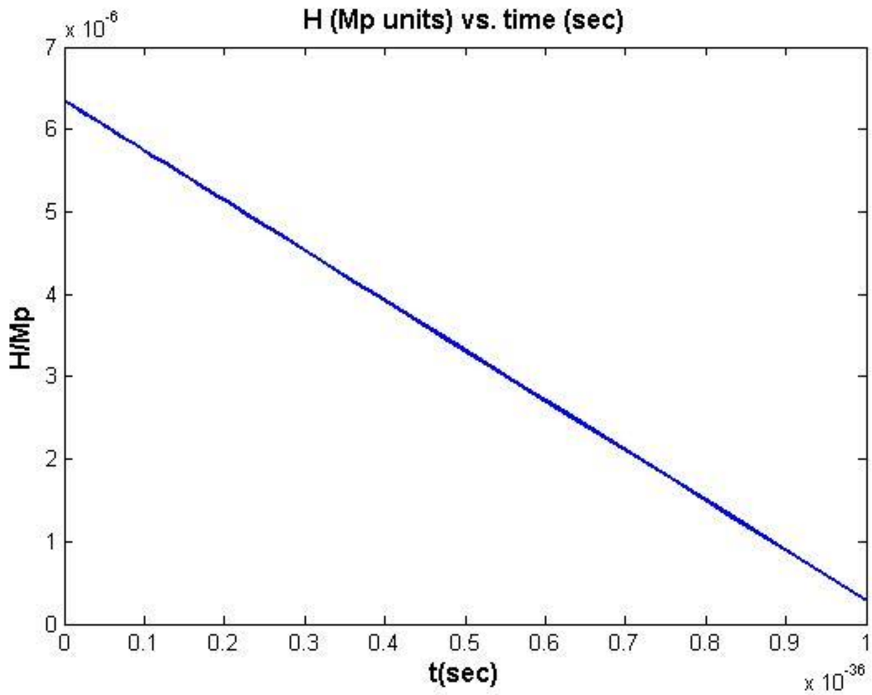

Figure 7: Behavior of the Hubble parameter as a function of time in the simple slow roll scenario.

The values of a(t) as a function of time are shown in Fig. 8. The exponential increase appears as expected. The value of a(0) is set rather arbitrarily to be scaled from the Planck length using the Planck time, Eq.1, scaled to the initial time of inflation, 10^{-40} sec, or 3 x 10^{-32} m. The increase in a(t) by a factor of 10^{26} over the inflationary period supplies 59.9 e folds by design. However, this choice is somewhat arbitrary. The size of N is largely controlled by the choice of the initial field magnitude.

