Selected Topics in Microwave Instabilities and Linacs

K.Y. Ng

Fermilab

June, 2013

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Selected Topics in Microwave Instabilities and Linacs

K.Y. Ng

Fermilab, Batavia, IL 60510

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Abstract

The Shanghai Institute of Applied Physics (SINAP) is embarking on its first X-ray free-electron laser (FEL) project. It is a cascading high-gain harmonic generation FEL. Microwave instabilities driven by various effects, especially the space-charge force, will degrade the quality of the electron beam before entering into the undulator. However, inside the undulator, the occurrence of microbunching becomes an utmost important ingredient for the generation of coherent radiation. In short, controlled and uncontrolled microwave instabilities must be fully understood in such a project. These are the slides of a series of eight-hour lectures given at the SINAP in June of 2013, with the intention of a fully understanding of the microbunching phenomenon. The sections of wake field and impedance theory are added as an introduction for those who are not familiar with the subject.

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2. Coupling Impedances
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Introduction

- A particle interacts with the vacuum chamber produces EM fields.
- The motion of a particle following is perturbed.

$$(\vec{E},\vec{B})_{\substack{\text{seen by}\\ \text{particle}}} = (\vec{E},\vec{B})_{\substack{\text{external, from}\\ \text{magnets, rf, etc.}}} + (\vec{E},\vec{B})_{\substack{\text{wake}\\ \text{fields}}}$$

where

$$(\vec{E},\vec{B})_{\substack{\text{wake}\\ \text{fields}}} \begin{cases} \propto & \text{beam intensity} \\ \ll & (\vec{E},\vec{B})_{\text{external}} \end{cases}$$

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- Perturbation breaks down when potential-well distortion is large. Then, distortion has to be included into non-perturbative part.

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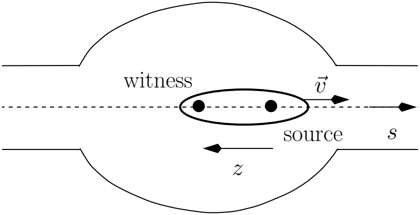
- Perturbation breaks down when potential-well distortion is large. Then, distortion has to be included into non-perturbative part.
- What we need to compute are the EM wake fields at a distance z behind the source particle.
- The computation of the wake fields is nontrivial.
- Two approximations lead to a lot of simplification.

1. Rigid-Bunch Approximation [1]

- Motion of beam not affected during traversal through discontinuities.



Source particle at $s = \beta ct$

Witness particle at $s = z + \beta ct$

$z < 0$ for particle following.

This does not imply no syn. motion.

Just require z and βc do not change after traversing the discontinuity.

- Rigidity implies beam at high energies.

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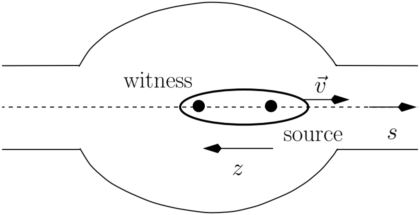

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2. Impulse Approximation

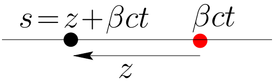
- We do not care about the wake fields $\vec{E}$, $\vec{B}$, or the wake force $\vec{F}$.
- We only care about the impulse

$$\Delta\vec{p} = \int_{-\infty}^{\infty} dt\ \vec{F} = \int_{-\infty}^{\infty} dt\ q(\vec{E} + \vec{v} \times \vec{B})$$

q is charge of witness particle

- We will see how the simplification evolves.

Panofsky-Wenzel Theorem [2]



- Maxwell equation for witness particle at (x, y, s, t) with $s = z + \beta ct$: and z constant

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss's law for electric charge}$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\beta c\rho\hat{s} \qquad \text{Ampere's law}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \text{Gauss's law for magnetic charge}$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \text{Faraday's \& Lenz law}$$

- Want to write Maxwell equation for the impulse $\Delta\vec{p}(x, y, z, t)$.

  First compute

  with $\vec{F}(x, y, z, t) = q(\vec{E} + \vec{v}\times\vec{B})$

$$\vec{\nabla}\cdot\vec{F} = \frac{q\rho}{\epsilon_0\gamma^2} - \frac{q\beta}{c}\frac{\partial E_s}{\partial t},$$

$$\vec{\nabla}\times\vec{F} = -q\left(\frac{\partial}{\partial t} + \beta c\frac{\partial}{\partial s}\right)\vec{B}.$$

The Wake Force Equations

- Wake force: $\vec{F}(x,y,z,t) = q(\vec{E} + \vec{v} \times \vec{B})$
- Maxwell's equations: $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0}, \quad \vec{\nabla}\cdot\vec{B} = 0,$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu_0\beta c\rho\hat{s}$$

- Divergent:

$$\vec{\nabla}\cdot\vec{F} = q\left(\vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{v}\times\vec{B}\right) = \frac{q\rho}{\epsilon_0} - q\vec{v}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \mu_0\beta c\rho\hat{s}\right)$$

$$= \frac{q\rho}{\epsilon_0\gamma^2} - \frac{q\beta}{c}\frac{\partial E_s}{\partial t} \qquad \boxed{c = \frac{1}{\sqrt{\epsilon_0\mu_0}}}$$

- Curl:

$$\vec{\nabla}\times\vec{F} = q\vec{\nabla}\times\vec{E} + q\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)$$

$$= -q\frac{\partial\vec{B}}{\partial t} + q\vec{v}\left(\vec{\nabla}\cdot\vec{B}\right) - qv\frac{\partial\vec{B}}{\partial s} = q\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial s}\right)\vec{B} = q\frac{d\vec{B}}{dt}$$

$$\underset{\uparrow}{\vec{\nabla}}\times\Delta\vec{p}\,(x,y,z)=\int_{-\infty}^{\infty}dt\;\Big[\underset{\uparrow}{\vec{\nabla}}\times\vec{F}(x,y,s,t)\Big]_{s=z+\beta ct}.$$

this $\vec{\nabla}$ refers
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We obtain

$$\vec{\nabla}\times\Delta\vec{p}=-q\int_{-\infty}^{\infty}dt\,\left[\left(\frac{\partial}{\partial t}+\beta c\frac{\partial}{\partial s}\right)\vec{B}(x,y,s,t)\right]_{s=z+\beta ct}$$

$$=-q\int_{-\infty}^{\infty}dt\,\frac{d\vec{B}}{dt}=-q\vec{B}(x,y,z+\beta ct,t)\Big|_{t=-\infty}^{\infty}=0,$$

$$\underset{\substack{\uparrow\\ \text{this } \vec{\nabla} \text{ refers}\\ \text{to } x,y,z}}{\vec{\nabla}} \times \Delta\vec{p}\,(x,y,z) = \int_{-\infty}^{\infty} dt \; \Big[\underset{\substack{\uparrow\\ \text{this } \vec{\nabla} \text{ refers}\\ \text{to } x,y,s}}{\vec{\nabla}} \times \vec{F}(x,y,s,t)\Big]_{s=z+\beta ct}.$$

We obtain

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- Dot product with $\hat{s} \Longrightarrow \hat{s}\cdot\left(\vec{\nabla}\times\Delta\vec{p}\right)=0 \Longrightarrow \dfrac{\partial\Delta p_x}{\partial y} = \dfrac{\partial\Delta p_y}{\partial x}$

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- Cross product with $\hat{s} \Longrightarrow \boxed{\dfrac{\partial}{\partial z}\Delta\vec{p}_\perp = \vec{\nabla}_\perp\Delta p_s}$ ⟵ P-W Theorem

$$\underset{\substack{\uparrow \\ \text{this } \vec{\nabla} \text{ refers} \\ \text{to } x, y, z}}{\vec{\nabla}} \times \Delta\vec{p}\,(x,y,z) = \int_{-\infty}^{\infty} dt \;\left[\underset{\substack{\uparrow \\ \text{this } \vec{\nabla} \text{ refers} \\ \text{to } x, y, s}}{\vec{\nabla}} \times \vec{F}(x,y,s,t)\right]_{s=z+\beta ct}.$$

We obtain

$$\vec{\nabla} \times \Delta\vec{p} = -q \int_{-\infty}^{\infty} dt \;\left[\left(\frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial s}\right) \vec{B}(x,y,s,t)\right]_{s=z+\beta ct}$$

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- Cross product with $\hat{s} \Longrightarrow \boxed{\dfrac{\partial}{\partial z}\Delta\vec{p}_\perp = \vec{\nabla}_\perp \Delta p_s}$ $\longleftarrow$ P-W Theorem
- P-W theorem gives strong restriction between long. and trans.
- But it is very general. Does not depend on any boundary conditions. Even do not require $\beta = 1$.

Supplement to Panofsky-Wenzel Theorem

$$\beta = 1 \implies \vec{\nabla}_\perp \cdot \Delta\vec{p}_\perp = 0.$$

Proof:

$$\vec{\nabla}\cdot\Delta\vec{p} = \int_{-\infty}^{\infty} dt \left[\vec{\nabla}\cdot\vec{F}(x,y,s,t)\right]_{s=z+ct} = q\int_{-\infty}^{\infty} dt \left[\frac{\rho}{\epsilon_0\gamma^2} - \frac{\beta}{c}\frac{\partial E_s}{\partial t}\right]_{s=z+ct}$$

$$\xrightarrow{\gamma\to\infty} q\int_{-\infty}^{\infty} dt \left[\frac{\partial E_s}{\partial s}\right]_{s=z+ct} = \frac{\partial}{\partial z}\Delta p_s$$

Use has been made of

1. Space-charge term $\dfrac{q\rho}{\epsilon_0\gamma^2}$ omitted because $\beta \to 1$.
2. $\dfrac{\partial}{\partial t}E_s(s,t) = \dfrac{d}{dt}E_s(s,t) - \dfrac{ds}{dt}\dfrac{\partial}{\partial s}E_s(s,t).$

Maxwell equations now become

$\boxed{\vec{\nabla}\times\Delta\vec{p} = 0}$ and $\boxed{\vec{\nabla}\cdot\Delta\vec{p} = \dfrac{\partial}{\partial z}\Delta p_s}$ without any source terms.

Cylindrical Symmetric Vacuum Chamber

$$\begin{cases} \dfrac{\partial}{\partial r}(r\Delta p_\theta) = \dfrac{\partial}{\partial \theta}\Delta p_r \\ \dfrac{\partial}{\partial z}\Delta p_r = \dfrac{\partial}{\partial r}\Delta p_s \\ \dfrac{\partial}{\partial z}\Delta p_\theta = \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\Delta p_s \\ \dfrac{\partial}{\partial r}(r\Delta p_r) = -\dfrac{\partial}{\partial \theta}\Delta p_\theta \quad (\beta = 1) \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial}{\partial r}(r\Delta \tilde{p}_\theta) = -m\Delta \tilde{p}_r \\ \dfrac{\partial}{\partial z}\Delta \tilde{p}_r = \dfrac{\partial}{\partial r}\Delta \tilde{p}_s \\ \dfrac{\partial}{\partial z}\Delta \tilde{p}_\theta = -\dfrac{m}{r}\Delta \tilde{p}_s \\ \dfrac{\partial}{\partial r}(r\Delta \tilde{p}_r) = -m\Delta \tilde{p}_\theta \quad (\beta = 1) \end{cases}$$

- Cylindrical symmetry $\Longrightarrow$ expansion in terms of $\cos m\theta$ or $\sin m\theta$.

  We write $\Delta p_s = \Delta\tilde{p}_s \cos m\theta, \quad \Delta p_r = \Delta\tilde{p}_r \cos m\theta, \quad \Delta p_\theta = \Delta\tilde{p}_\theta \sin m\theta,$

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- For $m=0$, $\Delta\tilde{p}_r = \Delta\tilde{p}_\theta = 0$, otherwise $\propto \frac{1}{r}$, singular at $r=0$, $\therefore$ $\boxed{\text{only } p_s}$

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- For $m\neq 0$, $\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(r\Delta\tilde{p}_r)\right] = m^2\Delta\tilde{p}_r \Longrightarrow \boxed{\Delta p_r(r,\theta,z) \sim mr^{m-1}\cos m\theta}$

Definition of Wake Functions

- Formal solution can be written as

$$\begin{cases} v\Delta\vec{p}_\perp = -q\mathcal{Q}_m W_m(z) m r^{m-1}\left(\hat{r}\cos m\theta - \hat{\theta}\sin m\theta\right) \\ v\Delta p_s = -q\mathcal{Q}_m W_m'(z) r^m \cos m\theta \end{cases}$$

dimension is energy

- Defn: $\begin{cases} W_m(z) & \longrightarrow \text{transverse wake function of azimuthal } m \\ W_m'(z) & \longrightarrow \text{longitudinal wake function of azimuthal } m \end{cases}$

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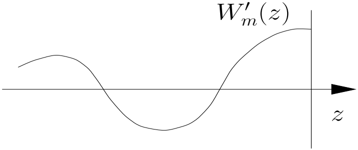
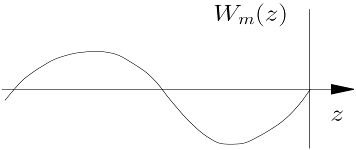
- Recall that solution of $\vec{E}$ and $\vec{B}$ reduces to solution of $W_m(z)$ only. Simplification comes from P-W theorem or rigid-bunch and impulse approximations.

- negative sign in front is a convention to make $W_m'(z) > 0$, since witness particle loses energy from impulse.

Some Properties of Wake Functions





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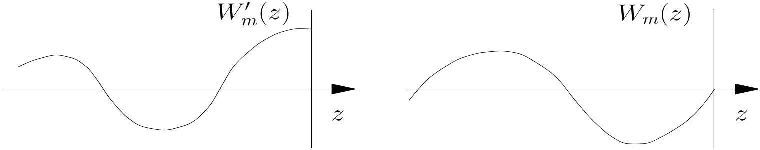

Fundamental Theorem of Beam Loading (P. Wilson)

A particle sees half of its wake, or $\frac{1}{2}W'_m(0_-)$.

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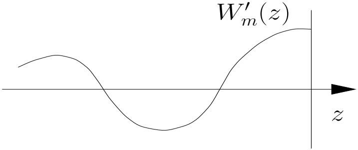

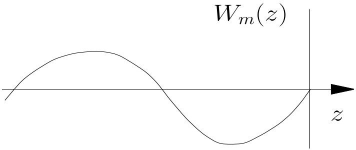

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Proof:

A particle of charge q passes a thin lossless cavity, excites cavity.

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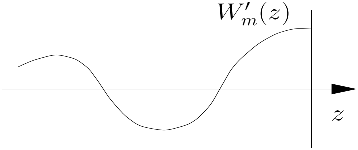

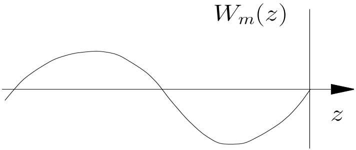

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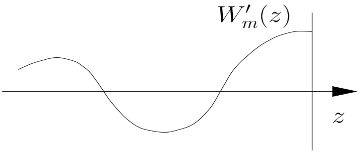
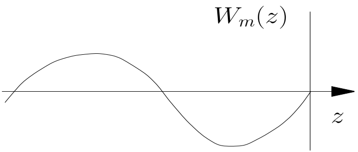
Half cycle later, a 2nd particle of same charge passes the cavity.

Energy gained $\Delta\mathcal{E}_2 = -fq^2 W'_m(0_-) + q^2 W'_m(0_-)$.

Some Properties of Wake Functions





Fundamental Theorem of Beam Loading (P. Wilson)

A particle sees half of its wake, or $\frac{1}{2}W_m'(0_-)$.

Proof:

A particle of charge q passes a thin lossless cavity, excites cavity.

Energy gained $\Delta\mathcal{E}_1 = -fq^2W_m'(0_-)$, i.e., sees fraction f of own wake.

Half cycle later, a 2nd particle of same charge passes the cavity.

Energy gained $\Delta\mathcal{E}_2 = -fq^2W_m'(0_-) + q^2W_m'(0_-)$.

Field inside cavity is completely cancelled.

$$\Delta\mathcal{E}_1 + \Delta\mathcal{E}_2 = -2fq^2W_m'(0_-) + q^2W_m'(0_-) = 0 \Longrightarrow f = \tfrac{1}{2}.$$

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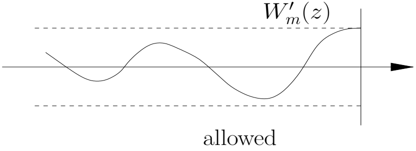

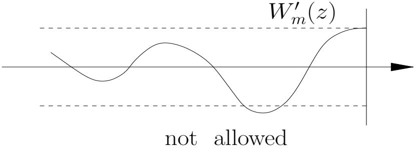

Properties of Wake Functions

- $W_m'(z) = 0$ for $z > 0$. (causality)
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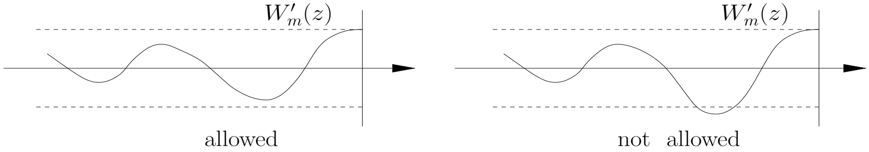

Pr

1st particle of charge q loses energy $\frac{1}{2}q^2W'(0_-)$.

2nd particle of charge q loses energy $\frac{1}{2}q^2W'(0_-) + q^2W_0'(-z)$.

Total loss $q^2W'(0_-) + q^2W_0'(-z) \geq 0$. Or $-W_0'(-z) \leq W_0'(0_-)$.

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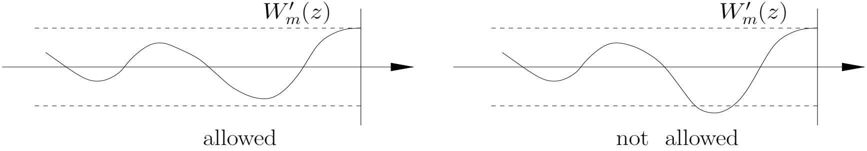

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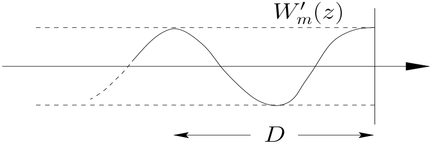
2nd particle of charge $-q$ loses energy $\frac{1}{2}q^2 W'(0_-) - q^2 W'_0(-z)$.

Total loss $q^2 W'(0_-) - q^2 W'_0(-z) \geq 0$. Or $W'_0(-z) \leq W'_0(0_-)$.

Properties of Wake Functions (cont.)

- $W'_m(-D) = W'_m(0_-)$ for some $D > 0 \Longrightarrow$ wake is of period D.



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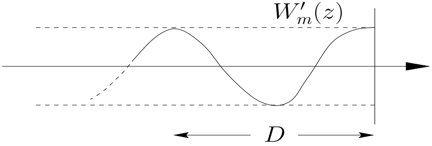

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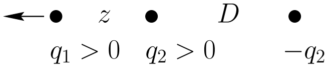

Energy loss:

1. $\frac{1}{2}q_1^2 W'_0(0_-)$.
2. $\frac{1}{2}q_2^2 W'_0(0_-) + q_1 q_2 W'_0(-z)$.
3. $\frac{1}{2}q_2^2 W'_0(0_-) - q_1 q_2 W'_0(-z-D) - q_2^2 W'_0(-D)$.

Since total must be ≥ 0 and q_1 arbitrary, $W'_0(-z) \geq W'_0(-z-D)$.

Change 3 charges to $(q_1,\, -q_2,\, q_2)$ to get $W'_0(-z) \leq W'_0(-z-D)$.

- Area under $W'_m(z)$ is non-negative.

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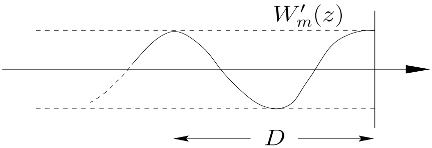

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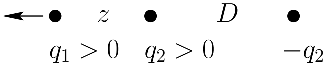

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Consider a dc beam current I.

For a particle of charge q in the beam, energy loss is $q\displaystyle\int W'_0(z)\, I \frac{dz}{v} \geq 0$.

Properties of Wake Functions (cont.)

- $$\begin{cases} v\Delta\vec{p}_\perp = -q\mathcal{Q}_m W_m(z) m r^{m-1}\left(\hat{r}\cos m\theta - \hat{\theta}\sin m\theta\right) \\ v\Delta p_s = -q\mathcal{Q}_m W_m'(z) r^m \cos m\theta \end{cases}$$
- For longitudinal, lowest azimuthal is $m = 0$ or $W_0'(z)$.
- For transverse, lowest azimuthal is $m = 1$ or $W_1(z)$.
- Higher azimuthals can be important for large transverse beam size compared with pipe radius.

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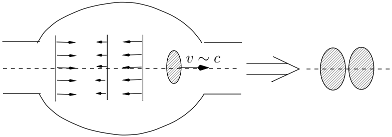
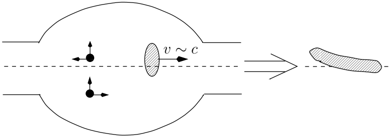
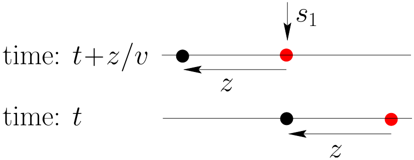


Particles in same vertical slice see same impulse. Can lead to longitudinal micro-bunching or microwave instability.



Particles in same vertical slice receive same vertical impulse independent of vertical position. Can lead to beam breakup.

Coupling Impedances



- Beam particles form current. Component with freq. ω is $I(s,t)=\hat{I}e^{-i\omega(t-s/v)}$.
- A test particle of charge q crossing a narrow discontinuity at s_1 gains energy from wake left by particles $-z$ in front $(z<0)$.
- From $v\Delta p_s=-q\mathcal{Q}_m W'_m(z)r^m\cos m\theta$, voltage gained is $(m=0)$

$$V(s_1,t)=-\int_{-\infty}^{\infty}[W'_0(z)]_1\left[\hat{I}e^{-i\omega[(t+z/v)-s_1/v]}\frac{dz}{v}\right]$$

$$=-I(s_1,t)\int_{-\infty}^{\infty}[W'_0(z)]_1 e^{-i\omega z/v}\frac{dz}{v}\equiv -I(s_1,t)\left[Z_0^{\|}(\omega)\right]_1$$

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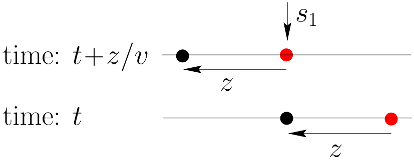

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- **Defn:** $Z_0^{\|}(\omega)=\int_{-\infty}^{\infty}W_0'(z)e^{-i\omega z/v}\frac{dz}{v}$ (summing over all continuities)
- Unlike a current in a circuit, a beam has transverse dimension and therefore higher multipoles.
- When the beam is off-center by amount a, the current mth multipole is $\mathcal{Q}_m(s,t)=I(s,t)a^m=\hat{\mathcal{Q}}_m e^{-i\omega(t-s/v)}$.

Higher Azimuthal Impedances

- Source particle has transverse density $\dfrac{\delta(r-a)}{a}\delta(\theta)$.