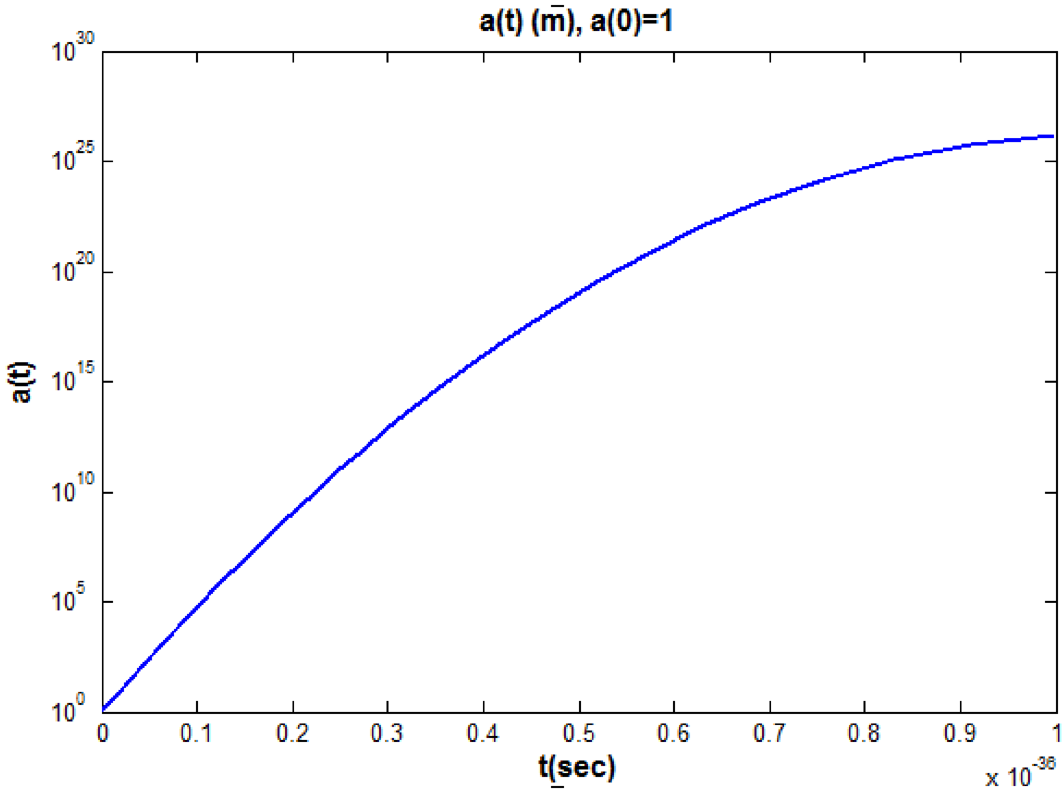

Figure 8: Time dependence of the scale factor a(t) during the inflationary period, where a(0) is here taken to be 1.

The values of a(t) and H(t) allow a calculation of the curvature as a function of time, Eq.9. The result is shown in Fig. 9 for the previously defined parameters. During the inflationary period the increase of a(t) by a factor 10^{26} and the approximate constancy of H means that the curvature is decreased by a factor about 10^{52} . After the scalar field has "decayed" into Standard Model (SM) particles, a radiation dominated curvature was assumed where $(1/Ha)^2$ scales as t (Table 2) since a decelerating Universe always has the curvature increasing with time. In contrast, for the BB cosmology the flatness, $|\Omega\text{-}1|$ would have to be less than 10^{-64} at the Planck time. The actual value for the curvature depends on the square of the initially assumed value of a(t) , which was chosen rather arbitrarily as mentioned above.

The Hubble length, L_H = c/H, defines the radius of the causal event horizon. Processes separated by physical distances greater than that length are not causally connected. In a static Universe, the length is ct. Because of decelerated expansion, the power law behavior of a(t) leads to a Hubble length of ct/n for a(t) ~ t^n. In a period of inflation, the Hubble length is approximately constant (Table 2) as is L_H. In the simple inflationary model the Hubble length can be displayed as a function of t. as observed in Fig. 10 where an abrupt transition from inflation to radiation dominance is assumed.

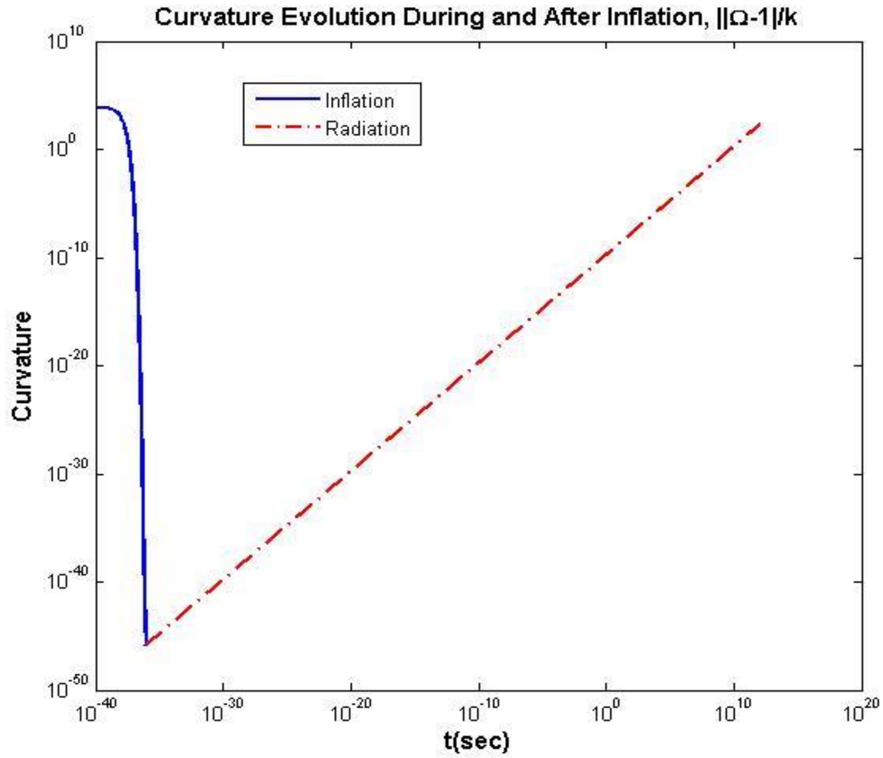

Figure 9: Time dependence of the spatial curvature during inflationary and an assumed radiation dominated period. Inflation resets the initial unknown value of the curvature to be a value about 10^{52} times smaller at the end of inflation. After that a linear increase with t is assumed until teq.

During inflation a physical scale, in this case a(t), (solid blue in Fig. 10), grows rapidly and is larger than the Hubble length, c/H (dashed red) which is roughly constant. After inflation the Hubble length grows as ct/n or 2ct in a radiation dominated regime where n is ½. During the time that any physical scale is greater than the Hubble length that scale is outside of causal influence. As seen in Fig. 10 at a later time a scale, exemplified here by a(t), becomes less than the Hubble length and then can become causally active again. The Hubble length, has a delayed increase with time which occurs only after inflation. In the hot BB phase a(t), scaling as the square root of t, is overtaken by the horizon, L_H, which scales as t.

With the resetting of clocks to time zero before inflation, the conformal time during inflation covers about 80 units followed by the radiation dominated era. The time development of the conformal time during inflation is shown in Fig. 11.The range of τ of about 80 leaves sufficient time for the CMB regions to be causally connected before the time of inflation. In BB cosmology, Fig.5, the conformal time without the intervention of inflation is small during radiation domination and then grows by about 3.6 units during matter domination..

One should be careful to distinguish the events separated by conformal distances χ greater than τ. They could never have been in communications but if the event separation is greater than c/(aH), the conformal Hubble length, or horizon these events cannot communicate now but are not precluded from having communicated in the past. Compared to Fig.6 with a different time origin chosen, τ would there be negative with a value about -80.

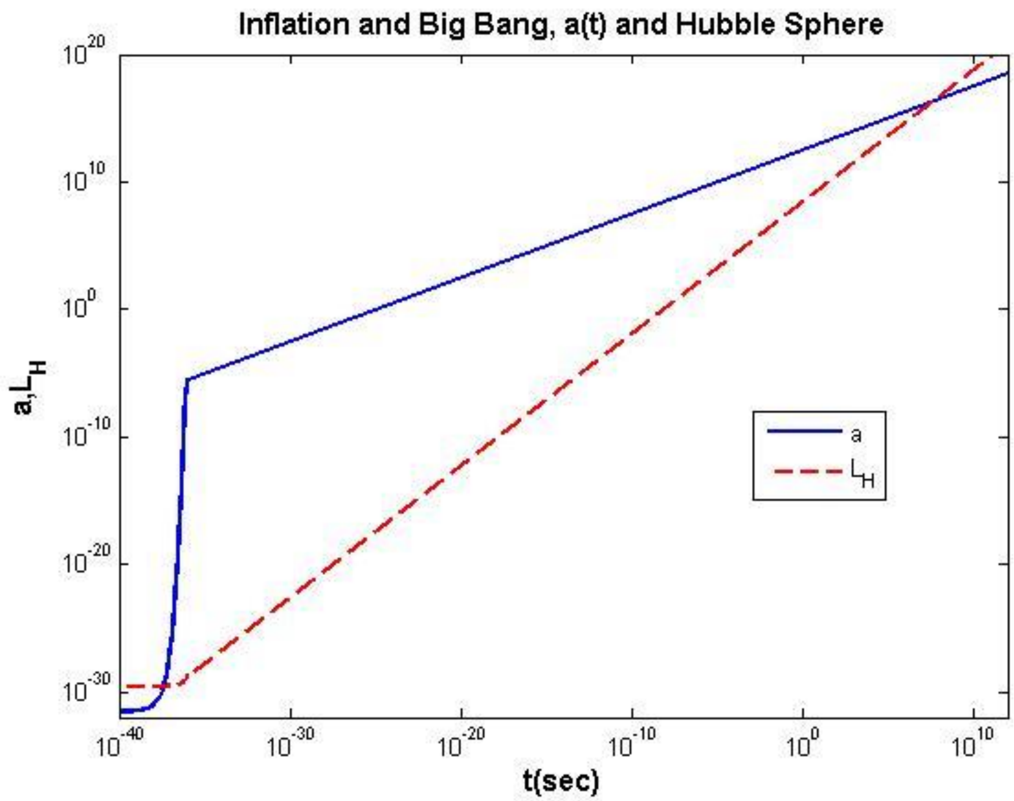

Figure 10: Time dependence of the physical scale factor a(t) and the Hubble length, c/H (m) for an inflationary period followed by a period of radiation dominance.

The conformal Hubble horizon, c/(aH), which is dimensionless, decreases dramatically during inflation , since H is roughly constant and a(t) is growing exponentially. The time dependence of the quantity c/aH appears in Fig. 12. As c/aH decreases, a commoving scale will exit the horizon and then re-enter it later during the hot BB phase.