  Subject to the mth multipole element $\mathcal{Q}_m\left(s_i, t+\dfrac{z}{v}\right)\dfrac{dz}{v}$ passes location i $-z$ earlier, voltage gained by test particle is

$$V(s_i,t) = -\int \frac{dz}{v}\mathcal{Q}_m(s_i,t+z/v)[W_m'(z)]_i \int r dr d\theta\, r^m \cos m\theta \frac{\delta(r-a)\delta(\theta)}{a}$$

$$= -\int \frac{dz}{v}\hat{\mathcal{Q}}_m e^{-i\omega[(t+z/v)-s/v]}[W_m'(z)]_i a^m$$

$$= -\frac{\mathcal{P}_m}{q}\mathcal{Q}_m(s_i,t)\int_{-\infty}^{0}\frac{dz}{v}[W_m'(z)]_i e^{-i\omega z/v} \qquad [\mathcal{P}_m = qa^m]$$

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- Identify mth multipole longitudinal impedance across location i as

$$\left[Z_m^{\|}(\omega)\right]_i = -\frac{q\hat{V}}{\mathcal{P}_m\hat{\mathcal{Q}}_m} = \int_{-\infty}^{\infty}\frac{dz}{v}[W_m'(z)]_i e^{-i\omega z/v}.$$

- Summing up around the vacuum chamber: $Z_m^{\|}(\omega) = \sum_i\left[Z_m^{\|}(\omega)\right]_i$.

Transverse Impedances

- General defn. for long. imp.: $Z_m^{\parallel}(\omega) = \int_{-\infty}^{\infty} \frac{dz}{v} W_m'(z) e^{-i\omega z/v}$.
- If we replace W_m' by W_m, we obtain transverse impedances

  Defn. $Z_m^{\perp}(\omega) = \frac{i}{\beta} \int_{-\infty}^{\infty} \frac{dz}{v} W_m(z) e^{-i\omega z/v}$ $[W_m(z) = 0 \text{ when } z > 0]$
- Long. and transverse imp. are then related by $Z_m^{\parallel}(\omega) = \frac{\omega}{c} Z_m^{\perp}(\omega)$,

  so that both $\mathcal{R}e\, Z_m^{\parallel}$ and $\mathcal{R}e\, Z_m^{\perp}$ represent energy loss or gain.
- Transverse force, $F_\perp \propto -W_m$, must lag $\mathcal{Q}_m$ by $\frac{\pi}{2}$ in order for $\mathcal{R}e\, Z_m^{\perp}$ to dissipate energy. Hence the factor i.
- The factor β is to cancel β in Lorenz force, just a convention.

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$$\begin{aligned}\langle F_1^\perp(s,t)\rangle &= -\frac{q}{L}\int_{-\infty}^{\infty} W_1(z)a\hat{I}e^{-i\omega[(t+z/v)-s/v]}\frac{dz}{v}\\ &= -\frac{qaI(s,t)}{L}\int_{-\infty}^{\infty} W_1(z)e^{-i\omega z/v}\frac{dz}{v}=\frac{i\beta qI(s,t)a}{L}Z_1^\perp(\omega).\end{aligned}$$

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- $\langle\cdots\rangle$ implies averaged over all preceding particles.
- For transverse: $Z_1^\perp(\omega)=-\dfrac{i}{q\hat{I}a\beta}\langle\hat{F}_1^\perp\rangle$.
- For longitudinal: $Z_0^\parallel(\omega)=-\dfrac{1}{q\hat{I}}\langle\hat{F}_0^\parallel\rangle$.
- Other than from wake fcns, these are formulas employed to compute imp. directly from the long. and trans. forces seen by test particle.

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$$W_m(z)=-\frac{i\beta}{2\pi}\int_{-\infty}^{\infty}Z_m^{\perp}(\omega)e^{i\omega z/v}d\omega$$

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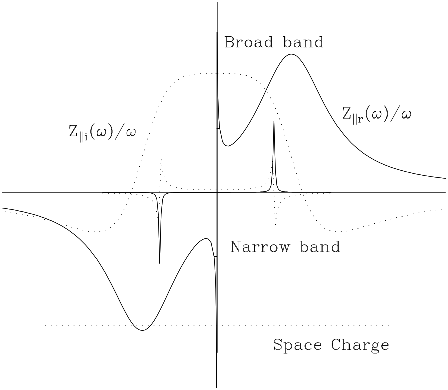
5. $\displaystyle\int_0^{\infty} d\omega\, \mathcal{I}m\, Z_m^{\perp}(\omega) = 0$ and $\displaystyle\int_0^{\infty} d\omega\, \frac{\mathcal{I}m\, Z_m^{\parallel}(\omega)}{\omega} = 0.$

General Behavior of Impedances



- Sharp resonances from cavities.
  Broad-band from bellows,
  BPM's, steps, etc
  Large values near $\omega = 0$ from resistive walls.
- $-\dfrac{\mathcal{I}m\, Z_0^{\|}(\omega)}{\omega}$ inductive at low freq and capacitive at high freq.
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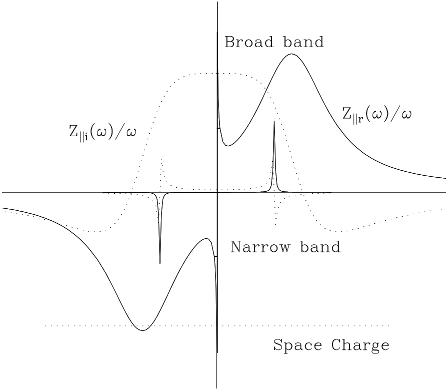

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- $Z_1^{\perp}(\omega)$ behaves similar to $\dfrac{Z_0^{\|}(\omega)}{\omega}$
- $\dfrac{\mathcal{R}e\, Z_0^{\|}(\omega)}{\omega}$ and $\mathcal{R}e\, Z_1^{\perp}(\omega)$ vanish at $\omega = 0$.
  Mathematically because of analyticity.
  Physically because of no dc loss.
- At $\omega = 0$, there is no Faraday's law. $\vec{E}$ and $\vec{B}$ are not related.
  No image current created and no impedance.

General Comments

- $W_m(z)=0$, $W'_m(z)=0$ when $z>0$ because of causality.
- It is awkward to deal with negative z. Some like to use $z>0$ for particle following. Then $W_m(z)=0$, $W'_m(z)=0$ when $z<0$.

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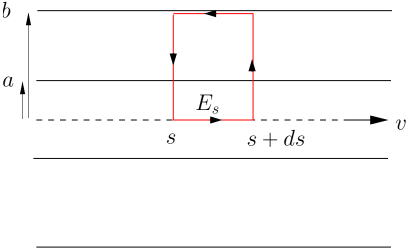
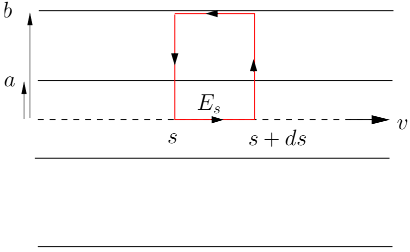
- All properties of the impedances remain unchanged, including no singularity in upper half ω-plane.
- Some may like to use j instead of i to denote imaginary value. Most of the time $j=-i$. Then $Z_m^{\parallel}$ and $Z_m^{\perp}$ have no singularity in lower half ω-plane instead.

Space-Charge Impedances

- Sp-ch imp. comes from EM fields of beam even when beam pipe is smooth and perfectly conducting.
- Want to compute E_s due to variation of linear density $\lambda(s-vt)$. Assume small variation of long. dist.



Space-Charge Impedances



- Sp-ch imp. comes from EM fields of beam even when beam pipe is smooth and perfectly conducting.
- Want to compute E_s due to variation of linear density $\lambda(s-vt)$. Assume small variation of long. dist.
- Faraday law: $\oint \vec{E}\cdot\overrightarrow{d\ell} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{A}$.

$$\oint \vec{E}\cdot\overrightarrow{d\ell} = E_s ds - \frac{e\lambda(s-vt)}{2\pi\epsilon_0}\left[\int_a^b \frac{dr}{r} + \int_0^a \overset{\substack{\text{uniform dist. assumed}\\ \downarrow}}{\frac{rdr}{a^2}}\right] + \left\{s \to s + ds\right\}$$

- Geometric factor $g_0 = 2\left[\int_a^b \frac{dr}{r} + \int_0^a \frac{rdr}{a^2}\right] = 1 + 2\ln\frac{b}{a}$.

- Electric field or left side: $\oint \vec{E}\cdot\overrightarrow{d\ell} = E_s ds + \dfrac{eg_0}{4\pi\epsilon_0}\dfrac{\partial\lambda}{\partial s}\,ds.$
- Magnetic field or right side:

$$-\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{A} = -\frac{\partial}{\partial t}\frac{\mu_0 e\lambda(s-vt)v}{2\pi}\left[\int_0^a \frac{rdr}{a^2}+\int_a^b \frac{dr}{r}\right]ds = v^2\frac{e\mu_0 g_0}{4\pi}\frac{\partial\lambda}{\partial s}ds.$$

- Long. field seen by particles on-axis: $E_s = -\dfrac{eg_0}{4\pi\epsilon_0\gamma^2}\dfrac{\partial\lambda}{\partial s}.$

$$\boxed{c = \frac{1}{\sqrt{\epsilon_0\mu_0}}}$$

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- Consider a long. harmonic wave $\lambda_1(s;t) \propto e^{i(ns/R-\Omega t)}$ perturbing a coasting beam of uniform linear density λ_0.
- Voltage drop per turn is $V = E_s 2\pi R = \dfrac{ineZ_0 cg_0}{2\gamma^2}\lambda_1 = \dfrac{inZ_0 g_0}{2\gamma^2\beta}I_1.$
- The wave constitutes a perturbing current of $I_1 = e\lambda_1 v$.
- Imp. is $\left.\dfrac{Z_0^{\|}}{n}\right|_{\rm sp\ ch} = \dfrac{iZ_0 g_0}{2\gamma^2\beta}$ with $g_0 = 1 + 2\ln\dfrac{b}{a}.$

$$\left[Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\epsilon_0 c} = \mu_0 c\right]$$

Comments

- $\left.\frac{Z_0^\parallel}{n}\right|_{\rm sp\ ch} = i\frac{Z_0 g_0}{2\beta\gamma^2}$ is independent of freq., but rolls off when $\omega \gtrsim \frac{\gamma c}{b}$.
- $\left.Z_0^\parallel\right|_{\rm sp\ ch} \propto \omega$, resembling a neg. inductive imp. rather than a cap. imp.
- For a freq.-independent reactive imp. $\left.\frac{Z_0^\parallel}{n}\right|_{\rm sp\ ch}$, corresponding wake is

$$W_0'(z) = -\delta'(z)\left[iRc\beta\frac{Z_0^\parallel}{n}\right]_{\rm reactive} = \delta'(z)\frac{Z_0 cRg_0}{2\gamma^2}.$$

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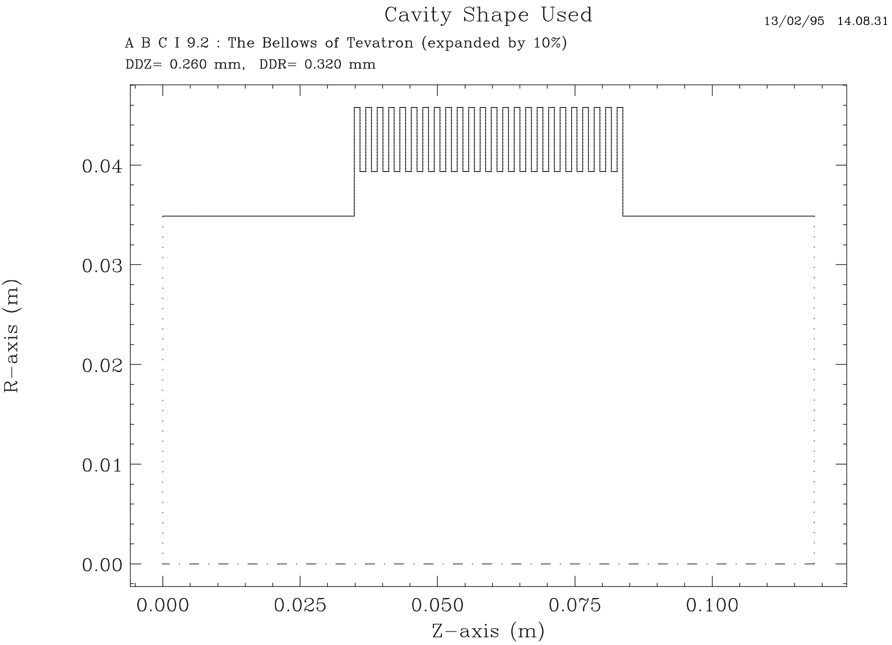
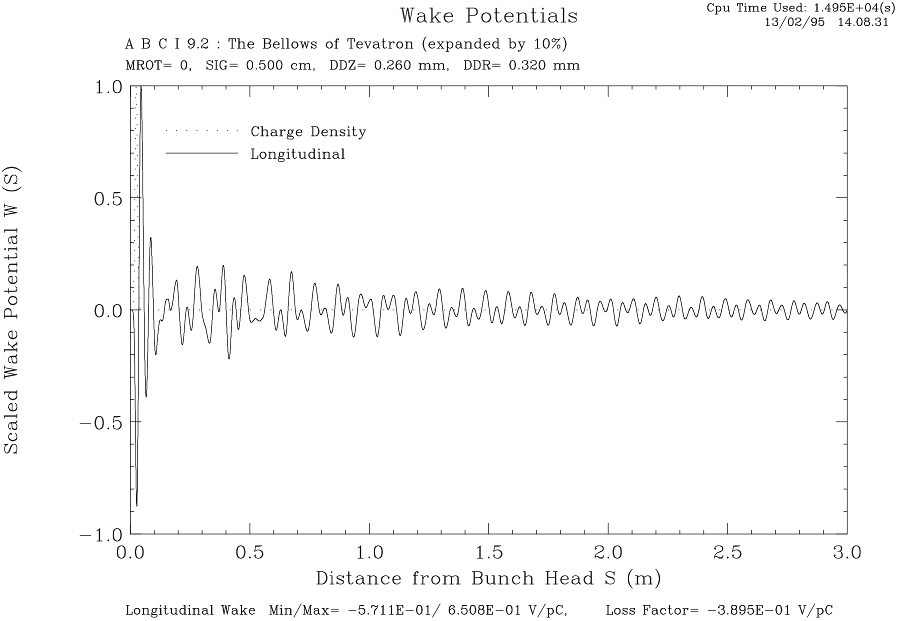
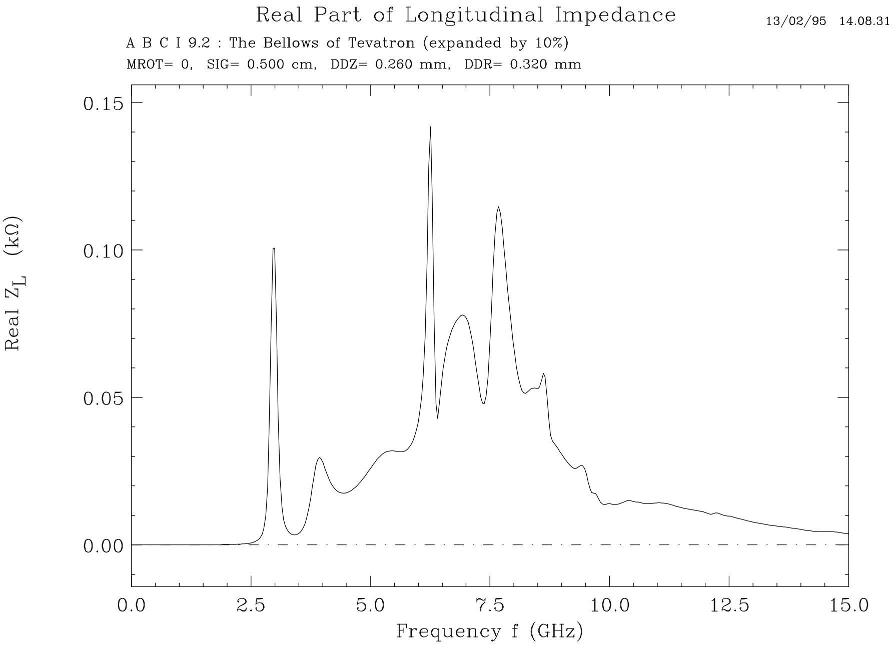
- Longitudinal reactive impedance results from a longitudinal reactive force $F_0^{\parallel}(s,t) = \frac{ie^2 v}{2\pi}\left.\frac{Z_0^{\parallel}}{n}\right|_{\rm reactive}\frac{\partial\lambda(s,t)}{\partial s}$.
- This force modifies the bunch shape, called *potential-well distortion.* Below/above transition, capacitive force lengthens/shortens the bunch.
- Below/above transition, inductive/capacitive force can generate micro-bunching and eventual microwave instabilities.

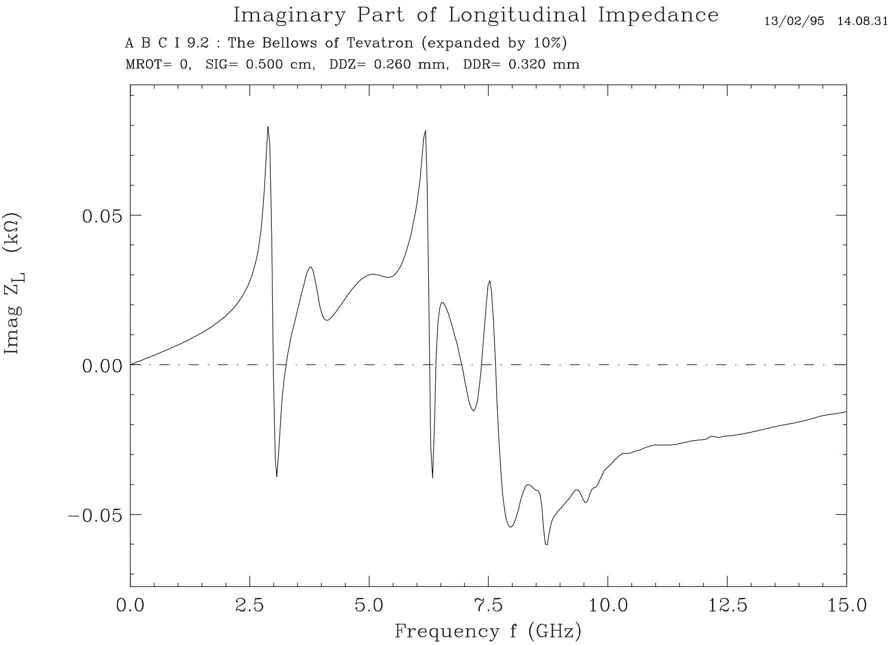
Numerical Computation of Wakes and Impedances

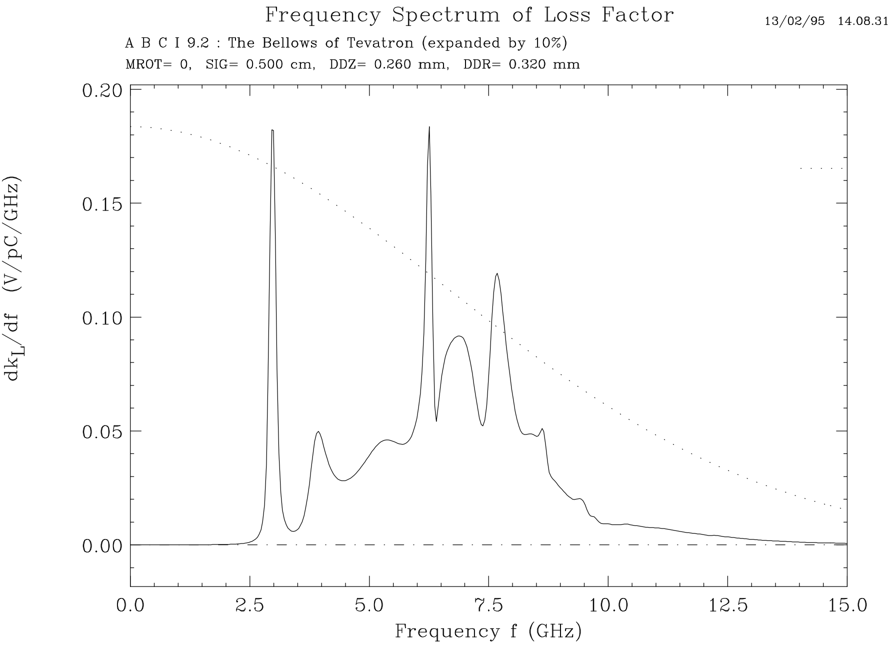
- There are numerical codes to compute wake potential of discontinuities in the vacuum chamber.
- For a cylindrical symmetric structure, there are the TBCI by T. Weiland and ABCI by Y. Chin, — they are called 2D codes.
  Both $m = 0$ and $m = 1$ modes can be computed.
- Without cylindrical symmetry, there is MAFIA by T. Weiland, with a very expensive license fee.
- The physical space is divided into grids, and the discontinuity is drawn along grid lines.
- The source cannot be a single particle.
  It can be approximated by a narrow bunch, for example, Gaussian truncated with 5 σ's.
- One need to specify the length of the wakes to be calculated.
  If one needs to compute impedances from the wakes by Fourier transformation, a rather long wake will be required.

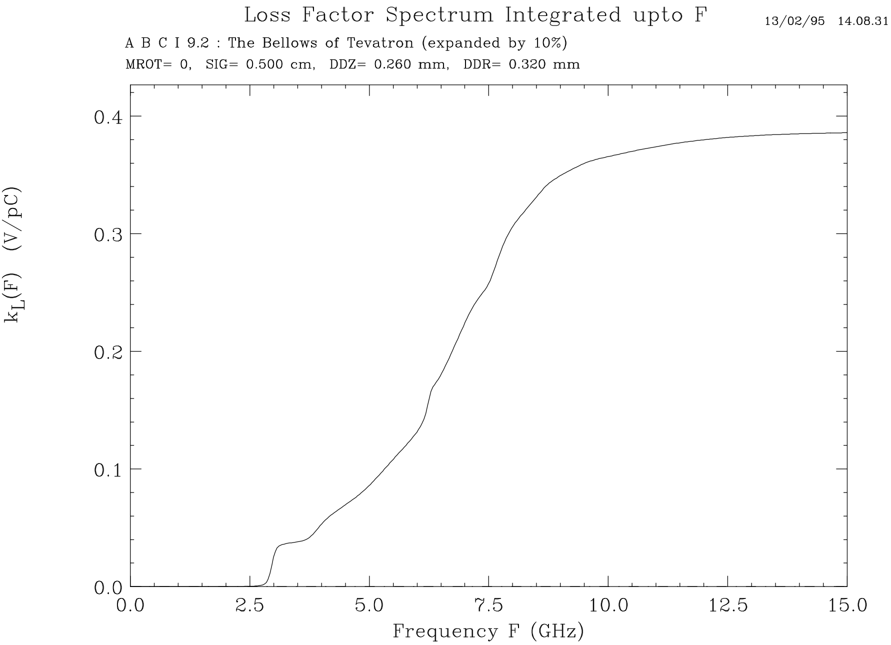

Cavity Shape Used
13/02/95 14.08.31
A B C I 9.2 : The Bellows of Tevatron (expanded by 10%)
DDZ= 0.260 mm, DDR= 0.320 mm
0.04
0.03
0.02
0.01
0.00
R−axis (m)
0.000
0.025
0.050
0.075
0.100
Z−axis (m)


Wake Potentials
Cpu Time Used: 1.495E+04(s)
13/02/95 14.08.31
A B C I 9.2 : The Bellows of Tevatron (expanded by 10%)
MROT= 0, SIG= 0.500 cm, DDZ= 0.260 mm, DDR= 0.320 mm
Charge Density
Longitudinal
Scaled Wake Potential W (S)
1.0
0.5
0.0
−0.5
−1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Distance from Bunch Head S (m)
Longitudinal Wake Min/Max= −5.711E−01/ 6.508E−01 V/pC, Loss Factor= −3.895E−01 V/pC


Real Part of Longitudinal Impedance
13/02/95 14.08.31
A B C I 9.2 : The Bellows of Tevatron (expanded by 10%)
MROT= 0, SIG= 0.500 cm, DDZ= 0.260 mm, DDR= 0.320 mm
Real Z_L (kΩ)
0.15
0.10
0.05
0.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
Frequency f (GHz)


Imaginary Part of Longitudinal Impedance
13/02/95 14.08.31
A B C I 9.2 : The Bellows of Tevatron (expanded by 10%)
MROT= 0, SIG= 0.500 cm, DDZ= 0.260 mm, DDR= 0.320 mm
Imag Z_L (kΩ)
0.05
0.00
−0.05
0.0
2.5
5.0
7.5
10.0
12.5
15.0
Frequency f (GHz)


Frequency Spectrum of Loss Factor
13/02/95 14.08.31
A B C I 9.2 : The Bellows of Tevatron (expanded by 10%)
MROT= 0, SIG= 0.500 cm, DDZ= 0.260 mm, DDR= 0.320 mm
dk_L/df (V/pC/GHz)
0.20
0.15
0.10
0.05
0.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
Frequency f (GHz)


Loss Factor Spectrum Integrated upto F
13/02/95 14.08.31
A B C I 9.2 : The Bellows of Tevatron (expanded by 10%)
MROT= 0, SIG= 0.500 cm, DDZ= 0.260 mm, DDR= 0.320 mm
$k_L(F)$ (V/pC)
0.0
0.1
0.2
0.3
0.4
0.0
2.5
5.0
7.5
10.0
12.5
15.0
Frequency F (GHz)

Landau Damping

- Wake force excites a number of collective waves in a beam and displaces it from its equilibrium position.
- These waves exchange energy among themselves, some grow and some are damped.
- Spread in oscillation frequency accelerates damping and decelerates growths.
- This is called *Landau Damping* [7, 8]

Illustration of Landau Damping [9]

- Consider transverse oscillation of a particle in an unbunched beam

$$\left[\left(\frac{\partial}{\partial t}+\omega_0\frac{\partial}{\partial\theta}\right)^2+\omega_\beta^2\right]y(\theta,t)=-\frac{eI_0}{\gamma mC_0}\int_{-\infty}^{\infty}dt'\,W_1(t'-t)\langle y(\theta,t')\rangle,$$

with $\langle y(\theta,t)\rangle=\int\rho(\omega_\beta)y(\theta,t;\omega_\beta)d\omega_\beta$

$$\boxed{\bar\omega=\int\omega_\beta\rho(\omega_\beta)d\omega_\beta}$$

- Go to frequency domain by introducing

$$\tilde y_n(\Omega)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}dt\int d\theta\,y(\theta,t)e^{-in\theta+i\Omega t}$$

snap-shot view

- Here, we solve as an *initial-value problem.* For $t<0$, $y(\theta,t)=0$.