Before leaving this section, the arbitrary nature of the plots should be mentioned. As stated above, the initial time for inflation and the time span of inflation are chosen, in concert with the scalar mass to give sixty e-folds. These choices are arbitrary and are meant to be illustrative only. For example, changing only the initial value of the scalar field to 3.5 times the Planck mass rather than 3.1 times, the scale factor increases by 10^{33} rather than 10^{26} and the number of e-folds increases to seventy-five. The curvature value is then driven down by a factor 10^{65} rather than $10^{52}$and at a teq the value only recovers to 10^{-30} compared to the roughly full recovery seen in

Fig.9. Clearly, there is a sufficient range of the parameters of the inflationary model to be chosen so as to satisfy all the constraints imposed by the data as explored so far.

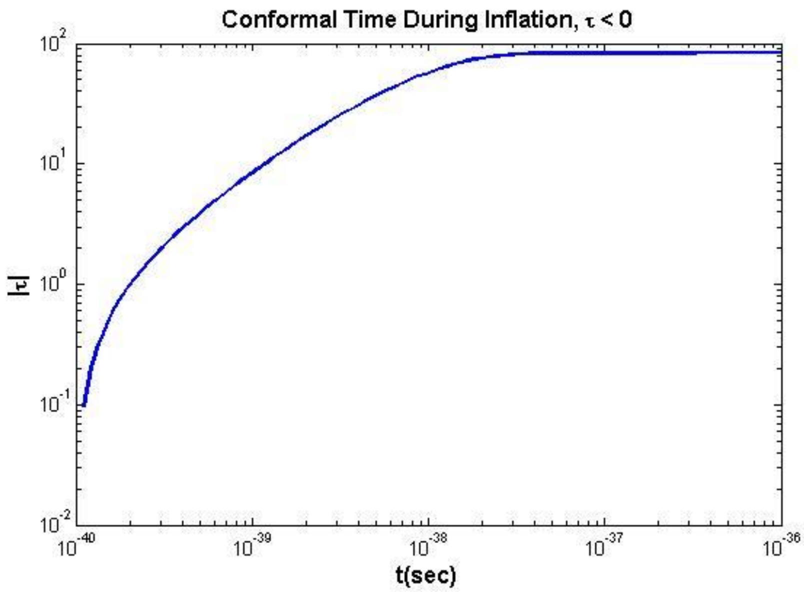

Figure 11: Conformal time (dimensionless) as a function of coordinate time during inflation. There is a sufficient range in conformal time to allow causal communication at very early times.

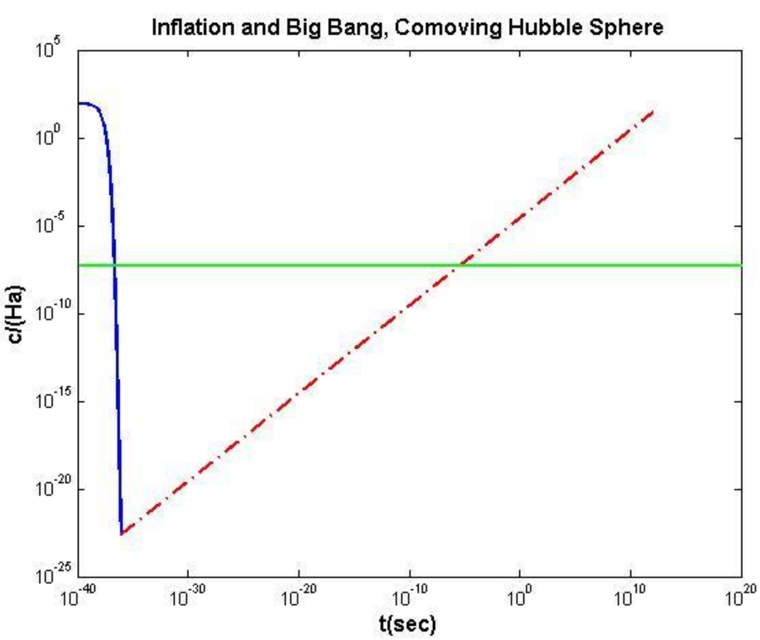

Figure 12: Time dependence of the comoving event horizon c/aH during inflation (blue solid) and radiation dominated epochs (red dash-dot) when it scales as the square root of t. The solid green horizontal line indicates the constant behavior of an arbitrary commoving scale.

For example, with an initial value of 3.1 for the scalar field with respect to the Planck mass, after inflation, extrapolation to teq and then extrapolation to the present time t_o yields a present scale a_o of roughly 10^{22} m, less than the value a_o appearing in Table 1. However a small change of φ_i to 3.5 scales to a present a_o value of10^{29} m. Clearly the plots appearing in this note are only illustrative, not definitive. In particular, after inflation ends an abrupt transition to a radiation dominated Universe is assumed. Since the fundamental physics of this early period, 10^{-36} to 10^{-12} sec, is not understood, this is, at best, a simplifying assumption.

9. Inflation and CMB Fluctuations

The inflationary hypothesis solves some outstanding issues in BB cosmology, but at the cost of introducing a scalar field unknown to particle physics. However, it also makes additional testable predictions which are a critical advantage. Specifically it predicts a power magnitude and a spectrum of the temperature fluctuations in the CMB which are nearly independent of the scale of those fluctuations.

During inflation there are irreducible zero point quantum fluctuations of the field. Although inflation is smoothing out all quantities such as the spatial curvature, quantum fluctuations are intrinsic. There are also metrical fluctuations due to the strong gravity waves during. The subject is complex [17], and simple order of magnitude arguments are made here in the interest of a short exposition of structure formation.

The commoving Hubble horizon, c/Ha, decreases rapidly during inflation, Fig. 12. A scale (coordinate), wavelength λ or wave number k, which is initially less than c/aH and can be casually active, provides the uniformity and small quantum fluctuations ultimately seen in the CMB. The physical wavelength, aλ, is red-shifted while the coordinate wavelength λ is comoving and constant. During inflation the physical scale goes outside the Hubble horizon, c/H, and falls out of causal contact, Fig.10. Therefore when the coordinate wavelength is less than 1/aH or coordinate wave number k, $\lambda = 2\pi / k$, is greater than aH casual connection is lost, Fig.12. After inflation, aH increases and the wave number crosses the commoving Hubble horizon and becomes accessible again. However, due to inflation, these wavelengths are of classical, not quantum size and can serve as seeds to macroscopic structure formation. The spectrum is defined in natural units, k/(Ha) in what follows. Quantum mechanics plus inflation naturally leads to observable density perturbations in the CMB which can be confronted with data.

The quantum fluctuations are formed on sub-horizon scales prior to exiting the horizon. The fluctuations are created on a time scale or wavelength scale of 1/H. The scalar field fluctuations are then;

$$\delta\varphi = \hbar(H / 2\pi) \quad . \qquad (17)$$

The factor $\hbar$ is here kept explicitly to highlight the quantum nature of the fluctuation. The field fluctuations cause a local spread in the time of the end of inflation, $\delta t = \delta\varphi / \dot{\varphi}$. A fluctuation in density occurs due to the fluctuation in the time of the end of inflation, which means the end of inflation varies locally in space. Density fluctuations, δ_H , are created at the end of inflation, $\delta_H \sim \delta\rho / \rho \sim \delta t / t \sim H\delta t$ **.** The dimensionless fractional density fluctuation depends on the square of the scalar field fluctuation and is;

$$\delta_H = \hbar H^2 / 2\pi\dot{\varphi} = 2\pi(\delta\varphi)^2 / \hbar\dot{\varphi} \qquad . \qquad (18)$$

Initially, when H is large and the time derivative of the field is small the fluctuation, is almost constant and applies to a large range of wavelengths. Therefore, the inflationary model predicts an almost flat spectrum of fluctuations in commoving wavelength and wave vector.

For a scalar field, Eq.11, Eq.12, $H^2 = (8\pi / 3M_p^2)V(\phi), 3H\dot{\varphi} = -dV / d\varphi$ in the slow roll approximation. A small local value of dV/dφ means a longer time delay or a larger density in that region and a larger δ_H. Substituting into Eq.18 for H;

$$\delta_H = \sqrt{512\pi / 75}[V^{3/2} / M_p^3(dV / d\varphi)] \qquad . \qquad (19)$$

The fluctuations depend on the shape of the potential. In the simplest case of a harmonic scalar field, adopted previously, Eq. 15 and Eq. 16 give the fluctuations;

$$\delta_H = \sqrt{64\pi / 75}[(m / M_p)(\phi_i / M_p)^2] \qquad . \qquad (20)$$

The dimensionless fractional density fluctuation depends on the square of the initial value of the field and also upon the scalar field mass, m, because that sets the scale for the time rate of decrease of the field, Eq.15.

It is well beyond the scope of this note to find the metrical fluctuations due to gravity waves. Suffice it to say that there is a dimensionless fractional amplitude, A_G , for such fluctuations [17], $A_G = \sqrt{32 / 75}[V^{1/2} / M_p^2]$ which can easily be evaluated in the simplest scalar model. The ratio of A_G to δ_H is the dimensionless slow roll parameter ε, $A_G = \delta_H \sqrt{\varepsilon}$. Alternatively;

$$A_G = \sqrt{16 / 75}[(m / M_p)(\varphi_i / M_p)] \qquad . \qquad (21)$$

This amplitude also depends on both the initial value of the scalar field and on the mass of the scalar. The initial value of the field is relevant because of the initial steep drop in the comoving horizon c/aH, Fig. 12, which causes different scales to exit the horizon at almost the same time. The small spread in the time when different scales exit the horizon is another prediction of the inflationary model. Scales which can now be observed crossed the comoving horizon very near to the start of inflation. In principle the fluctuation is evaluated when a physical wave length λa exits the horizon c/H at λ=c/(aH) or k = aH. The approximation used here is that the spread in times when of the horizon is initially crossed is small.