  Beam receives a kick at $t=0$, or $\dfrac{\partial y(\theta,0)}{\partial t}=\sum_n\dot y_{n0}e^{in\theta}$.

- R.S. $=\dfrac{iecI_oZ_1^\perp(\Omega)}{E_0T_0}\langle\tilde y_n\rangle\equiv-2\bar\omega(\Delta\omega)_0\langle\tilde y_n\rangle$

$$\boxed{(\Delta\omega)_0 \text{ is } \begin{cases}\text{freq shift w/o}\\ \text{Landau damping}\end{cases}}$$

- L.S. $=(\omega_\beta^2-\hat\omega^2)\tilde y_n(\Omega)+\dfrac{1}{4\pi^2}\displaystyle\int d\theta\left[\frac{\partial y}{\partial t}-i(\Omega-2n\omega_0)y\right]e^{-in\theta+i\Omega t}\Bigg|_0^\infty$

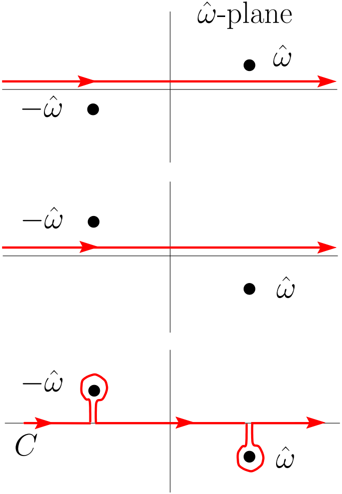
  $=(\omega_\beta^2-\hat\omega^2)\tilde y_n(\Omega)-\dfrac{\dot y_{n0}}{2\pi}$ ← for $\mathcal{I}m\,\Omega>0$ only.

$$\boxed{\hat\omega=\Omega-n\omega_0}$$

- Soln. for $\mathcal{I}m\,\Omega > 0$ (upper Ω-plane): $\tilde{y}_n(\Omega) = -\dfrac{2\bar{\omega}(\Delta\omega)_0}{\omega_\beta^2 - \hat{\omega}^2}\langle\tilde{y}_n\rangle + \dfrac{\dot{y}_{n0}}{2\pi(\omega_\beta^2 - \hat{\omega}^2)}$
- Now average over the whole beam by $\displaystyle\int d\omega_\beta\,\rho(\omega_\beta) \times \cdots$
- $$\left[1 + 2\bar{\omega}(\Delta\omega)_0 \int d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta^2 - \hat{\omega}^2}\right] \langle\tilde{y}_n\rangle = \frac{\dot{y}_{n0}}{2\pi} \int d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta^2 - \hat{\omega}^2}$$

  $\uparrow\ H(\Omega)$
- Thus $$\langle\tilde{y}_n(\Omega)\rangle = \frac{\dfrac{\dot{y}_{n0}}{2\pi}\displaystyle\int d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta^2 - \hat{\omega}^2}}{H(\Omega)} \qquad \boxed{\hat{\omega} = \Omega - n\omega_0}$$
- The inverse Fourier transform $\displaystyle\langle y(\theta, t)\rangle = \sum_n \int d\Omega\, \langle\tilde{y}_n(\Omega)\rangle e^{-i(\Omega t - n\theta)}$ gives solution in θ-t space.
- However, need to know $\langle\tilde{y}_n(\Omega)\rangle$ over the whole Ω-plane.
- Notice that in above $\langle\tilde{y}_n(\Omega)\rangle$ is discontinuous across $\mathcal{I}m\,\Omega = 0$
- $$\hat{\omega} = \hat{\omega}_R \pm i\epsilon \longrightarrow \int \frac{\rho(\omega_\beta)d\omega_\beta}{\omega_\beta - \hat{\omega}} = \int \frac{\rho(\omega_\beta)d\omega_\beta}{\omega_\beta - \hat{\omega}_R \mp i\epsilon} = \wp \int d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta - \hat{\omega}_R} \pm i\pi\rho(\hat{\omega}_R)$$

Analytic Continuation



- Reason of discontinuity is path of integration.
- Top: $\mathcal{I}m\,\Omega > 0$

  We know how the contour goes.
- Middle: $\mathcal{I}m\,\Omega < 0$ and $\langle \tilde{y}_n(\Omega) \rangle$ not defined.

  If same contour is followed, result will be different.

  Discontinuity occurs across $\mathcal{I}m\,\Omega = 0$.
- Bottom: If we follow Landau contour C,

  there will not be any discontinuity.
- Contour C is always below pole at $\hat{\omega}$ and above pole at $-\hat{\omega}$.

  This is what we mean by analytic continuation.

- $\langle y(\theta,t)\rangle = \sum_n \int_W d\Omega \frac{\dot{y}_{n0}\int_C d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta^2-\hat{\omega}^2}}{2\pi\, H(\Omega)} e^{-i\Omega t+in\theta}$

  where W is path above all singularities so that $\langle y(\theta,t)\rangle = 0$ at $t<0$.

- $\langle y(\theta,t)\rangle = \sum_k \mathrm{Res}\left[\sum_n \frac{-i\dot{y}_{n0}\int_C d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta^2-\hat{\omega}^2}}{H(\Omega)}\right]_{\Omega=\Omega_k} e^{-i\Omega_k t+in\theta}$,

  where Ω_k is k^{th} zero of $H(\Omega)$.

1. The poles of $H(\Omega)$ gives all waves excited
2. A pole with $\mathcal{I}m\,\Omega_k > 0$ implies growing
3. A pole with $\mathcal{I}m\,\Omega_k < 0$ implies damping
4. $\therefore$ it is essential to solve $H(\Omega) = 0$, which is known as the *dispersion relation*.

Dispersion Relation

- $H(\Omega) = 1 + 2\bar{\omega}(\Delta\omega)_0 \int_C d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta^2 - \hat{\omega}^2} = 0$ with $\hat{\omega} = \Omega - n\omega_0$
- Since $\pm\hat{\omega}$ are far apart, can linearize to get
  $$1 + (\Delta\omega)_0 \int_C d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta - \Omega + n\omega_0} = 0$$
- It gives excitation freq. Ω as a fcn. of wave number n, hence called *dispersion relation*.
- Can also write $1 + (\Delta\omega)_0 \int_C d\omega_\beta \frac{\rho(\omega_\beta)}{(\omega_\beta - \bar{\omega}) - (\Omega - n\omega_0 - \bar{\omega})} = 0$
- When spread in ω_β is small, $\omega_\beta - \bar{\omega} \to 0$, get $(\Delta\omega)_0 = \Omega - n\omega_0 - \bar{\omega}$, which is just betatron freq. shift *in the absence of Landau damping*.
- Note that this is *not* a coherent shift. It is the *shift driven by the wake or imp.* — called *dynamic shift* actually $\boxed{\text{dynamic shift} = \text{coherent shift} - \text{incoherent shift}}$

Coherent, Incoherent, and Impedance Forces

- Vertical force on a beam particle $\dfrac{d^2y}{ds^2}+\dfrac{\nu_{0y}^2}{R^2}\,y=\dfrac{F(y,\bar{y})}{\gamma mv^2}.$

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- Thus $\Delta\nu_{y\,\mathrm{inc}}\propto\dfrac{\partial F}{\partial y}\bigg|_{\bar{y}=0}$

$$\Delta\nu_{y\,\mathrm{coh}}\propto\frac{\partial F}{\partial y}\bigg|_{\bar{y}=0}+\frac{\partial F}{\partial \bar{y}}\bigg|_{y=0}$$

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  $\Delta\nu_{y\,\mathrm{coh}}\propto\dfrac{\partial F}{\partial y}\bigg|_{\bar{y}=0}+\dfrac{\partial F}{\partial \bar{y}}\bigg|_{y=0}$
- But $Z_1^\perp\propto\dfrac{\partial F}{\partial \bar{y}}\bigg|_{y=0},$
- $\therefore$ Impedance Shift = Coherent Shift − Incoherent Shift.

  ↑ also called *dynamic* shift

- $\Delta\nu_{y\,\mathrm{coh}}$: result of all forces acting on center of beam at $\bar{y}$.
- $Z_1^\perp$: force generated by center motion of beam on individual particle.

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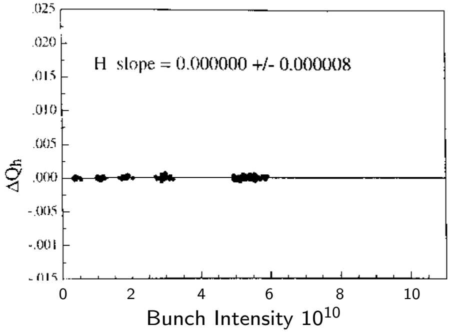
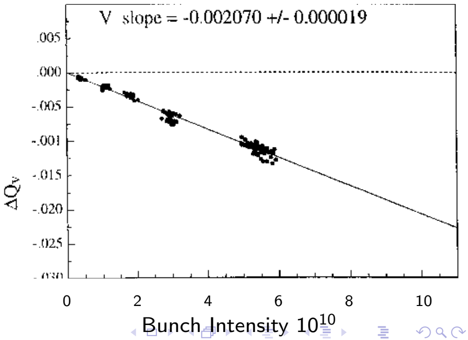
- $Z_1^{\perp}$: force generated by center motion of beam on individual particle.
- Example: a beam between two infinite horizontal conducting planes.
- Horizontal translational invariance $\Longrightarrow$ horizontal image force acting at center of beam vanishes independent of whether beam is oscillating horizontally or vertically. $\therefore \Delta\nu_{x\,\mathrm{coh}} = 0$, but $Z_1^{H} \neq 0$.

- $\Delta\nu_{y\,\mathrm{coh}}$: result of all forces acting on center of beam at $\bar{y}$.
- $Z_1^{\perp}$: force generated by center motion of beam on individual particle.
- Example: a beam between two infinite horizontal conducting planes.
- Horizontal translational invariance $\Longrightarrow$ horizontal image force acting at center of beam vanishes independent of whether beam is oscillating horizontally or vertically. $\therefore\ \Delta\nu_{x\,\mathrm{coh}} = 0$, but $Z_1^{H} \neq 0$.
- Single bunch tune shift measurement at CERN SPS. [5]





Decoherence vs Landau Damping

- Start with a displacement.

  Because particles oscillate with slightly different frequencies,

  very soon oscillations will not be in phase,

  and average displacement decays to zero.

- For decoherence, we do not need any force.

- This problem is easy to solve.

  However, let's go back to our derivation, and see what happen when the force is zero.

- $$\langle y(\theta,t)\rangle = \sum_n \int_W d\Omega \frac{\dot{y}_{n0} \int_C d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta^2 - \hat{\omega}^2}}{2\pi\, H(\Omega)} e^{-i\Omega t + in\theta}$$

  $$\langle y(\theta,t)\rangle = \sum_n \frac{\dot{y}_{n0}}{2\pi} e^{in(\theta-\omega_0 t)} \int_C d\omega_\beta \int_W d\hat{\omega} \frac{\rho(\omega_\beta)}{\omega_\beta^2 - \hat{\omega}^2} e^{-i\hat{\omega} t}$$

- We integrate over $\hat{\omega}$ by picking up the residue of two poles.

- For $t \geq 0$, $\langle y(\theta,t)\rangle = \sum_n \frac{\dot{y}_{n0}}{2\pi} e^{in(\theta-\omega_0 t)} \int d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta} \sin\omega_\beta t$
- We see that at the impact of a velocity,

  the average displacement increases from zero to a maximum,

  then decays $\longrightarrow$ decoheres.
- Want to check result at $t = 0$. For $t \geq 0$,

$$\langle \dot{y}(\theta,t)\rangle = \sum_n \frac{\dot{y}_{n0}}{2\pi} e^{in(\theta-\omega_0 t)} \int d\omega_\beta \rho(\omega_\beta) \cos\omega_\beta t$$

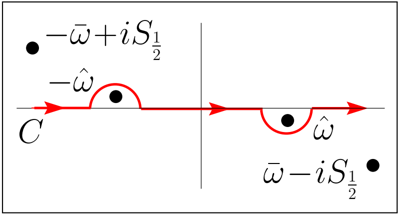
$$- \sum_n \frac{in\omega_0 \dot{y}_{n0}}{2\pi} e^{in(\theta-\omega_0 t)} \int d\omega_\beta \frac{\rho(\omega_\beta)}{\omega_\beta} \sin\omega_\beta t.$$

- Thus $\langle \dot{y}(\theta,t)\rangle = \sum_n \frac{\dot{y}_{n0}}{2\pi} e^{in\theta} = \dot{y}(\theta,0)$ as required.
- Notice that Landau damping requires a dynamic force.

  Here, it is the wake force or impedance.
- So decoherence is kinematic, while Landau damping is dynamic.

A Solvable Example [9]

- Consider the Lorentzian distribution $\rho(\omega_\beta)=\frac{S_{\frac{1}{2}}}{\pi}\frac{1}{(\omega_\beta-\hat{\omega})^2+{S_{\frac{1}{2}}}^2}$
- $\int_C\frac{\rho(\omega_\beta)d\omega_\beta}{\omega_\beta^2-\hat{\omega}^2}=\frac{S_{\frac{1}{2}}}{\pi}\int_C\frac{d\omega_\beta}{(\omega_\beta-\hat{\omega})(\omega_\beta+\hat{\omega})(\omega_\beta-\bar{\omega}+iS_{\frac{1}{2}})(\omega_\beta-\bar{\omega}-iS_{\frac{1}{2}})}$



- Can choose closing contour at top or bottom of ω_β-plane.
- Pick up only 2 poles.
- The moral of this example is easy integration.
- $\int_C\frac{\rho(\omega_\beta)d\omega_\beta}{\omega_\beta^2-\bar{\omega}^2}=-\frac{\hat{\omega}+iS_{\frac{1}{2}}}{\hat{\omega}(\hat{\omega}-\bar{\omega}+iS_{\frac{1}{2}})(\hat{\omega}+\bar{\omega}+iS_{\frac{1}{2}})}$
- $H(\Omega)=1+2\bar{\omega}(\Delta\omega)_0\int_C\frac{\rho(\omega_\beta)d\omega_\beta}{\omega_\beta^2-\bar{\omega}^2}$

$$=\frac{\hat{\omega}(\hat{\omega}-\bar{\omega}+iS_{\frac{1}{2}})(\hat{\omega}+\bar{\omega}+iS_{\frac{1}{2}})-2\bar{\omega}(\Delta\omega)_0(\hat{\omega}+iS_{\frac{1}{2}})}{\hat{\omega}(\hat{\omega}-\bar{\omega}+iS_{\frac{1}{2}})(\hat{\omega}+\bar{\omega}+iS_{\frac{1}{2}})}$$

- $\langle y(\theta,t)\rangle = \sum_n \dot{y}_{n0}\int_W \frac{d\hat\omega}{2\pi}\frac{(\hat\omega+iS_{\frac{1}{2}})e^{-i(\hat\omega+n\omega_0)t+in\theta}}{\hat\omega(\hat\omega-\bar\omega+iS_{\frac{1}{2}})(\hat\omega+\bar\omega+iS_{\frac{1}{2}})-2\bar\omega(\Delta\omega)_0(\hat\omega+iS_{\frac{1}{2}})}$
- Approximation: freq. shift $(\Delta\omega)_0$ and freq. spread $S_{\frac{1}{2}}$ are small

  $\frac{|(\Delta\omega)_0|}{\bar\omega} \ll 1$ and $\frac{S_{\frac{1}{2}}}{\bar\omega} \ll 1$
- $$\begin{aligned}\text{denom} &= \hat\omega\left[\hat\omega-\bar\omega-(\Delta\omega)_0+iS_{\frac{1}{2}}\right]\left[\hat\omega+\bar\omega+(\Delta\omega)_0+iS_{\frac{1}{2}}\right]+\left[-2i\bar\omega(\Delta\omega)_0 S_{\frac{1}{2}}+\hat\omega(\Delta\omega)_0^2\right]\\ &\approx \hat\omega\left[\hat\omega-\bar\omega-(\Delta\omega)_0+iS_{\frac{1}{2}}\right]\left[\hat\omega+\bar\omega+(\Delta\omega)_0+iS_{\frac{1}{2}}\right]\end{aligned}$$
- 3 solutions:

  $$\Omega = \begin{cases} n\omega_0 \\ n\omega_0+\bar\omega+(\Delta\omega)_0-iS_{\frac{1}{2}} \\ n\omega_0-\bar\omega-(\Delta\omega)_0-iS_{\frac{1}{2}} \end{cases} = \begin{cases} n\omega_0 \\ n\omega_0+\bar\omega+(\Delta\omega)_{0R}-i(S_{\frac{1}{2}}-(\Delta\omega)_{0I}) \\ n\omega_0-\bar\omega-(\Delta\omega)_{0R}-i(S_{\frac{1}{2}}+(\Delta\omega)_{0I}) \end{cases}$$
- 1st soln. consists of higher-harmonic revolution waves

  without freq. perturbation and is of not much interest.
- 2nd and 3rd are the same physically.

  We will show their combined result.

- $\langle y(\theta,t)\rangle = \sum_n \dfrac{-i\dot{y}_{n0}}{2\bar{\omega}} e^{in\theta - i[n\omega_0+\bar{\omega}+(\Delta\omega)_{0R}]t} e^{-[S_{\frac{1}{2}}-(\Delta\omega)_{0I}]t}$

  with $(\Delta\omega)_{0R} = \dfrac{ecI_0 \mathcal{I}m\, Z_1^{\perp}(\Omega)}{2\bar{\omega}E_0 T_0}$, $(\Delta\omega)_{0I} = -\dfrac{ecI_0 \mathcal{R}e\, Z_1^{\perp}(\Omega)}{2\bar{\omega}E_0 T_0}$, $\Omega = n\omega_0 + (\Delta\omega)_0$
- Distr. spread $S_{\frac{1}{2}}$ always provides damping, but $(\Delta\omega)_{0I} \gtrless 0$ grows/damps.
- Since $\mathcal{R}e\, Z_1^{\perp}(\Omega) \gtrless 0$ for $\Omega \gtrless 0$,

  damping/growth is related to $\Omega \gtrless 0$ or fast/slow waves.
- If $S_{\frac{1}{2}} \gtrless |(\Delta\omega)_{0I}|$, slow waves damp/grow.
- This mechanism is called *Landau Damping*.

- $\langle y(\theta,t)\rangle = \sum_n \frac{-i\dot{y}_{n0}}{2\bar{\omega}} e^{in\theta - i[n\omega_0+\bar{\omega}+(\Delta\omega)_{0R}]t} e^{-[S_{\frac{1}{2}}-(\Delta\omega)_{0I}]t}$

  with $(\Delta\omega)_{0R} = \frac{ecI_0 \mathcal{I}m\, Z_1^{\perp}(\Omega)}{2\bar{\omega}E_0T_0}$, $(\Delta\omega)_{0I} = -\frac{ecI_0 \mathcal{R}e\, Z_1^{\perp}(\Omega)}{2\bar{\omega}E_0T_0}$, $\Omega = n\omega_0 + (\Delta\omega)_0$
- Distr. spread $S_{\frac{1}{2}}$ always provides damping, but $(\Delta\omega)_{0I} \gtrless 0$ grows/damps.
- Since $\mathcal{R}e\, Z_1^{\perp}(\Omega) \gtrless 0$ for $\Omega \gtrless 0$,

  damping/growth is related to $\Omega \gtrless 0$ or fast/slow waves.
- If $S_{\frac{1}{2}} \gtrless |(\Delta\omega)_{0I}|$, slow waves damp/grow.
- This mechanism is called *Landau Damping*.
- Note that even $\langle y(\theta,t)\rangle$ is damped to zero,

  displacements of individual particles are not. ⟵important
- In practice, any small initial ripples of the beam center will be damped

  if $\boxed{S_{\frac{1}{2}} > |(\Delta\omega)_{0I}|}$, so that $\mathcal{R}e\, Z_1^{\perp}$ *cannot* drive individual displacements.
- In other words, Landau damping nips any instability growth in the bud.

Stability Region Plot [10]

- Spread in particle distribution provides Landau damping.
- For a given particle distribution $\rho(\omega_\beta)$, wish to know what $Z_1^\perp$ will ensure beam stability — the so-called *stability contour*.
- For this, need to solve the *dispersion relation*

$$1+(\Delta\omega)_0\int_C d\omega_\beta\frac{\rho(\omega_\beta)}{\omega_\beta-\Omega+n\omega_0}=0 \qquad \boxed{\hat{\omega}=\Omega-n\omega_0}$$

- We believe the HWHM provides a better description of distribution than *half width* or *rms width*.

  Want to express all variables in terms of $S_{\frac{1}{2}}$.
- Let $u=\dfrac{\hat{\omega}-\bar{\omega}}{S_{\frac{1}{2}}}$, $\quad v=\dfrac{\omega_\beta-\bar{\omega}}{S_{\frac{1}{2}}}$, $\quad \hat{\rho}(v)=S_{\frac{1}{2}}\rho(\omega_\beta)$ so that $\displaystyle\int\hat{\rho}(v)dv=1$
- We are interested in the *stability contour* or the threshold of instability. So let $i\,\mathcal{I}m\,\hat{\omega}\longrightarrow +i\epsilon$ with $\epsilon=0^+$
- The same as letting $u\longrightarrow u+i\epsilon$ with u real.