The power of the scalar density fluctuations, P_s , follows from Eq.13 for the slow roll parameters and from Eq.19 for the fractional density fluctuation δ_H **.** Stated without proof;

$$\delta_H^2 \sim P_s = (4\pi / M_p^2 \varepsilon_i)[(H / 2\pi)^2 (k / aH)^{n_s - 1}] \qquad (22)$$

The scales, (k/aH) are assumed not to be fully scale independent but to have a spectral index n_s-1 slightly different from zero. In the case of the simplest scalar field the fluctuation power becomes, with $\varepsilon_i = 1/4\pi(M_p/\varphi_i)^2$ which explicitly labels the parameter ε to be evaluated at the start of inflation;

$$P_s \sim (16\pi / 3)[(m / M_p)^2 (\varphi_i / M_p)^4 (k / aH)^{n_s - 1}] \qquad (23)$$

Using Eq. 20 it is seen that the power is, in fact, directly related to the square of the density fluctuation, $P_s = (75/12)\delta_H^2 (k/aH)^{n_s - 1}$. Alternatively, the magnitude of the power goes as the square of (m/M_p) and as the fourth power of (φ_i/M_p).

Using the value 3.1 for the initial scalar field scaled to the Planck mass, Eq. 20 predicts that δ_H is 15.7 times the scalar mass scaled to the Planck mass and, Eq.21 shows that A_G is 1.01 times that ratio. The temperature fluctuations occur because photons lose energy climbing out of the gravitational potential of higher density regions. The CMB fractional temperature fluctuations [14] are about a part in 10^5 which implies that δ_H is about 2 x 10^{-5} or that, using Eq.20, m/M_p is about 6.4 x 10^{-7} . This value is near to that found in the slow roll time dependence for the field, Eq.15, of 10^{-6}. This is a useful consistency check for the specific model. The mass, m, could also be related to the scale where the three SM coupling constants intersect, the scale [14] of a Grand Unified Theory (GUT).

There are three distinct predictions of the inflationary paradigm; the magnitude of the fluctuations, the spectral index of the scalar fluctuations , and the ratio of the magnitudes of the tensor to scalar fluctuations. As for the spectral index, deviations from scale invariance arise from time spreads for exiting the horizon early in the inflationary period. In turn, they therefore also depend on the shapes of the potential that drives the inflation. The first few terms in the Taylor expansion for the shape are contained in the slow roll parameters, ε and η. Stated without proof;

$$n_s - 1 = M_p^2 / 8\pi[-3(dV / d\varphi)^2 + (d^2V / d^2\phi)] \qquad (24)$$

Ignoring metrical fluctuations for the moment, $n_s - 1 = -6\varepsilon_i + 2\eta_i$ using the slow roll definitions, Eq.13. Using the φ_i/M_p value of 3.1 the slow roll parameters are both 0.0083 which means a predicted spectral index of 0.967. This is close to the experimental CMB result of 0.96 [14] which has an error of about 0.01. The inflationary prediction is in good agreement with the data.

The power spectrum of the CMB [14] is shown in Fig.13. It is striking that there are strong structures in the CMB spectrum. They can exist if the Fourier components of the spectrum all coherently exit the horizon at almost the same time. As mentioned above this is also a prediction of the inflationary paradigm. A mode with a physical wavelength is inside the horizon at the start of inflation, then exits the horizon and is stretched in length with a constant fractional amplitude. When the mode re-enters the horizon it begins to evolve. The recombination which creates the visible CMB is approximately instantaneous which preserves the relative phases of the Fourier components.