- Dispersion relation becomes $1+\frac{(\Delta\omega)_0}{S_{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{\hat\rho(v)dv}{v-u-i\epsilon}=0$

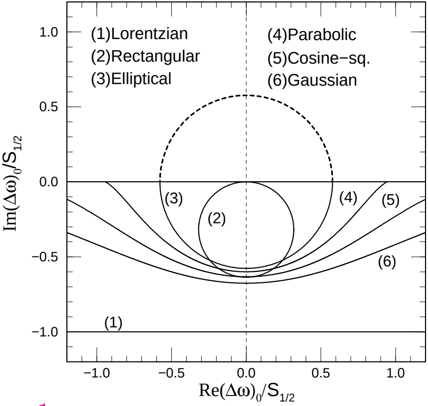
  or $1+\frac{(\Delta\omega)_0}{S_{\frac{1}{2}}}\left[\wp\int_{-\infty}^{\infty}\frac{\hat\rho(v)dv}{v-u}+i\pi\hat\rho(u)\right]=0$

1. Given $\hat\rho(v)$ and pick a u
2. Solve for $\frac{\mathcal{R}e(\Delta\omega)_0}{S_{\frac{1}{2}}}$ and $\frac{\mathcal{I}m(\Delta\omega)_0}{S_{\frac{1}{2}}}$
3. Vary u and trace out the contour.

One point on the contour is easy to solve. When $u=0$, $\wp\int=0\longrightarrow$

$\frac{\mathcal{R}e(\Delta\omega)_0}{S_{\frac{1}{2}}}=0,\ \frac{\mathcal{I}m(\Delta\omega)_0}{S_{\frac{1}{2}}}=-\frac{1}{\pi\hat\rho(0)}$

This is intercept on vertical axis and is roughly $-1/\sqrt{3}$

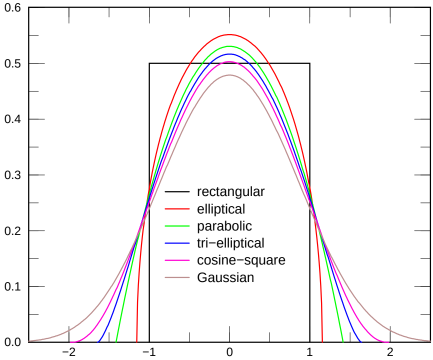


- Simplified stability criterion $\left|(\Delta\omega)_0\right|<\frac{1}{\sqrt{3}}S_{\frac{1}{2}}F_\perp$

  $F_\perp=1.103,\ 1,\ 1.040,\ 1.068,\ 1.097,\ 1.174$ for distr's (2), (3), (4), (5), (6)

Generalized elliptical distribution

- $\hat{\rho}(v) = \frac{A_n}{a_n}\left(1 - \frac{v^2}{a_n^2}\right)^n H(a_n - v)$
- By choosing various n's can reproduce nearly all distributions, from rectangular to Gaussian.
- Can approximate *cosine-square* with $n = 2.36$
- Choose $A_n = \frac{\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{\pi}\gamma(n+1)}$ for normalization.
- Choose $a_n = \frac{1}{\sqrt{1-2^{-1/n}}}$ so that $v=1$ is the HWHM.
- Trans. stability form factor $F_\perp = \frac{\sqrt{3}a_n}{\pi A_n}$



n	distribution	$F_\perp$
0	rectangular	1.103
$\frac{1}{2}$	elliptical	1.000
1	parabolic	1.040
$\frac{3}{2}$	tri-elliptical	1.068
2.36	cosine square	1.097
∞	Gaussian	1.174

Tune Spread Dependence on Momentum Spread

- Spread in betatron tune can come from mom. spread: $\Delta\nu = \xi\delta$
- However, rev. freq. also has a spread from mom. spread: $\Delta\omega_0 = -\eta\bar{\omega}_0\delta$
- Recall the dispersion relation $1 + (\Delta\omega)_0 \displaystyle\int \frac{\rho(\omega_\beta)d\omega_\beta}{\omega_\beta - (\Omega - n\omega_0) - i\epsilon} = 0$
- $\omega_\beta - (\Omega - n\omega_0) = (n+\nu)\omega_0 - \Omega = \left[(n+\nu)\omega_0 - \overline{(n+\nu)\omega_0}\right] - \left[\Omega - \overline{(n+\nu)\omega_0}\right]$

$$= \Delta\left[(n+\nu)\omega_0\right] - \Delta\Omega$$

- $\Delta\left[(n+\nu)\omega_0\right] = \xi\delta\bar{\omega}_0 - (n+\nu)\eta\bar{\omega}_0\delta = \left[\underset{\substack{\uparrow\\ \text{chromaticity}}}{\xi} - \eta(n+\bar{\nu})\right]\delta\bar{\omega}_0 \equiv \underset{\substack{\uparrow\\ \text{eff. chromaticity}}}{\xi_{\rm eff}}\delta\bar{\omega}_0$
- Let $v = \dfrac{\delta}{\delta_{\frac{1}{2}}}, \quad u = \dfrac{\Delta\Omega}{\xi_{\rm eff}\bar{\omega}_0\delta_{\frac{1}{2}}}$
- Dispersion relation becomes $1 + \dfrac{(\Delta\omega)_0}{\xi_{\rm eff}\bar{\omega}_0\delta_{\frac{1}{2}}} \displaystyle\int \frac{\hat{\rho}(v)dv}{v - u + j\epsilon}$ $\boxed{\xi_{\rm eff}\bar{\omega}_0\delta_{\frac{1}{2}} \leftrightarrow S_{\frac{1}{2}}}$
- Thus the stability contour will be exactly the same as before and the simplified stability criterion is

$$\left|\frac{(\Delta\omega)_0}{\xi_{\rm eff}\bar{\omega}_0\delta_{\frac{1}{2}}}\right| < \frac{1}{\sqrt{3}}F_\perp \quad \text{or} \quad \left|Z_1^\perp\right| < \frac{4\pi\beta E_0\xi_{\rm eff}}{\sqrt{3}eI_0\beta_\perp}\delta_{\frac{1}{2}}F_\perp$$

Transverse Oscillation of a Point Bunch [10]

- A bunch will only be affected by its own wake turn-after-turn ago.

$$\ddot{y}(t)+\omega_\beta^2 y(t)=-\frac{Ne^2}{\gamma m C_0}\sum_{k=1}^{\infty}\langle y(t-kT_0)\rangle W_1(-kT_0)$$

- Solve as initial-value problem in the freq. space by $\displaystyle\int_0^\infty \frac{dt}{2\pi}e^{i\Omega t}\times\cdots$
- For the upper Ω-plane ($\mathcal{I}m\,\Omega>0$),

$$\text{L.S.}=\left(-\Omega^2+\omega_\beta^2\right)\tilde{y}(\Omega)+\frac{\dot{y}_0-i\Omega y_0}{2\pi}\quad\text{with}\quad\tilde{y}(\Omega)\equiv\int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{i\Omega t}y(t)$$

- $\displaystyle \text{R.S.}=-\frac{Ne^2}{\gamma m C_0}\int\frac{dt}{2\pi}e^{i\Omega t}\sum_k\left[\int d\omega e^{i\omega(t-kT_0)}\langle\tilde{y}(\omega')\rangle\right]\left[-i\beta\int\frac{d\omega'}{2\pi}Z_1^\perp(\omega')e^{i\omega' kT_0}\right]$
- $\displaystyle \sum_k e^{-i\omega kT_0+i\omega' kT_o}=2\pi\sum_p\delta(\omega' T_0-\omega T_0-2\pi p)=\frac{2\pi}{T_0}\sum_p\delta(\omega'-\omega-p\omega_0)$
- $\displaystyle\int dt\,e^{i\Omega t-i\omega t}=2\pi\delta(\omega-\Omega)$ Next integrate over $d\omega'$ and $d\omega$
- For $\mathcal{I}m\,\Omega>0$, $\displaystyle\left(-\Omega^2+\omega_\beta^2\right)\tilde{y}(\Omega)-\frac{\dot{y}_0-i\Omega y_0}{2\pi}=\frac{i\beta Ne^2\mathcal{W}}{\gamma m C_0}\langle\tilde{y}(\Omega)\rangle$

- For the *upper* Ω-plane ($\mathcal{I}m\,\Omega > 0$),

  $\left(-\Omega^2 + \omega_\beta^2\right)\tilde{y}(\Omega) - \dfrac{\dot{y}_0 - i\Omega y_0}{2\pi} = \dfrac{i\beta Ne^2\mathcal{W}}{\gamma m C_0}\langle\tilde{y}(\Omega)\rangle \equiv -2\bar{\omega}(\Delta\omega)_0\langle\tilde{y}(\Omega)\rangle$, where

  $(\Delta\omega)_0 = -\dfrac{i\beta Ne^2\mathcal{W}}{2\bar{\omega}\gamma m C_0}$ is freq. shift without Landau damping

  and $\mathcal{W} = \dfrac{1}{T_0}\displaystyle\sum_p Z_1^\perp(p\omega_0 + \Omega)$
- So problem is the same as that of

  transverse oscillation of an unbunched beam.

  The dispersion relation is exactly the same.

  Only difference is the $(\Delta\omega)_0$.

Longitudinal Oscillation of an Extremely Short Bunch

- Let z be position of a particle ahead of some synchronous particle.

$$\frac{dz}{ds}=-\eta\delta, \qquad \frac{d\delta}{ds}=\frac{\omega_s^2}{\eta v^2}z-\frac{Ne^2}{\beta^2E_0C_0}\sum_k W_0'\left[-kC_0-\langle z(s-kC_0)\rangle+z(s)\right]$$

- Here we use s as indep. variable, because time for a rev. varies due to syn. oscillation. v is velocity of synchronous particle.

- $$\frac{d^2z(s)}{ds^2}+\frac{\omega_s^2}{v^2}z(s)=\frac{\eta Ne^2}{\beta^2E_0C_0}\sum_k W_0'\left[-kC_0-\langle z(s-kC_0)\rangle+z(s)\right]$$
$$\approx-\frac{\eta Ne^2}{\beta^2E_0C_0}\sum_k\left[\langle z(s-kC_0)\rangle-\underset{\uparrow}{z(s)}\right]W_0''(-kC_0)$$

pot.-well distortion, neglect here

- Solve as initial-value problem in freq. space via $\displaystyle\int_0^\infty\frac{ds}{2\pi}e^{i\Omega s/v}\times\cdots$.

- $$\mathcal{I}m\,\Omega>0\rightarrow\frac{\omega_s^2-\Omega^2}{v^2}\tilde z(\Omega)+\frac{z_0'-i\Omega z_0/v}{2\pi}=-\frac{i\eta Ne^2\mathcal{W}}{\beta^2E_0C_0^2}\langle\tilde z(\Omega)\rangle\equiv-\frac{2\bar\omega_s(\Delta\omega)_0}{v^2}\langle\tilde z(\Omega)\rangle$$

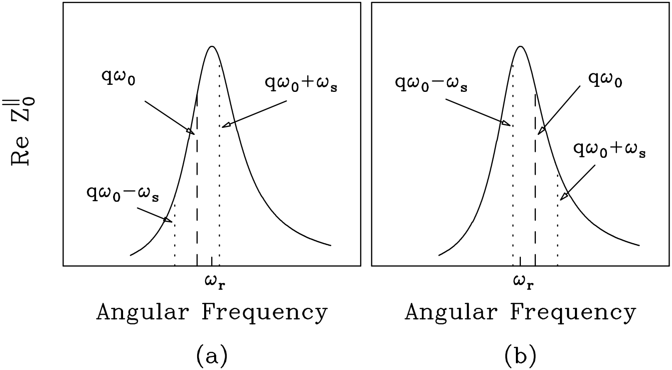
where $(\Delta\omega)_0=\dfrac{i\eta Ne^2\mathcal{W}}{2\bar\omega_s\beta^2E_0T_0^2}$ is freq. shift without Landau damping

and $\boxed{\mathcal{W}=\sum_p(p\omega_0+\Omega)Z_0^{\|}(p\omega_0+\Omega)}-\sum_p p\omega_0Z_0^{\|}(p\omega_0)$ ← *potential well distortion*

Robinsion Instabilities [16]

- When imp. is a narrow resonance centered at ω_r near $q\omega_0$, only 2 syn. sidebands at $q\omega_0 \pm \omega_s$ contribute.
- Without any spread in ω_s or Landau damping, growth rate is

$$\frac{1}{\tau} = \mathcal{I}m(\Delta\omega)_0 = \frac{\eta N e^2 \omega_r}{2\omega_s \beta^2 E_0 T_0^2}\left[\mathcal{R}e\, Z_0^{\parallel}(q\omega_0+\omega_s) - \mathcal{R}e\, Z_0^{\parallel}(q\omega_0-\omega_s)\right]$$



- Above transition ($\eta > 0$), (a) is anti-damped and (b) is damped. Need to detune ω_r to lower than $q\omega_0$ for stability.

- It is clear that growth rate will be severe when resonance is very narrow.
- Example, $\eta = 0.03$, $N = 10^{11}$, $E_0 = 1$ GeV, $\omega_0 = 9.4 \times 10^6\ \mathrm{s}^{-1}$, $\nu_s = 0.01$, $R_s = 1\ \mathrm{M}\Omega$, $h\omega_0/2\pi = 360$ MHz, $Q = 2000$. If rf is detuned to $(h\omega - \omega_r)/2\pi = -10$ kHz,

  growth time is $\tau = 1.2$ ms, which is rather strong.
- For stability, detune lower above transition,

  detune higher below transition.
- Electron rings has rf cavities of relatively higher Q, and larger η; Robinson stability is an important issue.
- Proton rings have rf cavities of relatively lower Q and smaller η; Robinson instability will be relative mild, even if cavities are detuned incorrectly in the wrong direction.
- For example, people were uncertain about the detuning of every rf cavity in the Tevatron at some time.

Transverse Robinson Instabilities

- Recall that for a point bunch, the soln. of equation of motion is:

  For the upper Ω-plane ($\mathcal{I}m\,\Omega > 0$),

  $$\left(-\Omega^2+\omega_\beta^2\right)\tilde{y}(\Omega)-\frac{\dot{y}_0-i\Omega y_0}{2\pi}=\frac{i\beta Ne^2\mathcal{W}}{\gamma m C_0}\langle\tilde{y}(\Omega)\rangle\equiv-2\bar{\omega}(\Delta\omega)_0\langle\tilde{y}(\Omega)\rangle,$$ where

  $$(\Delta\omega)_0=-\frac{i\beta Ne^2}{2\bar{\omega}\gamma m C_0 T_0}\sum_p Z_1^\perp(p\omega_0+\Omega)$$

- So in the absence of Landau damping or spread in ω_β,

  $\mathcal{I}m(\Delta\omega)_0$ is the growth rate.

- When the impedance is a narrow resonance near $q\omega_0$,

  only the sidebands $(q\pm[\nu])\omega_0$ contribute.

  Again stability is decided by whether the resonance peak ω_r is above or below $q\omega_0$.

- growth rate is

  $$\frac{1}{\tau}=\mathcal{I}m(\Delta\omega)_0=\frac{\beta Ne^2c^2\beta_\perp}{2E_0C_0^2}\Big[\mathcal{R}e\,\underset{\substack{\uparrow\\ \text{slow wave}}}{Z_1^\perp}(p\omega_0-\omega_\beta)-\mathcal{R}e\,\underset{\substack{\uparrow\\ \text{fast wave}}}{Z_1^\perp}(p\omega_0+\omega_\beta)\Big]$$

Longitudinal instabilities an unbunched beam [11]

- In the last 3 cases, there is an intrinsic oscillation to start with.

  However, for an unbunched beam there is no syn. oscillation to start with

  The derivation of dispersion relation and stability criterion will be very much different.

- We start with a beam with uniform linear density λ_0 Let us envision a small perturbation λ_1 of harmonic n and collective freq. Ω:

  $$\lambda(\theta, t) = \lambda_0 + \lambda_1 e^{-i(\Omega t - n\theta)}.$$

  $$\boxed{\frac{\lambda_1}{\lambda_0} \ll 1}$$

- Rev. freq. will also be perturbed:
  $\omega(\theta, t) = \omega_0 + \omega_1 e^{-i(\Omega t - n\theta)}.$

- Continuity eq.: $\dfrac{\partial \lambda}{\partial t} + \dfrac{\partial}{\partial \theta}(\lambda\omega) = 0 \longrightarrow \omega_1 = \left(\dfrac{\Omega}{n} - \omega_0\right)\dfrac{\lambda_1}{\lambda_0}.$

- The current $I = \lambda\omega$ is also perturbed to

  $I(\theta, t) = I_0 + I_1 e^{-i(\Omega t - n\theta)}$ with $I_1 = \dfrac{\Omega\lambda_1}{n}.$

  $\left\{\begin{array}{l}\text{just linear density } \lambda_1 \times \\ \text{angular velocity } \Omega/n\end{array}\right.$

- $\therefore$ λ_1 leads to the determination of ω_1 and I_1.

- I_1 interacts $Z_0^{\parallel}$, changes E by $\dfrac{dE}{dt}=-\dfrac{eI_1 Z_0^{\parallel}}{T_0}$, then ω by $\dfrac{d\omega}{dE}=-\dfrac{\eta\omega_0}{\beta^2 E}$.
- Riding on a particle, one sees only the rev. freq. change due to energy change, or $\dfrac{d\omega}{dt}=\dfrac{\partial\omega}{\partial t}+\dfrac{\partial\omega}{\partial\theta}\dfrac{\partial\theta}{\partial t}=\dfrac{d\omega}{dE}\dfrac{dE}{dt}$.
- Thus $\dfrac{d\omega}{dt}=-i(\Omega-n\omega_0)\omega_1=\dfrac{\eta eI_1 Z_0^{\parallel}}{2\pi\beta^2 E_0}\omega_0^2$, $\boxed{\omega_1=\left(\dfrac{\Omega}{n}-\omega_0\right)\dfrac{\lambda_1}{\lambda_0}}$

- I_1 interacts $Z_0^{\parallel}$, changes E by $\dfrac{dE}{dt}=-\dfrac{eI_1Z_0^{\parallel}}{T_0}$, then ω by $\dfrac{d\omega}{dE}=-\dfrac{\eta\omega_0}{\beta^2E}$.
- Riding on a particle, one sees only the rev. freq. change due to energy change, or $\dfrac{d\omega}{dt}=\dfrac{\partial\omega}{\partial t}+\dfrac{\partial\omega}{\partial\theta}\dfrac{\partial\theta}{\partial t}=\dfrac{d\omega}{dE}\dfrac{dE}{dt}$.
- Thus $\dfrac{d\omega}{dt}=-i(\Omega-n\omega_0)\omega_1=\dfrac{\eta eI_1Z_0^{\parallel}}{2\pi\beta^2E_0}\omega_0^2$, $\boxed{\omega_1=\left(\dfrac{\Omega}{n}-\omega_0\right)\dfrac{\lambda_1}{\lambda_0}}$
- **Physical meaning:**
  1. I_1 interacts with $Z_0^{\parallel}$ to give $V=I_1Z_0^{\parallel}$, creating n buckets along the ring.
  2. Inside a bucket, executes syn. freq.: $\omega_s^2=\dfrac{i\eta neI_1Z_0^{\parallel}}{2\pi\beta^2E_0}\omega_0^2$. $\boxed{-\dfrac{\eta heV_{\rm RF}\cos\phi_s}{2\pi\beta^2E_0}}$
  3. Voltage and current are 90° out of phase, therefore the factor i
  4. Particle inside a bucket moves azimuthal angle $\Delta\theta=\dfrac{\pi}{n}$ in $\Delta t=\dfrac{\pi}{\omega_s}$, or a perturbed rev. freq. $\Delta\omega\sim\dfrac{\omega_s}{n}e^{-i\omega_st}$ $\boxed{\text{bucket width }2\pi/n}$.
  5. Thus $\dfrac{d\omega}{dt}=-i\dfrac{\omega_s^2}{n}$.

- I_1 interacts $Z_0^{\parallel}$, changes E by $\dfrac{dE}{dt}=-\dfrac{eI_1Z_0^{\parallel}}{T_0}$, then ω by $\dfrac{d\omega}{dE}=-\dfrac{\eta\omega_0}{\beta^2E}$.
- Riding on a particle, one sees only the rev. freq. change due to energy change, or $\dfrac{d\omega}{dt}=\dfrac{\partial\omega}{\partial t}+\dfrac{\partial\omega}{\partial\theta}\dfrac{\partial\theta}{\partial t}=\dfrac{d\omega}{dE}\dfrac{dE}{dt}$.
- Thus $\dfrac{d\omega}{dt}=-i(\Omega-n\omega_0)\omega_1=\dfrac{\eta eI_1Z_0^{\parallel}}{2\pi\beta^2E_0}\omega_0^2,$ $\boxed{\omega_1=\left(\dfrac{\Omega}{n}-\omega_0\right)\dfrac{\lambda_1}{\lambda_0}}$
- **Physical meaning:**
    1. I_1 interacts with $Z_0^{\parallel}$ to give $V=I_1Z_0^{\parallel}$, creating n buckets along the ring.
    2. Inside a bucket, executes syn. freq.: $\omega_s^2=\dfrac{i\eta neI_1Z_0^{\parallel}}{2\pi\beta^2E_0}\omega_0^2.$ $\boxed{-\dfrac{\eta heV_{\rm RF}\cos\phi_s}{2\pi\beta^2E_0}}$
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    5. Thus $\dfrac{d\omega}{dt}=-i\dfrac{\omega_s^2}{n}$.
- From above, $(\Omega-n\omega_0)^2=\dfrac{in\eta eI_1Z_0^{\parallel}}{2\pi\beta^2E_0}\dfrac{\lambda_0}{\lambda_1}\omega_0^2=\dfrac{in\eta eI_0Z_0^{\parallel}}{2\pi\beta^2E_0}\omega_0^2.$

- Finally $(\Delta\omega)_0^2 \equiv (\Omega - n\omega_0)^2 = \dfrac{in\eta I_0 Z_0^\parallel}{2\pi\beta^2 E_0}\omega_0^2$.
- We identify $(\Delta\omega)_0$ as shift in rev. freq. due to $Z_0^\parallel$ in the absence of Landau damping.
- Dispersion relation is slightly different: $1 - (\Delta\omega)_0^2 \int_C \dfrac{\rho(\omega_0)d\omega_0}{(\Omega - n\omega_0)^2} = 0$.
- Landau damping comes from variation of ω_0 inside the unbunched beam.
- Let $v = \dfrac{\omega_0 - \bar{\omega}_0}{S_{\frac{1}{2}}}$, $\hat{\rho}(v) = S_{\frac{1}{2}}\rho(\omega_0)$

  so that $\displaystyle\int \hat{\rho}(v)dv = 1$ and $v = 1$ is at HWHM.
- Further let $u = \dfrac{\Omega/n - \bar{\omega}_0}{S_{\frac{1}{2}}}$.

  Dispersion relation becomes $1 - \dfrac{(\Delta\omega)_0^2}{n^2 {S_{\frac{1}{2}}}^2} \displaystyle\int_C \frac{\hat{\rho}(v)dv}{(v-u)^2} = 0$.
- Integration by parts: $1 - \dfrac{(\Delta\omega)_0^2}{n^2 {S_{\frac{1}{2}}}^2} \displaystyle\int_C \frac{\hat{\rho}'(v)dv}{v-u} = 0$.

- To obtain the stability contour, let $u \to u + i\epsilon$, with u real and $\epsilon = 0^+$.

  Then what needs to be solve is $$1 - \frac{(\Delta\omega)_0^2}{n^2 S_{\frac{1}{2}}{}^2} \int_C \frac{\hat{\rho}'(v)dv}{v - u - i\epsilon} = 0$$

- Spread in ω_0 usually comes from spread in momentum δ,

  then $S_{\frac{1}{2}} = |\eta|\bar{\omega}_0 \delta_{\frac{1}{2}}$.