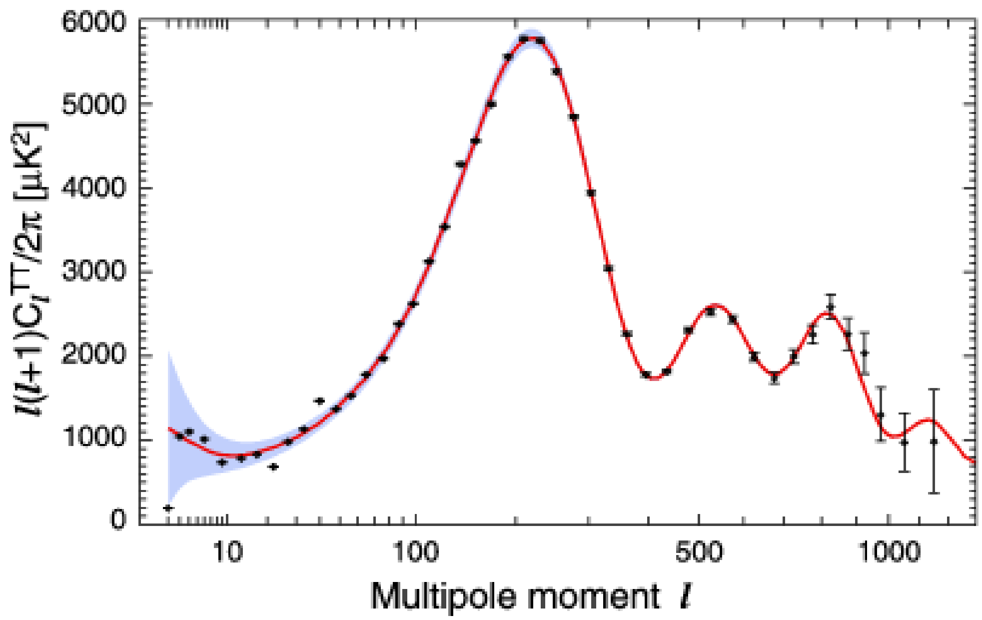

Figure 13: Power spectrum of the CMB temperature fluctuations. Structure in the spectrum indicates a phase coherence in the Fourier components, k/(Ha) , of the fluctuations.

There is another prediction made by inflation models for tensor perturbations due to gravity waves which has not yet been tested. The spectral index is predicted to be $n_G = -2\varepsilon_i$. Quantum perturbations make only scalar perturbations while gravitational waves make both scalar and tensor perturbations. The dimensionless power in the scalar case, P_s, ignoring Fourier factors of (k/aH), is $H^2/\pi M_p{}^2\varepsilon_i$, Eq. 22, while the tensor power, P_t, is, stated without proof to be proportional to the square of the amplitude A_G and is equal to $16(H/M_p)^2/\pi$. The ratio of tensor to scalar power is predicted to be, $r = P_t/P_s = 16\varepsilon_i$ or 0.13 in the simplest model.

The CMB is predicted to be polarized due to Thompson scattering of the electrons and photons. The polarization magnitude is related to the temperature fluctuations. Specifically, there are so-called “B modes” which are rotational modes of the polarization which cannot be excited by scalar fluctuations. Therefore, detection of a B mode [10], [11] polarization of the CMB fluctuations would be a strong indication for inflation. The present experimental situation is not clear [10], [11].

Current limits on the scalar spectral index and on tensor to scalar power ratio are shown [12] in Fig.14. The simplest model prediction is, in fact, rather close to the present limits and more data will arrive very soon. Tensor fluctuation magnitudes are now only limits since definitive tensor detection is not in hand. It is an exciting time for the inflationary paradigm. The simple scalar results of $n_s = 0.967$ and $r = 0.13$ with $N = 60$ are close to the large black dot in Fig.14 obtained by more sophisticated calculations. Other, more complex models give a range of possible values in the (n_s,r) plane.