- Finally obtain $$1 - \frac{2i\,\mathrm{sgn}(\eta)}{\pi}(U + iV) \int \frac{\hat{\rho}'(v)dv}{v - u - i\epsilon} = 0$$

  with $$U + iV = \frac{eI_0}{4|\eta|\beta^2 E_0 \delta_{\frac{1}{2}}^2} \frac{Z_0^{\parallel}(n\bar{\omega}_0)}{n}, \quad v = \frac{\delta}{\delta_{\frac{1}{2}}}, \quad u = \frac{1}{|\eta|\delta_{\frac{1}{2}}}\left(\frac{\Omega}{n\omega_0} - 1\right)$$

- $$1 - \frac{2i\,\mathrm{sgn}(\eta)}{\pi}(U + iV)\left[\wp \int \frac{\hat{\rho}'(v)dv}{v - u} + i\pi\hat{\rho}'(u)\right] = 0.$$

- One point is easy to solve: $\hat{\rho}'(u) = 0$ when $u = 0$

  This is intercept on V-axis:

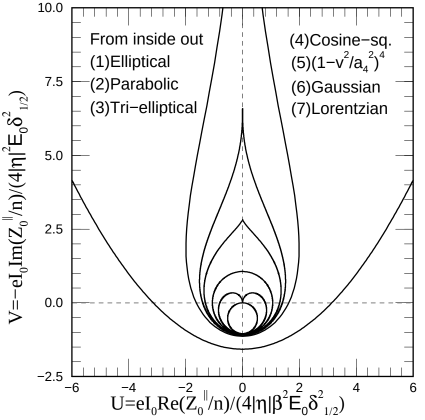
$$U = 0 \quad \text{and} \quad 1 + \frac{2\,\mathrm{sgn}(\eta) V}{\pi} \wp \int \frac{\hat{\rho}'(v)dv}{v} = 0$$

$$\boxed{(\Delta\omega)_0^2 = \frac{in\eta I_0 Z_0^{\parallel}}{2\pi\beta^2 E_0}\omega_0^2}$$

Stability Contours [12]

- Stability plot for $\eta < 0$: most contours cut V-axis at $V = -1$.
- Simplified stability criterion
$$\left|\frac{Z_0^{\parallel}}{n}\right| < \frac{4|\eta|\beta^2 E_0}{eI_0}\delta_{\frac{1}{2}}^2 F_{\parallel}$$
- For generalized elliptical model
$$\hat{\rho}(v) = \frac{A_n}{a_n}\left[1 - \frac{v^2}{a_n^2}\right]^n, \quad F_{\parallel} = \frac{\pi a_n^2}{4n+2}$$
- For distributions with long tails, system is stable below transition if $Z_0^{\parallel}$ is *pure capacitive.*
- As soon as transition is crossed, capacitive imp. excites instability because of small η

  — *negative-mass instability*



n	distribution	$F_{\parallel}$
$\frac{1}{2}$	elliptical	1.047
1	parabolic	1.047
$\frac{3}{2}$	tri-elliptical	1.061
2.36	cosine square	1.080
4	$(1 - v^2/a_4^2)^4$	1.097
∞	Gaussian	1.133

Comments

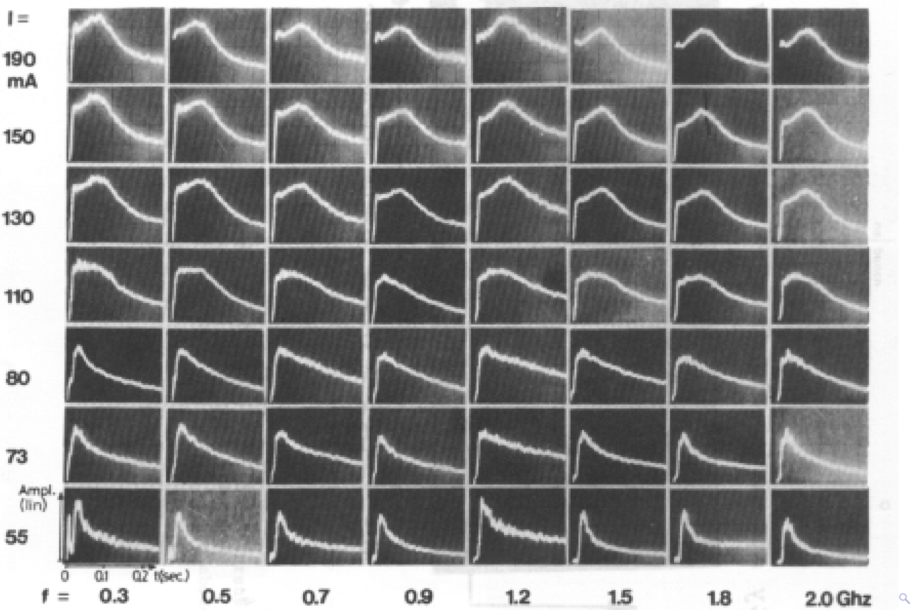
- For Gaussian distribution, the stability contour can be integrated in the closed form with $F_\parallel = \dfrac{\pi}{4\ln 2} = 1.133$. Stability criterion can also be expressed neatly as $\left|\dfrac{Z_0^\parallel}{n}\right| < \dfrac{|\eta|\beta^2 E_0}{eI_0}\sigma_\delta^2$
- Although for unbunched beam, criterion can be applied to *long* bunches provided that (suggestion by Boussard) [14]
  1. The perturbing wavelength $\lambda_1 \ll$ bunch length
  2. Growth time $\tau \ll T_s$, syn. period
  3. Substituting average current I_0 by peak local current $I_{\rm pk}$.
- Microwave instability is not catastrophic.

  When instability occurs, bunch will be lengthened and mom. spread increased so that stability is re-established.

- Since the growth is fast and stability becomes re-established,
  it poses difficulty in observing the microwave signals.
- If a network analyzer is used to view the bunch spectrum,
  high-freq. signals will often not be observed,
  because it takes time for the analyzer to process high freq. signals.
- Analyzer must be set at a narrow freq. span or *zero span*,
  and observe beam signal as a fcn. of time.
  The freq. span is next set to an adjacent narrow freq. interval
  and the observation repeated.

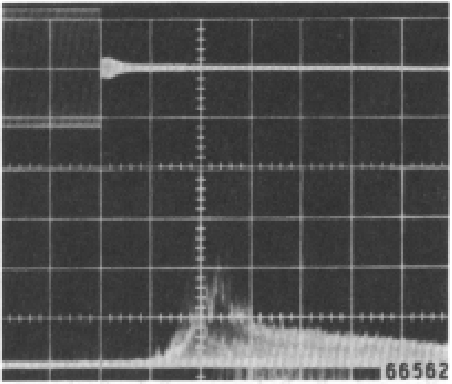
Microwave Signal Observation

- Pick-up signals after injection in CERN ISR [13]



Determination of $Z_0^{\parallel}$ from Instability [14]

- A very thick cable has to be used from beam monitor to network analyzer, because signal decays over distance.
- RF voltage lowered adiabatically $\Longrightarrow$ mom. spread will be reduced gradually until stability criterion is violated.
  Usually, it is hard to know the exact $V_{\rm rf}$ when it is low enough.
- RF voltage turned off abruptly, beam will be debunched. When local mom. spread is small enough, instability occurs.
- RF cavity must be mechanically shorted to avoid beam loading.
- It is hard to determine exact time when stability occurs.
- Fig: CERN CPS, signals between 1.5 to 1.8 GHz, 2 ms/division



Longitudinal Decoherence

- Dispersion relation for microwave instability depends on gradient of energy or frequency distribution, $\dfrac{df_0(\Delta E)}{d\Delta E}$ or $\dfrac{df_0(\Delta\omega_0)}{d\Delta\omega_0}$
- Thus flat distribution $\longrightarrow$ no Landau damping.
- Is there longitudinal decoherence for a flat beam?
- We study the response of an energy impulse to a beam at $t = 0$. [21]
- At $t = 0+$, $f(\theta, \Delta E; 0+) = f_0(\Delta E - \widehat{\delta E}\cos k\theta) = f_0(\Delta E) - \dfrac{df_0}{d\Delta E}\widehat{\delta E}\cos k\theta$
- A particle moves according to $\theta = \theta_0 + \omega_0 t$,

  Perturbed distribution at time is $f_1(\theta, \Delta E; t) = -\dfrac{df_0}{d\Delta E}\widehat{\delta E}\cos(k\theta - k\omega_0 t)$.
- Perturbed part of current jumps from zero to

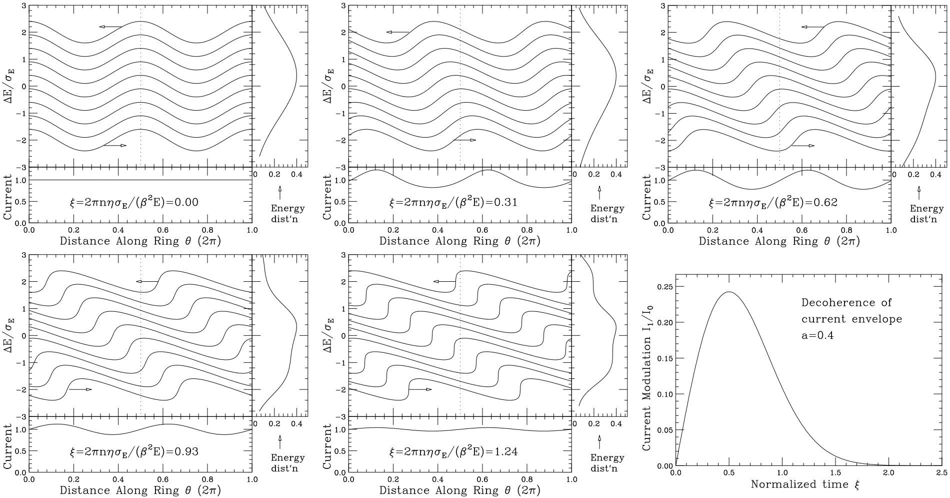
  $$I_1(\theta, t) = \frac{eN}{2\pi}\int \omega_0(\Delta E) f_1(\theta, \Delta E; t)\, d\Delta E$$

- $$I_1(\theta,t) = \underbrace{-H(t)\frac{eN\widehat{\delta E}}{2\pi}\int\frac{df_0}{d\Delta E}\,\omega_0\cos(k\theta-k\omega_0 t)\,d\Delta E}_{\uparrow\ \text{Response function}} \qquad H(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$
- So longitudinal decoherence does depends on gradient of distribution.
- Next introduce $\Delta\omega_0 = \omega_0 - \bar{\omega}_0$ and $g_0(\Delta\omega_0)$.

  Then $f_0(\Delta E)d\Delta E = g_0(\Delta\omega_0)d\Delta\omega_0$

  and $$\frac{df_0}{d\Delta E}\,d\Delta E = \frac{dg_0}{d\Delta E}\,d\Delta\omega_0 = -\frac{\eta\bar{\omega}_0}{\beta^2 E_0}\frac{dg_0}{d\Delta\omega_0}\,d\Delta\omega_0$$
- $$I_1(\theta,t) = \frac{eN\eta\bar{\omega}_0^2\widehat{\delta E}}{2\pi\beta^2 E_0}\int_{-\infty}^{\infty}\frac{dg_0}{d\Delta\omega_0}\Big[\cos(k\theta-k\bar{\omega}_0 t)\cos k\Delta\omega_0 t + \sin(k\theta-k\bar{\omega}_0 t)\sin k\Delta\omega_0 t\Big]d\Delta\omega_0$$
- If $g_0(\Delta\omega_0)$ is even,

  $$I_1(\theta,t) = \frac{eN\eta\bar{\omega}_0^2\widehat{\delta E}}{2\pi\beta^2 E_0}\sin(k\theta-k\bar{\omega}_0 t)\int_{-\infty}^{\infty}\frac{dg_0}{d\Delta\omega_0}\sin k\Delta\omega_0 t\,d\Delta\omega_0.$$



Long. decoherence of an energy disturbance of harmonic $k = 2$ and amplitude $a = \widehat{\delta E}/\sigma_E = 0.4$ in a coasting beam. First five plots are at 'times' $\xi = 0+$, 0.31, 0.62, 0.93, and 1.24. Shown in plots are equi-density curves at 0, ±0.5, ±1.0, ±1.5, and $\pm 2.0\sigma_E$'s. Right: Energy distribution recorded at $\theta = \pi$, initial Gaussian assumed. Bottom: perturbed current.

About the Plots

- $f(\theta, \Delta E; t) = f_0 \left[\Delta E - \widehat{\delta E} \cos\left(k\theta - k\bar{\omega}_0 t + \frac{k\eta\bar{\omega}_0 t}{\beta^2 E_0} \Delta E \right) \right]$
- Normalized everything to the rms energy spread σ_E
- $f(\theta, \Delta E; t) = f_0 \left[\varepsilon - a \cos(k\theta + k\xi\varepsilon) \right]$

  where $\varepsilon = \dfrac{\Delta E}{\sigma_E}, \quad a = \dfrac{\widehat{\delta E}}{\sigma_E}, \quad \xi = \dfrac{2\pi n\eta\bar{\sigma}_E}{\beta^2 E_0}$
- Equi-density curve at density x σ's $f_0(x)$ is obtained by solving for $\varepsilon(\theta)$ from $\quad \varepsilon - a\cos(k\theta + k\xi\varepsilon) = x$
- Initial distribution: $f_0(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ ⟵ Gaussian assumed
- Perturbed current: $I(\theta, \xi) = \displaystyle\int_{-\infty}^{\infty} f_0 \left[\varepsilon - a\cos(k\theta + k\xi\varepsilon) \right] d\varepsilon$

Transition

- For a particle in an accelerator ring, the period is $T = \dfrac{C}{v}$

  Relative to the synchronized particle, $\dfrac{\Delta T}{T_0} = \dfrac{\Delta C}{C_0} - \dfrac{\Delta v}{v_0}$

  or $\dfrac{\Delta T}{T_0} = \left(\alpha - \dfrac{1}{\gamma^2}\right)\dfrac{\Delta p}{p_0} = \eta\dfrac{\Delta p}{p_0}$
- Transition is crossed when frequency-slip parameter $\eta = 0$.
- Above transition, velocity decreases as more energy is put in, as if the *mass is negative*.
- Space-charge force is repulsive. But above transition, it is attractive. Particles tend to lump together at the same place.
  This is called *negative-mass instability*.
- Negative-mass instability is a type of microwave instability, which will also result in emittance growth and even possible breakup of the bunch.
- Instability occurs only when $|\eta|$ is small (close to transition).
- It is important to know the growth of emittance.

Negative-Mass Instability

- Growth rate without Landau damping at peak current I_p at the revolution harmonic k_c is $G(k_c,t)=\omega_0\left(\dfrac{\eta e k_c Z_I I_p}{2\pi\beta^2\gamma E_0}\right)^{1/2}$ where Z_I is $\mathcal{I}m\, Z_0^{\parallel}\big|_{\rm spch}=\dfrac{Z_I}{k_c}=\dfrac{Z_0 g}{2\beta\gamma^2}$
- At low freq., geometric factor $g\to g_0=1+2\ln\dfrac{b}{a}$, where a is beam radius and b is beam pipe radius.
- Thus the growth rate increases linearly with harmonic k_c.
- But at high harmonic, g rolls off at high frequencies roughly like $$g=\frac{g_0}{1+(k_c/k_{c\frac{1}{2}})^2}\qquad \frac{b}{a}\ \text{not too big,}\qquad k_{c\frac{1}{2}}\approx\gamma R\left(\frac{1.6}{b}+\frac{0.52}{a}\right)$$
- Therefore Z_I/k_c is almost constant below $k_{c\frac{1}{2}}$ and the growth rate is $G(k_c,t)\propto k_c$.
- With Landau damping, Hardt showed that the growth rate becomes $G(k_c,t)\propto k_c g^2$ with maximum at $k_{cp}=k_{c\frac{1}{2}}/\sqrt{3}$.

Fermilab Old Main Ring

- Fermilab Main Ring with $a = 5$ mm and $b = 35$ mm:

  this corresponds to 77.6 GHz.

- But cutoff frequency is ~ 1.5 GHz.

- Typical cycle: Growth of power spectral line is 1.5×10^6 times at 77.6 GHz and 1.6 times at 1.5 GHz.

- In Wei's simulation, [22] the bunch was divided into bins with bin width equal to the cutoff wavelength, so higher frequencies were not included.

- Lee and Teng [23] were the first to demonstrate negative-mass instability by simulating transition crossing in the Fermilab Booster.

  They divided the bunch into cutoff wavelengths also.

- Lee said that as the bins become smaller, the observed growth is larger.

  But because of limited speed of computer in the old days, he could not reduce bin size by too much.

Schottky-Noise Model

- Hardt [24] assumed that the seeds of the negative-mass growth come from statistical fluctuations or Schottky noise of the finite number of particles N_b within the bunch.
- The growth across transition is to be derived analytically.
- We measure rf phase offset $\Delta\phi$ from the synchronous particle.
- The bunch is supposed to have an expected smooth distribution $F(\Delta\phi)$, which is normalized to $2\widehat{\Delta\phi}$, the total bunch length.
  In other words, $\dfrac{1}{2\widehat{\Delta\phi}}\displaystyle\int F(\Delta\phi)d\Delta\phi=1$.
- The bunch is divided into M bins in the rf phase
- Number of particles in mth bin is $\dfrac{N_bF(\Delta\phi_m)}{M}$.
- Due to statistical fluctuation, the mth bin contains δN_m extra particles.
- Define a fluctuation step function: $f(\Delta\phi,t)=\dfrac{\delta N_m}{\Delta N}$
  where $\Delta N=\dfrac{N_b}{M}$ is average number of particles in a bin.

- Expand in a Fourier series $f(\Delta\phi,t)=\displaystyle\sum_{k_b=-\infty}^{\infty}c_{k_b}(t)e^{i2\pi k_b\Delta\phi/(2\widehat{\Delta\phi})}$

  where $c_{k_b}(t)=\dfrac{1}{2\widehat{\Delta\phi}}\displaystyle\int_{-\widehat{\Delta\phi}}^{\widehat{\Delta\phi}}f(\Delta\phi,t)e^{-i2\pi k_b\Delta\phi/(2\widehat{\Delta\phi})}d\Delta\phi$

- In above, k_b is bunch mode number, or number of wavelengths can reside in the bunch.

  We are using periodic boundary conditions. Therefore k_b can be all integers, positive and negative.

  Relation of k_b to harmonic number n or k_c is $\dfrac{k_b}{k_c}=\dfrac{2\widehat{\Delta\phi}}{2\pi h}$.

- $E\left[|c_{k_b}(0)|^2\right]=\dfrac{1}{(2\widehat{\Delta\phi})^2}\displaystyle\int_{-\widehat{\Delta\phi}}^{\widehat{\Delta\phi}}d\Delta\phi\int_{-\widehat{\Delta\phi}}^{\widehat{\Delta\phi}}d\Delta\phi' E\left[\frac{\delta N_m\delta N_n}{(\Delta N)^2}\right]e^{i2\pi k_b(\Delta\phi-\Delta\phi')/(2\widehat{\Delta\phi})}$

- Since $E\Big[\delta N_m\delta N_n\Big]=\delta_{mn}\,\Delta NF(\Delta\phi)$,

$$E\left[|c_{k_b}(0)|^2\right]=\frac{1}{(2\widehat{\Delta\phi})^2}\int_{-\widehat{\Delta\phi}}^{\widehat{\Delta\phi}}\frac{F(\Delta\phi)}{\Delta N}\,\frac{2\widehat{\Delta\phi}}{M}\,d\Delta\phi=\frac{1}{N_b}$$

  independent of mode number k_b and the number of bins M,

  otherwise model will be meaningless.

- The growth rate of each mode amplitude c_{k_b} can be derived from Vlasov equation, and the evolution is $|c_{k_b}(t)| \approx \dfrac{1}{\sqrt{N_b}} \exp \displaystyle\int_0^t G(k_c, t)dt$
- Hardt's assertion: no blowup if $\sum_{k_b} |c_{k_b}(t_0)|^2 < 1$

  where t_0 is end of nonadiabatic time or when stability is regained.
- Comment 1: Critical condition implies $\dfrac{1}{2\widehat{\Delta\phi}} \displaystyle\int |f(\Delta\phi, t_0)|^2 d\Delta\phi = 1$

  or the average fluctuation in each bin is comparable to the average number of particles in each bin.

  However, below critical condition, there is still emittance blowup.
- Comment 2: Perturbation expansion is

  $$F(\Delta\phi) + f(\Delta\phi, t) = F(\Delta\phi) + \sum_{k_b=-\infty}^{\infty} c_{k_b}(t) e^{i2\pi k_b \Delta\phi/(2\widehat{\Delta\phi})}$$

  The perturbation is justified when $\sum_{k_b} |c_{k_b}(t_0)|^2 < 1$.
- Remaining problem: compute $G(k_c, t)$ in the presence of Landau damping.

Dispersion Relation

- To obtain growth rate in the presence of Landau damping, we need to solve the dispersion relation

$$1 = -\left(\frac{\Delta\Omega_1}{n}\right)^2 \int \frac{dF(\Delta\omega)/d\Delta\omega}{\Delta\Omega/n - \Delta\omega}\, d\Delta\omega$$

  where the frequency shift without Landau damping is

$$\left(\frac{\Delta\Omega_1}{n}\right)^2 = \frac{ieI_{\rm local}\eta\omega_0^2 \left[Z_0^\parallel(\Omega)/n\right]_{\rm spch}}{2\pi\beta^2\gamma E_{\rm rest}}$$

- What Hardt did:

  1. Assume elliptical initial distribution

$$\psi(\Delta\phi,\Delta E) = \frac{3}{2\pi\widehat{\Delta\phi}\widehat{\Delta E}}\sqrt{1-\frac{\Delta\phi^2}{\widehat{\Delta\phi}^2}-\frac{\Delta E^2}{\widehat{\Delta E}^2}}$$

  which becomes parabolic in linear distribution.

  2. The region of maximum growth is center of bunch.
  3. Substitute in dispersion relation, and solve for $\mathcal{I}m\,\Delta\Omega$.

Analytic Solution

- Growth per unit $x = t/T_c$

$$G(n,x) = T_c\,\mathcal{I}m\,\Omega = \frac{n\eta_N}{h}\sqrt{\frac{S_c|\tan\phi_s|\beta_t}{\pi\dot{\gamma}_t T_c}}\,\frac{1-\dfrac{x}{\eta_N\theta}}{\sqrt{\dfrac{2\eta_N\theta}{x}-1}}$$

  T_c is nonadiabatic time

  $\eta_N \propto g(n)$ is space charge parameter

  $S_c = \pi\widehat{\beta\gamma}\widehat{\Delta\phi}$ is long bunch emittance

  θ is normalized half bunch length

  ϕ_s is synchronous rf phase

- The growth increases from zero at $x = 0$ to a maximum and decreases to zero at $x = \eta_N\theta$.
- Integrated growth decrement

$$E_{\rm acc}(n) = \int_0^{\eta_N\theta} G(n,x)\,dx = \frac{n\eta_N^2\theta}{h}\left(1-\frac{\pi}{4}\right)\sqrt{\frac{S_c|\tan\phi_s|\beta_t}{\pi\dot{\gamma}_t T_c}}$$

- Recall that the space charge geometric factor $g(n)$ rolls off at high freq.
- $E_{\rm acc}(n) \propto \dfrac{n}{\left(1+\dfrac{n^2}{n_{\frac{1}{2}}^2}\right)^2}$
- The maximum occurs at $n = n_{\rm max} = n_{\frac{1}{2}}/\sqrt{3}$:

$$E_{\rm max} = \frac{3\sqrt{3}\, n_{\frac{1}{2}} \eta_{N0}^2 \theta}{16h} \left(1-\frac{\pi}{4}\right) \sqrt{\frac{S_c |\tan\phi_s| \beta_t}{\pi \dot{\gamma}_t T_c}}$$

- Up to now, we have everything derived to compute the growth from Schottky noise, i.e.,

$$f(\Delta\phi, t) = \sum_{k_b=-\infty}^{\infty} c_{k_b}(t) e^{i2\pi k_b \Delta\phi/(2\widehat{\Delta\phi})}$$

$$|c_{k_b}(t)| \approx \frac{1}{\sqrt{N_b}} \exp \int_0^t G(k_c, t)dt = \frac{1}{\sqrt{N_b}} \exp E_{\rm acc}(k_b)$$

- In a linac, we can follow this idea to compute the microwave instability growth with Schottky noise as seeds.

Final Touch of Transition Growth

- Hardt's blowup assertion:

$$\sum_{k_b=-\infty}^{\infty} |c_{k_b}(t_0)|^2 \lesssim 1 \quad \Longrightarrow \quad \sum_{k_b=-\infty}^{\infty} \exp\Big[2E_{\rm acc}(k_b)\Big] \lesssim N_b$$

- Because of sharp peak at $n_{\rm max} = n_{\frac{1}{2}}/\sqrt{3}$, can transform $\sum_{k_b}$ to an integral and employ method of steepest descend for evaluation.

- Final result: $E_{\rm crit} \approx \dfrac{1}{2}\left[\ln N_b - \ln\left(\dfrac{2k_{b\frac{1}{2}}}{3}\sqrt{\dfrac{\pi}{\ln N_b}}\right)\right]$

- Can introduce a parameter c: [25]

$E_{\rm acc} = cE_{\rm crit}$ or

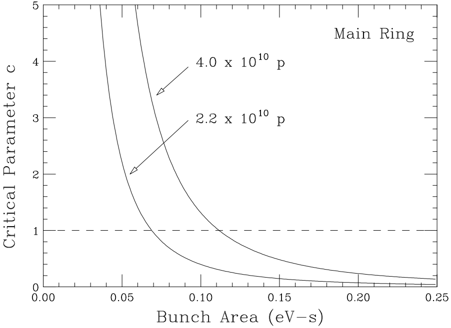
$$2.45\, n_{\rm max}\left(\frac{r_p}{R}\right)^2\left(\frac{E_{\rm rest}^{5/2}\beta_t^{7/6}}{h^{1/3}\omega_0^{4/3}\gamma_t^{2/3}}\right)\left(\frac{N_b^2 g_0^2|\tan\phi_s|^{1/3}}{S^{5/2}\dot{\gamma}_t^{7/6}}\right) = cE_{\rm crit}$$

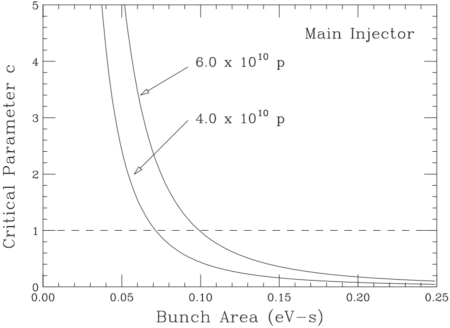
Thus $c < 1$ implies no blowup.

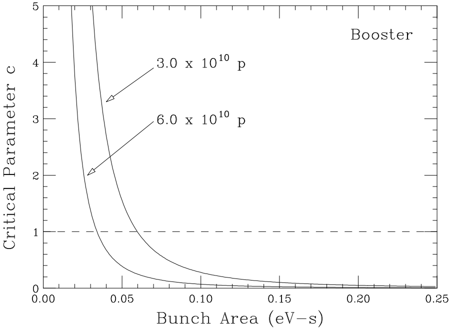
Numerical Results

Bunch area (eV-s)	Half bunch width (ns)	$N_b = 2.2 \times 10^{10}$		$N_b = 4.0 \times 10^{10}$	
		c	$E_{\rm crit}$	c	$E_{\rm crit}$
0.040	0.439	3.84	10.23	12.70	10.54
0.050	0.490	2.21	10.18	7.31	10.48
0.060	0.537	1.41	10.13	4.65	10.44
0.070	0.580	0.96	10.09	3.18	10.40
0.080	0.620	0.69	10.06	2.28	10.36
0.100	0.693	0.40	10.00	1.31	10.31
0.120	0.760	0.25	9.96	0.84	10.26
0.140	0.820	0.17	9.92	0.57	10.22
0.160	0.877	0.12	9.89	0.41	10.19
0.180	0.930	0.09	9.86	0.31	10.16
0.200	0.981	0.07	9.83	0.24	10.13
0.220	1.028	0.06	9.81	0.19	10.11
0.240	1.074	0.05	9.78	0.15	10.09

Critical parameter c for negative-mass instability for a proton bunch in the Fermilab Main Ring with $N_b = 2.2 \times 10^{10}$ or 4.0×10^{10} particles. The ramp rate across transition is $\dot{\gamma}_t = 90.0\ \mathrm{s}^{-1}$.







Ramp rates across transition:

$\dot{\gamma}_t = 90.0\ \mathrm{s}^{-1}$ Main Ring

$\dot{\gamma}_t = 161.1\ \mathrm{s}^{-1}$ Main Injector

$\dot{\gamma}_t = 406.7\ \mathrm{s}^{-1}$ Booster

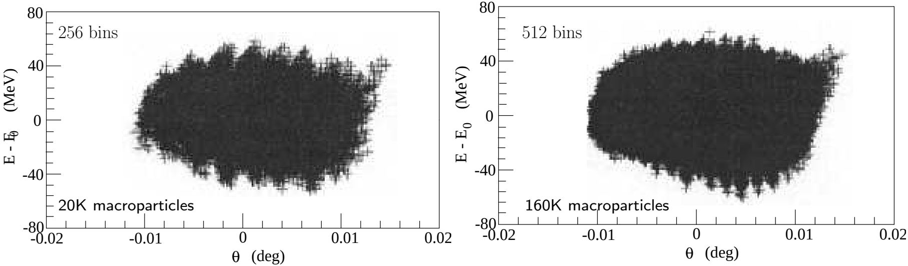
Growths at Cutoff and High Frequencies

- At cutoff freq., growth rate small but seeds from beam are large.
  At high freq., growth rate huge but seeds from Schottky noise tiny.
- So which one gives higher final growth in amplitude or power.
- We analyze this problem analytically.
- Results are given for the old Fermilab Main Ring.

$\dot{\gamma}_t$ (s^{-1})	N_b (10^{10})	Initial Bunch Emittance (eV-s)	Final Power Spectrum of Fluctuation		
			at $n_{\rm cutoff}$	at $n_{\rm max}$	sum
90	2.2	0.05	3.70	1.50×10^{9}	4.03×10^{10}
90	2.2	0.06	2.21	1.08×10^{2}	3.97×10^{3}
90	2.2	0.07	1.67	1.19×10^{-2}	5.74×10^{-1}
90	2.2	0.08	1.41	4.86×10^{-5}	2.93×10^{-3}
90	2.2	0.09	1.26	1.41×10^{-6}	1.06×10^{-4}
120	4.0	0.06	7.44	4.37×10^{18}	1.00×10^{20}
120	4.0	0.07	3.80	1.94×10^{9}	5.83×10^{10}
120	4.0	0.08	2.54	4.40×10^{3}	1.67×10^{5}
120	4.0	0.09	1.95	1.02×10^{0}	4.76×10^{1}
120	4.0	0.10	1.64	3.57×10^{-3}	2.00×10^{-1}

Difficulties in Simulation

- For Fermilab Main Ring, half-value space-charge roll-off harmonic $n_{\frac{1}{2}}$ corresponds to 134 GHz.
- In simulations need bin size $1/(2 \times 134) = 0.00373$ ns.
- RF wavelength or bucket width 18.8ns $\longrightarrow$ need at least 4096 bins.
- Simulation results across transition using ESME. [26]



- Because space charge force $\propto$ gradient of distribution. To maintain same accuracy, 3-in-1 rule says that macroparticle number must increase by 2^3 when bin width is reduced by a factor of 2. [27]
- Thus tracking time will increase by a factor of 2^4.

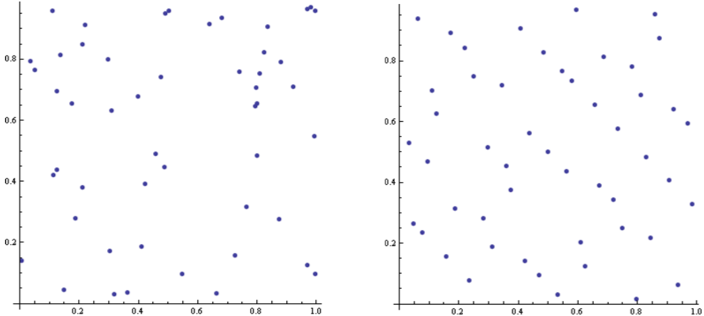
Right Amount of Schottky Noise

- In simulation of microwave instability, there is usually ample time for instability to develop to saturation. Therefore, we do not care so much about the size of the initial excitation or seed of the growth.
- Across transition, however, the bunch is negative-mass unstable only for a short time until the slip factor η becomes large enough to provide enough Landau damping.

  This time is typically of the order of the nonadiabatic time, about 3 ms for the Fermilab Main Ring. Therefore right amount of seeds is very important.
- Relative size of Schottky noise $\propto \dfrac{1}{\sqrt{N_M}}$, where N_M is no. of macroparticles.
- But the right amount of Schottky noise is $\sim \dfrac{1}{\sqrt{N_b}}$, where $N_b \sim 2 \times 10^{10}$.
- Since it is impossible to use so many particles in simulation, most reported simulations across transition are incorrect.

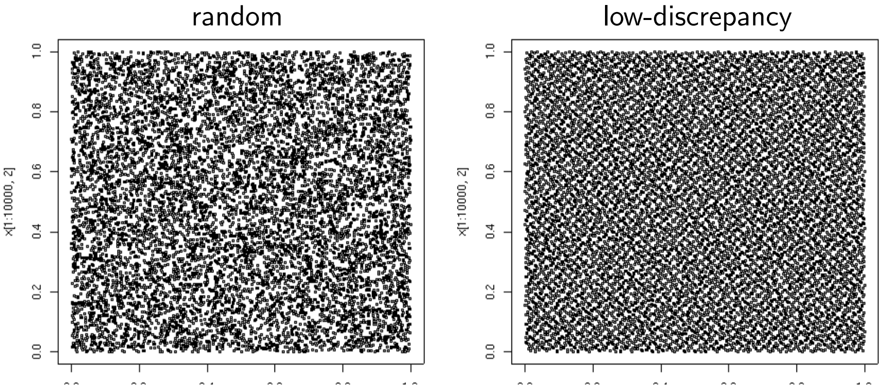
Low-Discrepancy Sequence [28]

- Since we must use macroparticles of number much less than number of the microparticles, Schottky noise is very much larger.
- But if we populate the macroparticles in more orderly way, Schottky noise can be reduced.
- This way of population is according to a *low-discrepancy sequence*.
- Population of 50 particles randomly (left) and according to Hammersley sequence (right).



- Fluctuation of N particle becomes $\mathcal{O}(1)$ instead of $\mathcal{O}(\sqrt{N})$.

Population of 10,000 particles:



Relative noise level:

$$\frac{\sqrt{N}}{N} = 3.2 \times 10^{-3}$$

$$\frac{1}{N} = 1 \times 10^{-5}$$

Improvement with Hammersley Sequence

	With Microparticles	With Macroparticles Hammersley Sequence
average no. per bin	$\Delta N = \dfrac{N_b}{M}$	$\Delta N_{_M} = \dfrac{N_{_M}}{M}$
fluctuation function	$f(\Delta\phi_n) = \dfrac{\delta N_n}{\Delta N}$	$f(\Delta\phi_n) = \dfrac{\delta N_n}{\Delta N_{_M}}$
Expectation of errors	$E(\delta N_n \delta N_m) = \delta_{nm}\Delta N F(\Delta\phi_n)$	$E(\delta N_n \delta N_m) = \delta_{nm}$
	$E\left(\lvert c_{k_b}(0)\rvert^2\right) = \dfrac{1}{N_b}$	$E\left(\lvert c_{k_b}(0)\rvert^2\right) = \dfrac{1}{M(\Delta N_{_M})^2} = \dfrac{M}{N_{_M}^2}$

- N_b is no. of microparticles, $N_{_M}$ is no. of macroparticles
- Thus number of macroparticles required for the same Schottky noise is $N_M = (MN_b)^{\frac{1}{2}}$

  For Fermilab MR, $N_M \sim 2.4 \text{ to } 3.6 \times 10^6$, instead of $N_b = 2.2 \times 10^{10}$

Two Difficulties

- Formerly $E\left[f^2(\Delta\phi_m,0)\right] = E\left[\dfrac{\delta N_m^2}{\Delta N^2}\right] = \dfrac{F(\Delta\phi_m)}{\Delta N} = \dfrac{M}{N_b}\,F(\Delta\phi_m)$
- Now $E[f^2(\Delta\phi_m,0)] = \dfrac{1}{\Delta N_M^2} = \dfrac{M^2}{N_M^2}$ which is *not proportional* to $F(\Delta\phi_m)$.

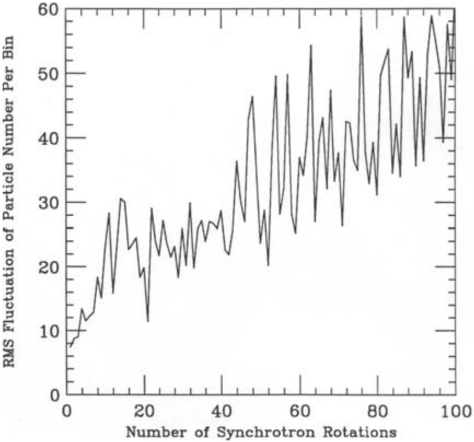
  Difficulty 1: Thus relative fluctuations in bins have changed.
- Difficulty 2: Fluctuation of particles may change in time.
- Because of space charge, rf bucket will be modified.
- Usual way of population is to populate as if no space charge.

  Then turn on space charge and let particles fit the rf bucket

  after a number of synchrotron oscillations.
- But synchrotron tune is amplitude dependent.
- Try simulation of 2×10^5 particles in 20 bins of equal size.

  Assume syn. tune to decrease linearly by 1% from bunch's center to edge.

- Distribution is projected onto time or phase axis.
- Number of particles in excess of a smooth Gaussian is recorded for each bin
- Result: rms fluctuation starts from 7, increases rapidly to 20 after 20 turns, and 50 after 100 turns.

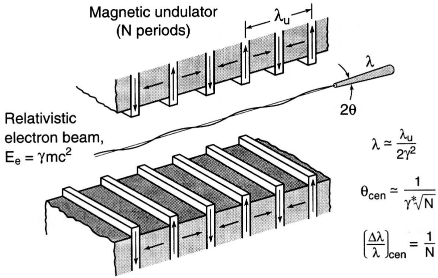


- Thus the advantage of a smaller noise level by using a Hammersley sequence can be lost rapidly when particles are shuffled.

  Care must be taken and many tests have to be made when such population is used.

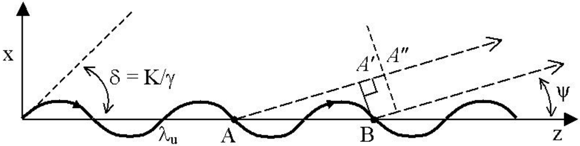
Undulator Radiation [29, 30, 31]

- Consider a planar undulator with sinusoidal magnetic field in y-direction.
  Suppose undulator period is λ_u, magnetic field is
  $$B_y(z) = B_0 \sin k_u z$$
  $$B_x = B_z = 0 \quad \text{with} \quad k_u = \frac{2\pi}{\lambda_u}$$



- Electron motion $m\gamma \dfrac{d^2x}{dt^2} = e(v_yB_z - v_zB_y) = -eB_0c\sin k_uz$
- Changing t to z and assume $v_z \approx c$, $x'' = -\dfrac{eB_0}{\gamma mc}\sin k_uz$
- Since γ is constant in a static magnetic field,
  $$x' = \frac{eB_0}{\gamma mck_u}\cos k_uz \equiv \frac{K}{\gamma}\cos k_uz, \qquad x = \frac{K}{\gamma k_u}\sin k_uz$$
  where $K = \dfrac{eB_0}{mck_u} = 0.9337B_0[\mathrm{Tesla}]\lambda_u[\mathrm{cm}]$, $\quad \dfrac{K}{\gamma} \sim$ deflection angle
  is the dimensionless deflection parameter or *undulator parameter*

Undulator Radiation

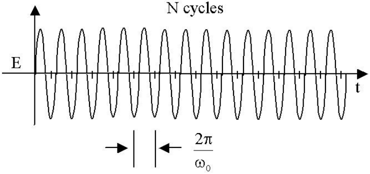


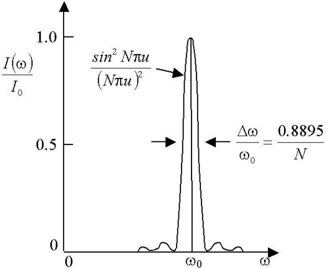
- Electron moves along arc length $\widetilde{AB}$, radiation along straight line $\overline{AA'}$. Their time difference is one radiation wavelength λ_1/c.
- $\dfrac{\lambda_1}{c}=\dfrac{\widetilde{AB}}{v}-\dfrac{\overline{AA'}}{c}$

  with $\widetilde{AB}=\displaystyle\int_0^{\lambda_u}dz\sqrt{1+x'^2}\approx\int_0^{\lambda_u}dz\left[1+\frac{K^2}{2\gamma^2}\cos^2 k_uz\right]=\lambda_u\left(1+\frac{K^2}{4\gamma^2}\right)$

  and $\overline{AA'}=\lambda_u\cos\psi\approx\lambda_u\left(1-\dfrac{\psi^2}{2}\right)$
- Get $\dfrac{\lambda_1(\psi)}{\lambda_u}=\dfrac{1+K^2/4\gamma^2}{\beta}-\left(1-\dfrac{\psi^2}{2}\right)\approx\dfrac{1+K^2/2+\gamma^2\psi^2}{2\gamma^2}$

- Example $\lambda_u = 5$ cm, $B_0 = 0.35$ T, $K = 0.934 \times 0.35 \times 5 = 1.6$
- At $E = 5$ GeV, $\lambda_1(0) = 5.7\ \mathring{A}$
- For one electron, emission at one undulator magnet is in phase with emission at the next magnet. Thus for N undulator periods, emission adds up in phase.

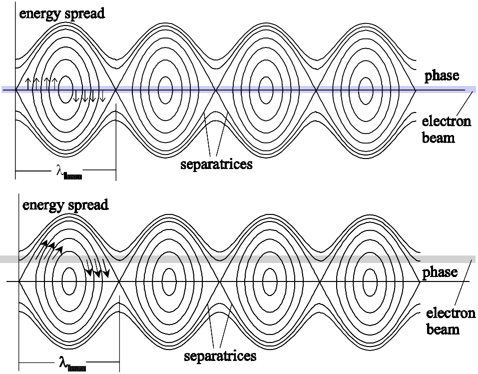




- However, emission from one electron usually is not in phase from that of other electrons, unless they are within $\lambda_1/2$.
- Thus of beam is microbunched in wavelength of $\lambda_1/2$, electrons within each bunchlet will emit in phase, or emission will be coherent.

Pendulum Equation

- Emission freq depends on γ. which has a spread in beam.
- Thus there is a spread in λ_1.
- In presence of a laser beam of wavelength λ_1, electrons will be driven into synchrotron oscillation in buckets setup by E field of laser.
- This pendulum motion is first discovered by Colson. [32]



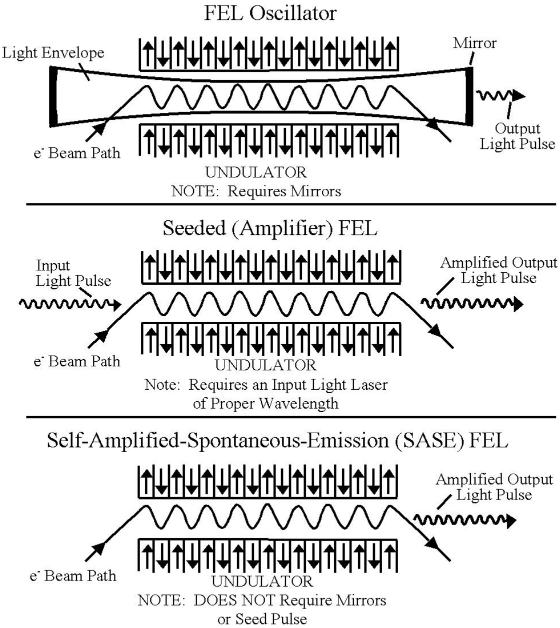
- Electron interacts with E of laser with power gain $\vec{E}\cdot\vec{v}$.
- This has to be a loss in order intensity of laser is increases.
- This can be accomplished to offsetting initial energy of electron beam.
- In $\frac{1}{4}$ of syn. period, beam will point in γ-direction and narrow in z-direction. Emission mostly coherent and power is peaked.

3 Types of FEL

1. Mirrors at right position so that reflected emission wave is in phase with undulator.
Good for low freq only, because there are no high freq mirrors.

2. There is an input laser pulse that induces stimulus emission at undulator, and microbunches electron beam.

3. Electron beam is microbunched by emission wave — called SASE.



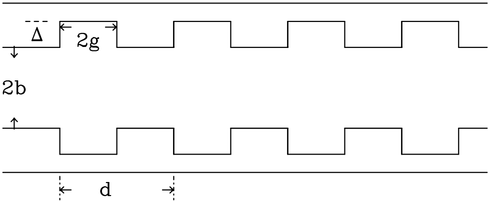
Low-Energy Electron Storage Ring at ICEEM

- Using the 4 dipoles left behind from the cooler injector (CIS) at ICEEM, a low energy electron storage ring has been built.
- Ring circumference 20 m, energy 50–200 MeV
- Dipoles: $\rho = 1.273$ m, $L = 2$ m, edge angle 12°.
- $\mathcal{J}_x < 0 \Longrightarrow$ horizontal motion unstable.
- $\mathcal{J}_x$ can be adjusted by two Robinson wigglers.
- Dispersion can be changed in main dipoles,

  so momentum compaction α is tunable.
- want to produce coherent 1 THz X-ray in the ring.
- Need micro-bunching of beam to 1 ps or $\lambda \sim 300\ \mu$m.
- Use impedance to create controlled microwave-like instability and microbunching.

Diffraction Grating [33]

- Beam through cavity-like structure generates wake.
- E_z of TM_{0np} modes provides controllable microbunching for the required frequency.



- $$Z_\parallel(\omega) = \frac{gZ_0}{\pi b I_0^2\left(\frac{kb}{\beta\gamma}\right) D}, \quad D = j\frac{R_0'(kb)}{R_0(kb)} - 2jk\left[\sum_{s=1}^{S}\frac{1}{\beta_s^2 b}\left(1-e^{-j\beta_s g}\frac{\sin\beta_s g}{\beta_s g}\right) - \sum_{s=S+1}^{\infty}\frac{1}{\alpha_s^2 b}\left(1-e^{-\alpha_s g}\frac{\sinh\alpha_s g}{\alpha_s g}\right)\right]$$
- $k = \omega/c$

  $R_0(kb) = J_0(kb)Y_0(ka) - J_0(ka)Y_0(kb)$, with $a = b + \Delta$

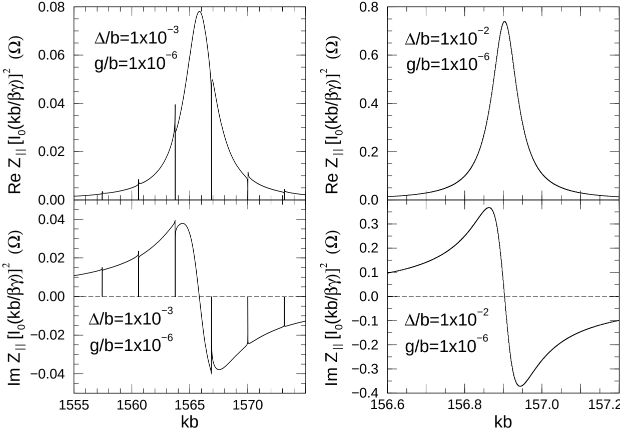
  $\beta_s b = \sqrt{k^2b^2 - j_{0s}^2}$ and $\alpha_s b = \sqrt{j_{0s}^2 - k^2b^2}$,

  where j_{0s} is sth zero of J_0 and j_{0S} is zero just larger than or equal to kb.