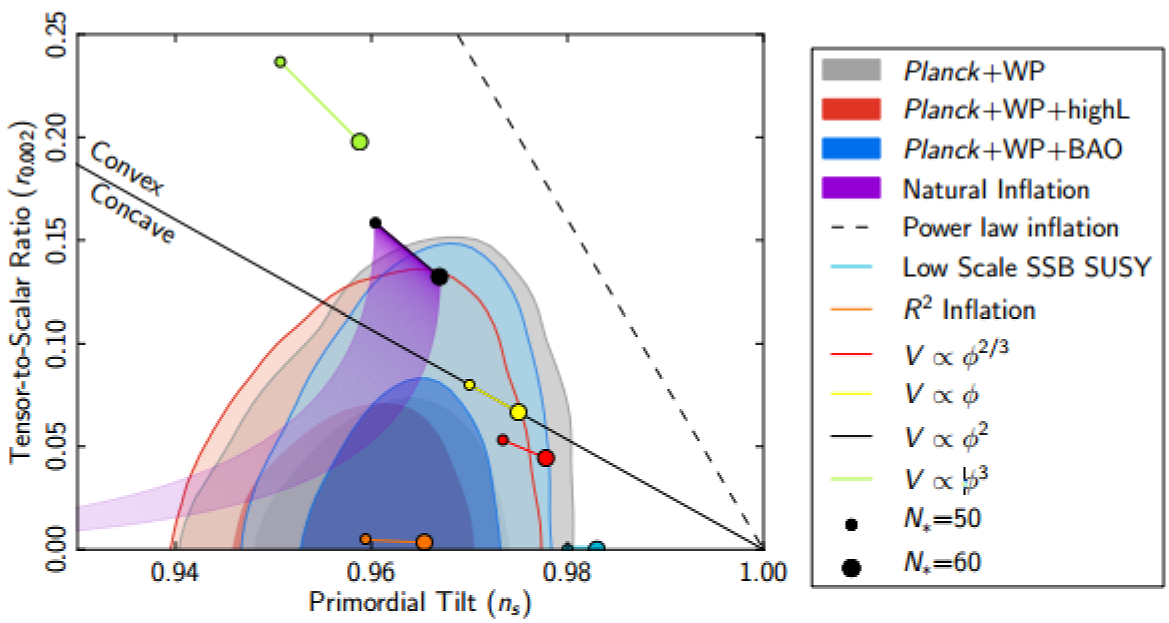

Figure 14: Constraints on n_s and r from the 2014 WMAP data set. Also shown are inflationary models. The simplest scalar model is consistent with, but near to, the present limits.

10. Reheating

The inflationary paradigm is so far successful and provides testable predictions for the future. However some obvious questions remain. Where is the scalar field now? How, in detail, did the period from the end of inflation, at the Planck mass scale, evolve to a regime, the TeV scale a factor ~ 10^{16} lower in mass where the physics is presently understood.

Recall that the scalar field, after the end of the rundown of the potential, acts like a SHO, Eq.12. The remaining potential decays and l becomes radiation and the Universe is thermalized. The equation of motion allowing for the addition of a decay width Γ to Eq.12 is;

$$\ddot{\varphi} + (3H + \Gamma)\dot{\varphi} + m^2\varphi = 0 \quad . \tag{25}$$

Dimensional argument concludes that the decay width, Γ, is proportional to the scalar mass, m, and some coupling constant, α, if the decays of the field are first order two body decays. The decay daughters, x, are assumed to be SM objects with masses much less than the scalar mass [14].

$$\Gamma(\varphi \to xx) \sim \alpha m \tag{26}$$

Since the Hubble parameter scales as H ~ 1/t in a radiation dominated phase the fraction of the contribution of produced particles to the total energy density is large when 3H decreases at large times and becomes comparable to Γ or $\Gamma^2 \sim (3H)^2 = (72\pi\rho / 3M_p^2)$, Eq. 3. Using the Stefan-Boltzmann law, with a number of degrees of freedom , $g_* \sim 100$ appropriate to the SM, leads to a thermalized re-heating temperature T_R. Note that if all the x particles indicated in Eq.26 are SM then they are all relativistic since the scalar mass m is so much larger than their masse. The T^4 behavior holds in this case;

$$\rho_{re-heat} \sim (\Gamma M_p)^2 / 24\pi = g_* \pi^2 T_R^4 / 30 \tag{27}$$

The grand unified results [14], assuming that supersymmetry (SUSY) intervenes at masses of about one TeV, are that the unified coupling constant is about 1/24 and the three SM forces come together at about 10^{16} GeV. Solving Eq.27 for the reheating temperature, $T_R = 0.14\sqrt{\Gamma M_p} = 9x10^{-5} M_p$. Extrapolating from teq ~ 10^{12} sec and Teq~ 1 eV, and scaling for radiation domination to 10^{-36} sec (factor 10^{24}) T rises to ~ 10^{15} GeV or ~ 10^{-4} M_p. This result is uncertain because the reheating decay mechanism is poorly understood. Obviously, there is much poorly understood physics between accelerator based data and the inflationary paradigm but the order of magnitude "guesstimates" are encouraging. The inflationary simplest model yields m/M_p values of about 10^{-6}. The extrapolation to the reheating temperature is within two orders of magnitude of this result and the GUT scale is within three. Clearly a more complex model of reheating is called for [18].

The scalar field responsible for inflation has a value near the Planck mass. The Higgs mass is 125 GeV, or a factor 10^{17} smaller and has a comparable magnitude for its' vacuum expectation value. The other possible scalar field is dark energy with a vacuum expectation value of 0.8 meV, a factor 10^{14} less than the Higgs field. The connection between these scalar fields, if there is one, is unknown. However, it is now known that fundamental scalar fields exist, so the questions of possible connections and the understanding of the early times for now inaccessible to accelerators will likely remain of great importance to both cosmologists and particle physicists.

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