I_0 Factor

- $I_0^2(\frac{kb}{\beta\gamma})$ occurs in the denominator.
- Image of a point charge on beam pipe has rms length $\sigma_\tau = \dfrac{b}{\sqrt{2}\beta\gamma}$.
- The Fourier spectrum of image current is $\dfrac{1}{I_0\left(\frac{kb}{\beta\gamma}\right)}$.
- $\dfrac{1}{I_0^2\left(\frac{kb}{\beta\gamma}\right)}$ can become very small at high frequency and low energy.
- This is the only particle-energy dependent factor in the impedance.
- It constitutes a measure of the efficiency at which the beam particle generates electromagnetic fields and excites a resonance in the cavity.
- Energy is low at the Indiana ring.
  This limits the highest frequency X-ray generated to $\lesssim 1$ THz

Illustration of Impedance



For 1 THz, $b = 4$ cm

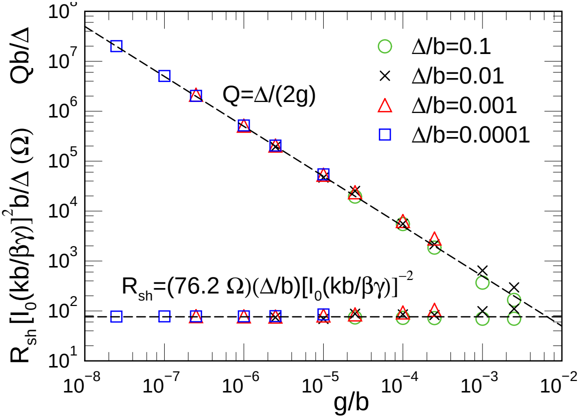
$kb = 838$

$\Delta = 75\ \mu$m

$\dfrac{\Delta}{b} = 1.88\times10^{-3}$

- Center freq of 1st broad-band of 1 cavity is at $k\Delta \approx \dfrac{\pi}{2}$ when $\dfrac{g}{b} \to 0$.
- It is shifted slightly downward as $\dfrac{g}{b}$ increases and $\dfrac{\Delta}{b}$ decreases.
- The center freq. is approx. $f_c \approx \dfrac{c}{4\Delta}$.

Quality Factor and Impedance



- Quality factor (Q-value) of resonance generated by the grating structure depends essentially on Δ/g
- Shunt impedance depends essentially on Δ/b.
- Impedance becomes $\left|\dfrac{Z_\parallel}{n}\right| \approx \dfrac{300\beta\Delta^2}{2\pi Rb}\ \Omega.$ [I_0^2 factor not included]

Microwave Instabilities

- With N cavities, diffraction phenomenon takes place by enhancing some resonances N-fold
  and shrinking width N-fold within the envelope of broadband resonance of a single cavity.
- This impedance is used to generate microbunching.
- Condition of controlled microwave instability (Keil-Schnell):

  $$eI_{\rm pk}\beta^2\left|\frac{Z_\parallel}{n}\right| \ge 2\pi E\sigma_\delta^2|\eta|F_{\rm dist} \approx 1.23\times 10^{-6}\frac{\gamma^3|\eta|}{\mathcal{J}_E\rho[{\rm m}]}\quad {\rm eV}$$

  $I_{\rm pk} = F_{\rm B}I_{\rm av}$ is the peak current,

  $I_{\rm av}$ is the average current,

  $F_{\rm B} = 2\pi R/\sqrt{2\pi}\sigma_z$ is the bunching factor,

  $\mathcal{J}_E \approx 2$ is the longitudinal damping partition number,

  $\sigma_\delta = \sqrt{\dfrac{3.83\times 10^{-13}\gamma^2}{\rho\mathcal{J}_E}}$ is natural momentum spread.

Application to ICEEM Ring

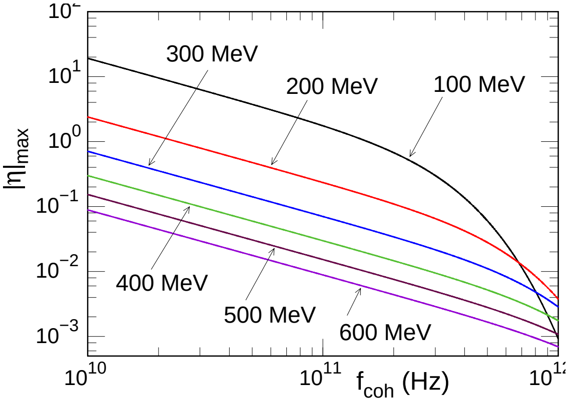
- Consider generation of $f_{\rm coh} = 1$ THz radiation.
- Minimum electron energy required is $\gamma = \left(4\pi f_{\rm coh}\rho/3c\right)^{1/3} \geq 26$.
- Consider 100-Mev electrons $(\gamma = 196)$.

  Get $e\hat{I}\beta^2 \left|\dfrac{Z_\parallel}{n}\right| \geq 3.37|\eta|$ eV.
- If $I_{\rm pk} = 1$ kA, requires $\left|\dfrac{Z_\parallel}{n}\right| \geq 3.37 \times 10^{-3}|\eta|\ \Omega$.
- 1 THz radiation $\Longrightarrow$ grating depth $\Delta \approx 75\ \mu$m

  and rev. harmonic $n = 6.7 \times 10^4$.
- Structure radius $b = 4$ cm gives $\left|\dfrac{Z_\parallel}{n}\right| = 1.02 \times 10^{-8}\ \Omega$

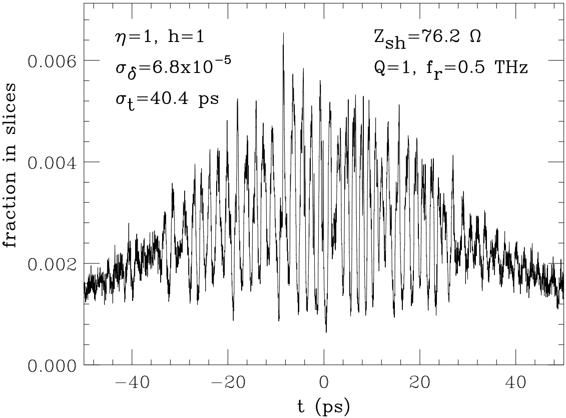
  $\longrightarrow N = 10000$ gratings required if $|\eta| \lesssim 0.003$.

Maximum Slip Factor Required



- For each coherent radiation frequency $f_{\rm coh}$, plot gives maximum $|\eta|$ required for Keil-Schell instability or microbunching.
- We use grating depth $\Delta = c/(4f_{\rm coh})$, $g = \Delta$ (groove half width equal depth), grating period $d = 4g$, total length of gratings 1 m, peak current $I_{\rm pk} = 100$ A.

Simulation



ICEEM ring
100 MeV
$h = 1$
$V_{\rm rf} = 2$ kV
$\longrightarrow$ head

- Bin width 0.04 ps, 80,000 macroparticles used.
- Keil-Schnell limit of stability is $I_{\rm pk} = 12.7$ A.
- However, $I_{\rm pk} = 90$ A was used so that microbunching develops fast before peaks are overlapped.

Higher-Frequency Coherent Radiation

- The grating method sets limit to coherent radiation freq. in ICEEM ring.

  There are 2 reasons:

  1. Since the ring can reach 200 Mev only,

     the $\dfrac{1}{I_0^2(kb/\beta\gamma)}$ factor becomes too small when $\gtrsim 1$ THz.
  2. The grating depth cannot be made too shallow technically.
- To achieve higher frequency coherent radiation, need another method.
- The method of inverse Compton scattering can be used.

Inverse Compton Scattering

- Compton scattering is for photon scattered by electron in rest frame.
- Now electron in the beam travels at high speed.
- Inverse Compton scattering is for a photon collide with the electron head-on and its direction is reverse,
- There is one Lorentz transformation to bring incident photon to electron rest frame.
- There is another Lorentz transformation to boost the direction-reverse photon back to the lab frame.
- Two Lorentz transformations increases photon freq. by the factor $\sim 4\gamma^2$.
- If we start from a laser beam of freq $f_L = 3 \times 10^{14}$ Hz (1000 Å), the back-scattered photon has frequency $f_s = 4\gamma^2 f_L = 1.15 \times 10^{19}$ Hz with $\gamma = 196$ or 100-MeV electrons. $(\lambda_s = 8.70 \times 10^{-20}$ s.)

Undulator Analog

- The electron traveling towards the head-on laser beam sees alternating horizontal E field and alternating vertical B field.
- Both fields steer the electron in an oscillatory path in horizontal plane.
- the effect of E and B fields add, because electron and photon are in opposite directions.
- This resembles the electron traversing an undulator, with undulator wavelength $\lambda_u \approx \dfrac{\lambda_L}{2}$.
- It can be shown that the undulator concept gives relatively the same intensity of emitted photon per electron per unit angular frequency and per unit solid angle, $\dfrac{d^2N}{d\omega d\Omega}$.

Microbunching

- The electron bunch passes through an undulator, it will be micro-bunched to the forward radiating photon wavelength.
- Now the electron bunch passes through the incident laser beam will also be micro-bunched to the inverse-scattered photon wavelength.
- Photons scattered by each micro-bunched slice of the electron bunch will be coherent.
- It takes time for the intensity of scattered photon to grow,

  the *growth length* $L_g = \dfrac{\lambda_u}{4\pi\sqrt{3}\rho}$, and to saturate,

  the *saturation length* $L_{\rm sat} = \dfrac{\lambda_u}{\rho}$. ($\rho$ is Pierce parameter)
- We need to make sure that during this time, the electron micro-bunched slice will not drift by more than one wavelength of the scattered photon or $\lambda_s = 8.70 \times 10^{-20}$ s.
- This means that the slip factor of the ring η must be kept very small.

FEL or Pierce Parameter [34]

- The Pierce parameter is an important parameter in FEL.
- It is dimensionless and is defined as $\rho = \left[\frac{r_e \lambda_u^2 K^2 [JJ]^2 n_e}{32\pi\gamma^3} \right]^{1/3}$

  r_e is electron classical radius,

  $$[JJ] = J_0(\xi) - J_1(\xi), \qquad \xi = \frac{K^2}{4+2K^2} = \frac{1}{2\,(1+2/K^2)} < \frac{1}{2}.$$

  $n_e = \frac{N_B}{\sqrt{2\pi}\sigma_\tau c\sqrt{2\pi}\sigma_x\sqrt{2\pi}\sigma_y} = \frac{I_p/e}{2\pi c\sigma_x\sigma_y}$ is peak electron density,

  I_p is peak current, and σ_x and σ_y are rms sizes of electron beam.

- 1. $\rho = \frac{\text{field energy generated}}{\text{e-beam kinetic energy}}$, $\therefore$ saturated power $\sim \rho \times$ e-beam power

  2. Saturated length $L_{sat} \sim \frac{\lambda_u}{\rho}$.

  3. Final saturated energy spread of e-beam $\sim \rho$.

  4. Transverse size of laser beam $\sigma_r \sim \sqrt{\frac{\lambda_1}{4\pi}\frac{\lambda_u}{4\pi\rho}}$.

σ_x (m)	σ_z (m)	σ_t (s)	σ_s (m)	λ_u (m)	q (coulumb)	N_0 (m^3)	fact	E_r (eV)	γ_r	ϵ
6.32E-05	6.32E-05	1.00E-11	3.00E-03	1.00E-06	1.00E-08	1.66E+21	4.48E-16	2.50E+07	48.92	8.00E-08
	1.26E-08	←area(m^2)	3.00E+14	← Hz						3.91E-06
C (m)	T_0 (s)	f_0 (Hz)	ρ_0 (m)	δ_0						
20	6.67E-08	1.50E+07	1.273	1.898E-05						

P (W)	I (W/m^2)	u (J/m^3)	E (V/m)	B (T)	K_w	ρ	L_G (m)	$L_{\rm sat}$ (m)	$N_{\rm path}$	$\|\eta\|$
1.00E+03	7.96E+10	2.65E+02	5.48E+06	1.83E-02	1.71E-06	1.05E-08	4.38E+00	8.76E+01	29213	9.42E-12
3.16E+03	2.52E+11	8.39E+02	9.74E+06	3.25E-02	3.03E-06	3.88E-08	1.18E+00	2.37E+01	7897	3.48E-11
1.00E+04	7.96E+11	2.65E+03	1.73E+07	5.78E-02	5.39E-06	5.69E-08	8.07E-01	1.61E+01	5380	5.11E-11
3.16E+04	2.52E+12	8.39E+03	3.08E+07	1.03E-01	9.59E-06	8.36E-08	5.50E-01	1.10E+01	3666	7.51E-11
1.00E+05	7.96E+12	2.65E+04	5.48E+07	1.83E-01	1.71E-05	1.23E-07	3.75E-01	7.49E+00	2497	1.10E-10
3.16E+05	2.52E+13	8.39E+04	9.74E+07	3.25E-01	3.03E-05	1.80E-07	2.55E-01	5.10E+00	1701	1.62E-10
1.00E+06	7.96E+13	2.65E+05	1.73E+08	5.78E-01	5.39E-05	2.64E-07	1.74E-01	3.48E+00	1159	2.37E-10
3.16E+06	2.52E+14	8.39E+05	3.08E+08	1.03E+00	9.59E-05	3.88E-07	1.18E-01	2.37E+00	790	3.48E-10
1.00E+07	7.96E+14	2.65E+06	5.48E+08	1.83E+00	1.71E-04	5.69E-07	8.07E-02	1.61E+00	538	5.11E-10
3.16E+07	2.52E+15	8.39E+06	9.74E+08	3.25E+00	3.03E-04	8.36E-07	5.50E-02	1.10E+00	367	7.51E-10
1.00E+08	7.96E+15	2.65E+07	1.73E+09	5.78E+00	5.39E-04	1.23E-06	3.75E-02	7.49E-01	250	1.10E-09
3.16E+08	2.52E+16	8.39E+07	3.08E+09	1.03E+01	9.59E-04	1.80E-06	2.55E-02	5.10E-01	170	1.62E-09
1.00E+09	7.96E+16	2.65E+08	5.48E+09	1.83E+01	1.71E-03	2.64E-06	1.74E-02	3.48E-01	116	2.37E-09
3.16E+09	2.52E+17	8.39E+08	9.74E+09	3.25E+01	3.03E-03	3.88E-06	1.18E-02	2.37E-01	79	3.48E-09
1.00E+10	7.96E+17	2.65E+09	1.73E+10	5.78E+01	5.39E-03	5.69E-06	8.07E-03	1.61E-01	54	5.11E-09
3.16E+10	2.52E+18	8.39E+09	3.08E+10	1.03E+02	9.59E-03	8.36E-06	5.50E-03	1.10E-01	37	7.51E-09
1.00E+11	7.96E+18	2.65E+10	5.48E+10	1.83E+02	1.71E-02	1.23E-05	3.75E-03	7.49E-02	25	1.10E-08
3.16E+11	2.52E+19	8.39E+10	9.74E+10	3.25E+02	3.03E-02	1.80E-05	2.55E-03	5.10E-02	17	1.62E-08
1.00E+12	7.96E+19	2.65E+11	1.73E+11	5.78E+02	5.39E-02	2.64E-05	1.74E-03	3.48E-02	12	2.37E-08
3.16E+12	2.52E+20	8.39E+11	3.08E+11	1.03E+03	9.59E-02	3.88E-05	1.18E-03	2.37E-02	8	3.48E-08
1.00E+13	7.96E+20	2.65E+12	5.48E+11	1.83E+03	1.71E-01	5.69E-05	8.07E-04	1.61E-02	5	5.11E-08
3.16E+13	2.52E+21	8.39E+12	9.74E+11	3.25E+03	3.03E-01	8.36E-05	5.50E-04	1.10E-02	4	7.51E-08
1.00E+14	7.96E+21	2.65E+13	1.73E+12	5.78E+03	5.39E-01	1.23E-04	3.75E-04	7.49E-03	2	1.10E-07

Microbunching

- It appears that the slip factor has to be controlled to 10^{-7}.
- The natural momentum spread is $\sigma_\delta = 1.90 \times 10^{-5}$.

  If we expand momentum compaction as

  $\alpha = \alpha_0 + \alpha_1\delta + \alpha_2\delta^2 + \alpha_3\delta^3 + \cdots$

  We must control α_0, α_1 and even α_2,

  using quadrupoles, sextupoles, and octupoles, respectively.
- Whether the slip factor can be controlled to such high accuracy remains to be seen.

Conclusions

- We would like to produce coherent synchrotron radiation at the 20-m electron storage ring under commission now at ICEEM.
- For coherent radiation of frequency up to 1 THz, a cavity-like grating structure can be used.
- The low-energy nature of the ring limits the frequency of such radiation.
- For freq, higher than 1 THz, inverse Compton scattering can be used.
- However, for the radiation to reach saturation and remain coherent, the slip factor seen by particles of all momenta in the bunch must be controlled to $|\eta| \lesssim 10^{-7}$.
- This implies the control of momentum compaction up to $\mathcal{O}(\delta^2)$ and is extremely difficult.
- Whether this method works at ICEEM remains an open question.

Linac

- Main difference from rings is no synchrotron oscillation because
  1. Total linac length is not long enough
  2. Bunch is placed at crest of rf wave.
- Thus there is no head-tail exchange.

  Tail particles constantly affected by head particles.
- Longitudinal and transverse wake effects will be different

  from accelerator rings.

Longitudinal Effects

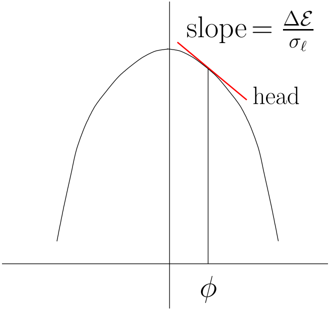
- Beam particles lose energy constantly because of radiation.
- But tail particles lose more because of wake left by head particles.
- Example of SLAC linac:
  Linac length $L = 3$ km, cell length $L_0 = 3.5$ cm.
  Wake: $W_0'(0) = 7.0\ \text{cm}^{-1}$ per cell
  $W_0'(\sigma_\ell) = 4.5\ \text{cm}^{-1}$ per cell, $\sigma_\ell = 1$ mm.
- Change from *Gaussian* units to *MKS* units:
  W_0' is in stat volts/stat coulomb
  1 stat volt $= 300$ V, 1 stat coulomb $= \frac{1}{3} \times 10^9$ coulombs.
  Thus $W_0'(0) = 7.0 \times 300 \times 3 \times 10^9 = 6.29 \times 10^{12}$ V/C.
- Two particle model: bunch represented by 2 macro-bunches σ_ℓ apart, each carrying charge $\frac{1}{2}eN$ with $N = 5 \times 10^{10}$.

head particle: $$\Delta\mathcal{E}_{\rm head} = -\frac{1}{2}\frac{1}{2}Ne^2W_0'(0)\frac{L}{L_0} = -1.08\ \text{GeV/electron}$$

tail particle: $$\Delta\mathcal{E}_{\rm tail} = -\left[\frac{1}{2}\frac{1}{2}Ne^2W_0'(0) + \frac{1}{2}Ne^2W_0'(\sigma_\ell)\right]\frac{L}{L_0}$$
$$= -1.08 - 1.39 = -2.47\ \text{GeV/electron}$$

Compensation

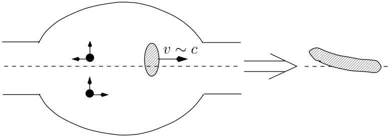
- If uncorrected, energy spread of bunch will increase. Unwanted chromatic effects may result.
- Usually bunch is placed at crest of rf wave to receive maximum acceleration. Can displace it so that head receives less energy than tail.



- SLAC linac: $V_{\rm rf} = 600$ kV per cell. For $L = 3$ km, total rf $V_t = V_{\rm rf}\dfrac{L}{L_0} = 51.4$ GV. $\dfrac{\omega_{\rm rf}}{2\pi} = 2.8$ GHz
- RF wave is $V = V_t \cos\left(\dfrac{\omega_{\rm rf} z}{c} + \phi\right)$ with $\phi = 0$ implying at crest.
- From 2-particle model, $\Delta\mathcal{E} = \Delta\mathcal{E}_{\rm tail} - \Delta\mathcal{E}_{\rm head} = -1.39$ GeV.
- We choose phase offset ϕ so that $V' = -V_t \sin\phi \dfrac{\omega{\rm rf}}{c} = \dfrac{\Delta\mathcal{E}}{\sigma_\ell} \longrightarrow \phi = 27.4^\circ$.
- Actual computation gives $\Delta\mathcal{E} = -0.9$ GeV and $\phi = 17.3^\circ$.

Transverse Effects

- Want to address the effect of transverse wake.
- In a linac, tail particles are constantly pushed sideway by head wake of particles.
- Deflection of tail accumulates along linac.
- When tail particles hit vacuum chamber ⟶ beam loss.
- This is called *beam breakup* (BBU)

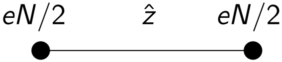


Particles in same vertical slice receive same vertical impulse independent of vertical position. Can lead to beam breakup.

- To avoid BBU, transverse wake must be suppressed.

Two-Particle Model

- A short bunch: represented by 2 macroparticles each of charge $\frac{1}{2}eN$.



$eN/2$ $\hat{z}$ $eN/2$

$\hat{z} =$ rms bunch length

- $$\frac{d^2y_1}{ds^2} + k_{\beta_1}^2 y_1 = 0 \qquad L = \text{length of a number of cavities}$$
  $$\frac{d^2y_2}{ds^2} + k_{\beta_2}^2 y_2 = -\frac{e^2N_bW_1(\hat{z})}{2LE_0}\,y_1 \qquad L = 2\pi R \text{ for a ring.}$$
- $k_{\beta_{1,2}} = \dfrac{\nu_{\beta_{1,2}}}{R} \sim \dfrac{1}{\beta_\perp}$ for a ring.

  For linac, $\nu_{\beta_{1,2}}$ is number of betatron oscillations in length L.
- Solution: $y_1(s) = y_{10}\cos k_{\beta_1}s$

  $$y_2(s) = y_{10}\cos\bar{k}_\beta s\cos\frac{\Delta k_\beta s}{2} - y_{10}\sin\bar{k}_\beta s\left[\frac{\Delta k_\beta}{2} + \frac{e^2N_bW_1(\hat{z})}{4LE_0\bar{k}_\beta}\right]\left[\frac{\sin\Delta k_\beta s/2}{\Delta k_\beta/2}\right],$$

  with $\bar{k}_\beta = \frac{1}{2}(k_{\beta_1} + k_{\beta_2})$, $\Delta k_\beta = k_{\beta_2} - k_{\beta_1}$
- As $\Delta k_\beta \to 0$, last term $\propto s$
- This is a resonance growth.

- $y_2(s) = y_1(s) - s\, y_{10} \sin k_{\beta_1} s \dfrac{e^2 N_b W_1(\hat{z})}{4E_0 L k_\beta}$

  Thus in a length L_0, tail's oscillation amp grows by Υ_1-fold:

  $\Upsilon_1 = -\dfrac{e^2 N_b W_1(\hat{z}) L_0}{4E_0 L k_\beta}$
- Let us look at the transverse wake. Assuming broadband,

  $W_1(z) = -\dfrac{\omega_r^2 Z_1^\perp}{Q\bar{\omega}}\, e^{-\alpha z/c} \sin \dfrac{\bar{\omega} z}{c}$

  $Z_1^\perp$ is transverse impedance at the resonant frequency ω_r
  $\bar{\omega} = \sqrt{\omega_r^2 - \alpha^2}$
  $\alpha = \omega_r/(2Q)$, Q being the quality factor.
- Introduce dimensionless variables

  $v = \dfrac{\omega_r \sigma_\ell}{c}, \quad t = \dfrac{z}{\sigma_\ell}, \quad \text{and} \quad \phi = vt\cos\phi_0 = \dfrac{\bar{\omega} z}{c}$

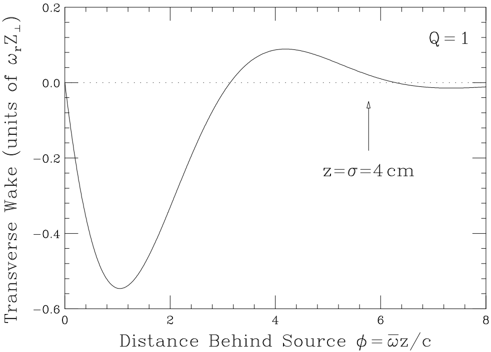
  with $\cos\phi_0 = \sqrt{1 - \dfrac{1}{4Q^2}} \quad \text{or} \quad \sin\phi_0 = \dfrac{1}{2Q}$
- For $Q > \frac{1}{2}$

  $W_1\big|_{\min} = -2\omega_r Z_1^\perp \tan\phi_0 \cos\phi_0\, e^{-\left(\frac{\pi}{2}-\phi_0\right)\tan\phi_0}$

- Minimum at $W_1\big|_{\min} = -2\omega_r Z_1^\perp \tan\phi_0 \cos\phi_0\, e^{-\left(\frac{\pi}{2}-\phi_0\right)\tan\phi_0}$
  at $\phi = \dfrac{\pi}{2} - \phi_0$ or $\dfrac{\alpha z}{c} = \left(\dfrac{\pi}{2} - \phi_0\right)\tan\phi_0$.

Plot of dipole wake W_1

with $\omega_r = 50$ GHz



- 2-particle model applicable only in linear part of wake, usually for only short bunches.
- Or valid only when $\phi = \dfrac{\bar{\omega} z}{c} \ll 1 \quad\longrightarrow\quad \sigma_\ell \ll \dfrac{1}{2}\,\dfrac{\lambda}{2\pi}$
- With $Q \sim 1$, resonant freq $f_r = 7.96$ GHz ($\omega_r = 50$ GHz), two-particle model works only when the rms bunch length $\sigma_\ell \ll 3$ mm.

Long Bunches

- Although long bunches are for proton or muon rings,

  will go over theory because it will be useful later.

- For linear density $\lambda(z)$, transverse motion of a particle is

$$\frac{d^2y(z,s)}{ds^2}+k_\beta^2 y(z,s)=-\frac{e^2N_b}{LE_0}\int_{-\infty}^{z}dz'\lambda(z')W_1(z-z')y(z',s).$$

- One solution is by iteration:

  equate LS to zero and solve for $y(z,s)$.

  substitute soln on RS, and solve again,

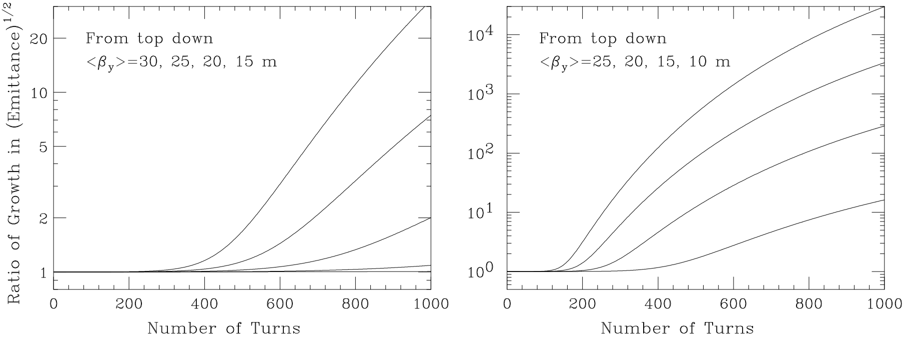
  iterate until soln is stable.

- When amplitude growth factor Υ_1 is large, soln gives growth as powers of Υ_1 or even exponential.

- Soln very sensitive to $\left[\beta_y Z_1^\perp\right]$, ω_r, as well as Q.

- Beam-breakup growth for 1000 turns of a muon bunch of intensity 4×10^{12} at 50 GeV interacting with a broadband impedance of $Q = 1$, $Z_1^{\perp} = 0.1$ MΩ/m at the angular resonant frequency of $\omega_r = 50$ GHz.
- Left: rms 13-cm bunch has total growths of 32.50, 7.4, 2.0, 1.09, 1.006, respectively for $\langle\beta_y\rangle = 30, 25, 20, 15, 10$ m.

  Right: rms 4-cm bunch has total growths of 29713, 3361, 287, 16.2, respectively for $\langle\beta_y\rangle = 25, 20, 15, 10$ m.



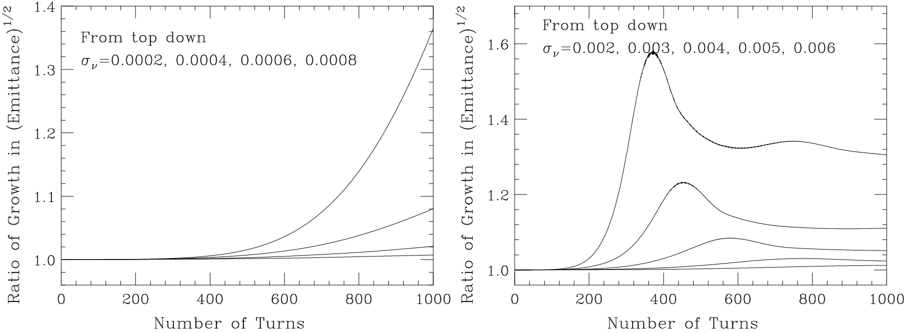
Balakin–Novokhatsky–Smirnov Damping (BNS)

- Kim, Wurtele, and Sessler suggested to suppress beam breakup by a small tune spread in the beam, coming either through chromaticity, amplitude dependency, or beam-beam interaction. [35]
- A beam particle will be resonantly driven by only a small number of particles in front that have the same betatron tune.

  This is a form of *Balakin–Novokhatsky–Smirnov* (BNS) damping suggested in 1983. [36]
- To implement this, add a detuning term $\Delta\nu_{\beta_i} = a[y_i^2 + (\langle\beta_y\rangle y_i')^2]$ to the *i*th particle, as if it is contributed by an octupole or sextupole.
- The rms tune spread becomes $\sigma_{\nu_\beta} = a\langle\sigma_y^2 + (\langle\beta_y\rangle\sigma_{y'})^2\rangle$.
- Continue the example of a muon bunch of intensity 4×10^{12}. Solve for the growth in 1000 turns, assuming $\langle\beta_\perp\rangle = 20$ m and $Z_1^\perp = 0.1$ MΩ/m.

- Left: growths of rms 13 cm bunch are 1.36, 1.08, 1.02, 1.007 for rms tune spread of $\sigma_{\nu_\beta} = 0.0002, 0.0004, 0.0006, 0.0008$.

  Right: growths of rms 4 cm bunch are 1.58, 1.23, 1.08, 1.03, 1.012 for rms tune spread of $\sigma_{\nu_\beta} = 0.002, 0.003, 0.004, 0.005, 0.006$.



- Notice that for 13-cm bunch, $\sigma_{\nu_\beta} = 0.0006$ damps growth to < 1.08. To do the same for 4-cm bunch, $\sigma_{\nu_\beta} = 0.004$ is needed.
- Thus BNS damping is good for long bunch only.

Autophasing [37]

- Let us look at the result of two-particle model again.

  $$y_2(s) = y_{10}\cos\bar{k}_\beta s\cos\frac{\Delta k_\beta s}{2} - y_{10}\sin\bar{k}_\beta s\left[\frac{\Delta k_\beta}{2} + \frac{e^2 N_b W_1(\hat{z})}{4LE_0\bar{k}_\beta}\right]\left[\frac{\sin\Delta k_\beta s/2}{\Delta k_\beta/2}\right],$$

- If we let the tune difference $\Delta k_\beta = -\dfrac{e^2 N_b W_1(\hat{z})}{2LE_0\bar{k}_\beta}$,

  the resonant growth term will be eliminated.

  Then $y_2(s) = y_{10}\cos\bar{k}_\beta s\cos\dfrac{\Delta k_\beta s}{2}$

- This implies we allow a tune difference along the linear bunch density.
- Can do better by letting $\Delta k_\beta = -\dfrac{e^2 N_b W_1(\hat{z})}{4LE_0\bar{k}_\beta} = \dfrac{\Upsilon_1}{L_0}$

  Then $y_2(s) = y_{10}\left[\cos\bar{k}_\beta s\cos\dfrac{\Delta k_\beta s}{2} - \sin\bar{k}_\beta s\sin\dfrac{\Delta k_\beta s}{2}\right] = y_{10}\cos k_{\beta 1}s$

  exactly same as head particle.

- Being in phase all the time, the tail cannot be driven by the head at all. This is another form of BNS damping known as *autophasing*.

Autophasing for Long Bunches

- Add Δk_β to k_β to get
$$\frac{d^2y(z,s)}{ds^2}+\left[k_\beta+\Delta k_\beta(z)\right]^2 y(z,s)=-\frac{e^2N_b}{LE_0}\int_{-\infty}^{z}dz'\lambda(z')W_1(z-z')y(z',s)$$
$\lambda(z)$ is linear bunch density.

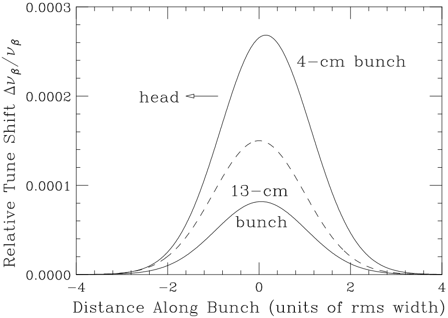
- In order to have $y(z,s)\sim\sin\left(k_\beta s+\varphi_0\right)$ independent of z, one needs
$$2k_\beta\Delta k_\beta+\Delta k_\beta^2(z)=-\frac{e^2N_b}{LE_0}\int_{-\infty}^{z}dz'\lambda(z')W_1(z-z')$$

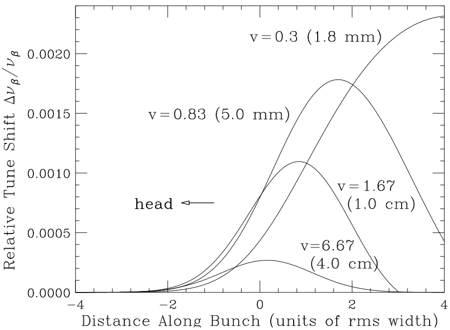
- For small compensation, $\displaystyle\frac{\Delta k_\beta(z)}{k_\beta}=-\frac{e^2N_bR}{2LE_0k_\beta^2}\int_{-\infty}^{z}dz'\lambda(z')W_1(z-z')$

- For Gaussian $\lambda(z)$ and resonant wake, can integrate to close form in terms of complex error function.

- Continue with example of muon bunches.

- For long bunches like muon bunches $v = \omega_r \sigma_\ell / c = 6.67$ and 21.7, compensation is mostly symmetric and Gaussian.
- For short electron bunches, compensation becomes linear. Can implement thru chromaticity by displacing bunch from crest.





- Autophasing is used mostly for short electron bunches in linac.
- For long bunches in a ring, need rf quadrupole to be pulsed according to compensation curve as bunch passing through.
- If frequency is high, one needs cavities having dipole oscillations.

Adiabatic Damping in Linacs

- In linacs, beam energy increases rapidly, 2-particle model should read:

$$\frac{1}{\gamma}\frac{d}{ds}\left(\gamma\frac{dy_1}{ds}\right)+k_\beta^2 y_1=0$$

$$\frac{1}{\gamma}\frac{d}{ds}\left(\gamma\frac{dy_2}{ds}\right)+k_\beta^2 y_2=-\frac{e^2N_bW_1(\hat{z})}{2L\gamma E_{\rm rest}}\,y_1$$

- Assume linear acceleration $\gamma(s)=\gamma_i(1+\alpha s)$, α constant.
- First equation becomes $\dfrac{d}{du}\left(u\dfrac{dy_1}{du}\right)+\dfrac{k_\beta^2}{\alpha^2}\,uy_1=0$, with $u=1+\alpha s$.

  Solution is $y_1(s)=\hat{y}J_0[k_\beta(1+\alpha s)/\alpha]\approx\dfrac{\hat{y}}{\sqrt{1+\alpha s}}\cos k_\beta s$,

  since $\alpha/k_\beta\ll 1$ usually.

  SLAC linac, $E_i=1$ GeV to $E_f=50$ GeV, $\alpha=0.0163\ \mathrm{m}^{-1}$,

  while the betatron wave number is $k_\beta=0.06\ \mathrm{m}^{-1}$.
- Tail particle: $\dfrac{d}{du}\left(u\dfrac{dy_2}{du}\right)+\dfrac{k_\beta^2}{\alpha^2}\,uy_2=-\dfrac{e^2N_bW_1(\hat{z})}{2LE_i\alpha^2}\,\dfrac{\hat{y}}{\sqrt{u}}\cos k_\beta s$
- Try $y_2=\dfrac{D\sin k_\beta s}{\sqrt{u}}$ with D a slowly varying function of u.

- Get $y_2(s) = \dfrac{\hat{y}}{\sqrt{1+\alpha s}}\left[\cos k_\beta s - \dfrac{e^2 N_b W_1(\hat{z})}{4LE_i\alpha k_\beta}\ln(1+\alpha s)\sin k_\beta s\right].$
- Since $E_f = E_i(1+\alpha L_0)$, $\alpha L_0 \approx \dfrac{E_f}{E_i}$.
- Growth for the whole length L_0 is

  $$\Upsilon_1 = -\frac{e^2 N_b W_1(\hat{z}) L_0}{4k_\beta E_f L}\ln\frac{E_f}{E_i}.$$
- Compare with former result, there is an extra factor of $\mathcal{F} = \dfrac{E_i}{E_f}\ln\dfrac{E_f}{E_i}$.
- For SLAC linac, $\mathcal{F} = 1/12.8 = 0.0782$ meaning that the tail will be deflected by 12.8 less with the acceleration.
- This effect is called *adiabatic damping*.

Detuned Cavities

- We shown before that a spread in betatron tune helps in alleviating bbu. Such spread can come from a detuned cavity structure.
- Dipole wake of a cavity structure is given by

  $$W_1(z) = -2\sum_n K_n \sin\frac{2\pi\nu_n z}{c}\, e^{-\pi\nu_v z/(cQ_n)} \qquad z > 0$$

  where $K_n = \dfrac{R_n c}{Q_n}$, ν_n, and Q_n are *kick factor*, resonant frequency, and quality factor of the nth eigenmode in the structure.
- To reduce bbu, it is important to reduce this dipole wake $W_1(z)$.
- One way to reduce $W_1(z)$ is to manufacture the cavity structure with cell dimension varying gradually so that each cell has a slightly different resonant frequency.
- Effect of the wake due to sharp resonant peak of each individual cell will not add together and wake of the whole structure will be reduced.
- Such a structure is called a *detuned cavity structure*.

Short-Range Part of Wake

- For short range, can assume all cells do not couple.

  Total wake is sum of wake of individual cells.

- Cell-to-cell variation is small, can replace sum over cells by an integral.

  $$W_1(z) \approx -2 \int d\nu \, K \frac{dn}{d\nu} \sin \frac{2\pi\nu z}{c} \qquad \text{one eigenmode included only}$$

- $W_1(z)$ defined as *dipole wake per cell* $\longrightarrow$ $\dfrac{dn}{d\nu}$ normalized to unity.

- Let $\nu = \bar{\nu} + x$, where $\bar{\nu}$ is average resonant frequency.

  $$W_1(z) \approx -2\,\mathcal{I}m \left[\underset{\substack{\uparrow \\ \text{rapidly varying}}}{e^{2i\pi\bar{\nu}z/c}} \underbrace{\int dx \, K(\bar{\nu}+x) \frac{dn}{d\nu}(\bar{\nu}+x) e^{2\pi i x z/c}}_{\substack{\uparrow \\ \text{slowly varying} \\ \text{envelope}}} \right]$$

- Slowly varying part or envelope is Fourier transform of $K\dfrac{dn}{d\nu}$.

Two Examples

- Uniform frequency distribution with *full* frequency spread $\Delta\nu$:

$$W_1(z) \approx -2\bar{K}\,\sin\frac{2\pi\bar{\nu}z}{c}\,\frac{\sin(\pi\Delta\nu z/c)}{\pi\Delta\nu z/c}$$

or flat distribution

- Gaussian freq distribution with rms width σ_ν:

$$W_1(z) \approx -2\bar{K}\,\sin\frac{2\pi\bar{\nu}z}{c}\,e^{-2(\pi\sigma_\nu z/c)^2}$$

Rapid decay as Gaussian and is therefore preferred.

Next Linear Collider (NLC)

$N = 206$ cells

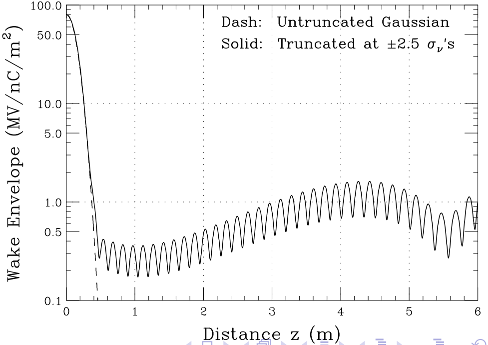
$\bar{\nu} = 15.25$ GHz

$\bar{K} = 40$ MV/nC/m^2

Detuned distribution:

Gaussian $\pm 2.5\sigma_\nu$

$\sigma_\nu = 2.5\%$ of $\bar{\nu}$.

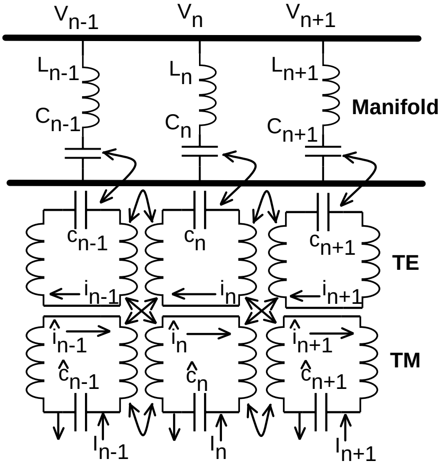


Comments on NLC Detuned Wake

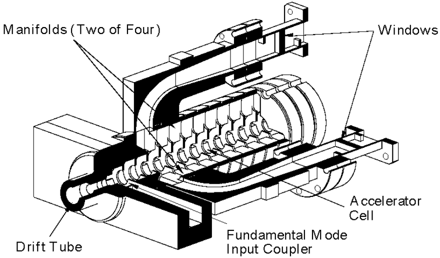
- The negative signs in the wake expressions are just convention.
- The detuned dipole wake actually starts from zero, increases linearly, reaches a maximum, and rolls off like a Gaussian.
- For larger distance, the roll-off stops, because the detuning is *not a true Gaussian*, but *truncated*. So we get $\frac{\sin x}{x}$-behavior.
- There are the 42-cm and 82-cm bunch-spacing scenarios. The dipole wake at the 2nd bunch has been suppressed by more than 2 orders of magnitude.
- This detuning method is very useful for long-range bunch-to-bunch bbu, but not useful for single bunch bbu.
- However, for long-range suppression, we cannot trust the above result, because cell-to-cell interaction has been neglected.

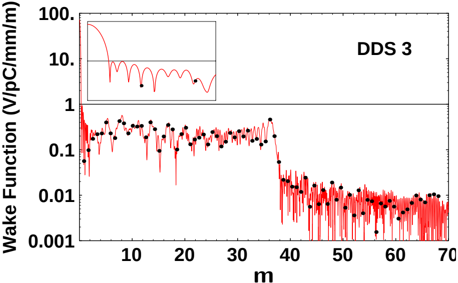
Circuit Model for DDS

- To incorporate interaction between cells, Bane and Gluckstern devised an equivalent circuit to represents cell structure. [38]
- Later Kroll and Jones *et al.* improved the model by introducing two circuits together with 4 damping manifolds, corresponding to 4 holes in the cells to carry away dipole wake. [39]



- There are $N = 206$ TE and TM resonant circuits connected together.
- The wake is computed by solving a 618×618 matrix.
  But the matrix is sparse, which makes solution much easier.





- The short-range part of detuned wake is almost same as earlier calculation.
- The long-range part of detuned wake is suppressed to less than 1 V/pC/mm/m.
- The dots represent the 82 bunches with 84-cm spacing.
- The wake was computed in the frequency domain and Fourier transformed to time domain.
  Thus very short-range part may not be accurate.
- Parameters used are obtained by fitting to measured freq vs phase advance dispersion curve.

Multi-Bunch BBU

- DDS suppresses the dipole wake by more than 2 orders of magnitude. For the 95-bunch 42-cm scenario of NLC, dipole wake per unit length is only $\sim 0.21\ \mathrm{MV/nC/m^2}$.
- Want to examine how much emittance growth and bbu will be driven by detuned wake, and how much energy spread is required for further suppression.
- 2-particle model is used by considering each bunch as a maga particles

  1st bunch: $\dfrac{d^2y_1}{ds^2} + k_\beta^2 y_1 = 0$

  2nd bunch: $\dfrac{d^2y_2}{ds^2} + k_\beta^2 y_2 = -\dfrac{e^2 N_b W_1(\hat{z})}{LE}\, y_1$

  where L is cavity length and $\hat{z}$ is bunch spacing.
- Soln: 1st bunch $y_1(s) = \mathcal{R}e\, \hat{y} e^{ik_\beta s}$

  2nd bunch $y_2(s) = \mathcal{R}e\, \hat{y}\, \Gamma s e^{ik_\beta s}$ $\qquad \Gamma = \dfrac{ie^2 N_b W_1(\hat{z})}{2k_\beta L E_0}$

  Keep only particular solution because it increases with s.

- 3rd bunch: $\dfrac{d^2y_3}{ds^2}+k_\beta^2 y_3=-\dfrac{e^2N_bW_1(2\hat z)}{LE_0}\,y_1-\dfrac{e^2N_bW_1(\hat z)}{LE_0}\,y_2$
- Keeping only the most divergent term, i.e., the last term, $y_3(s)=\mathcal{R}e\,\hat y\,\dfrac{1}{2}\,\Gamma^2s^2e^{ik_\beta s}$
- Continuing, get $y_m(s)=\mathcal{R}e\,\hat y\,\dfrac{\Gamma^{m-1}s^{m-1}}{(m-1)!}\,e^{ik_\beta s}$
- If we employ BNS damping on 2nd bunch, amount of tune spread is $\dfrac{\Delta k_\beta}{k_\beta}=-\dfrac{e^2N_bW_1(\hat z)}{2k_\beta^2E_f}\ln\dfrac{E_f}{E_i}$

  where adiabatic damping has been included.
- To damp n_b bunches, it is reasonable to assume n_b times of spread.
- Tune spread can come from chromaticity.

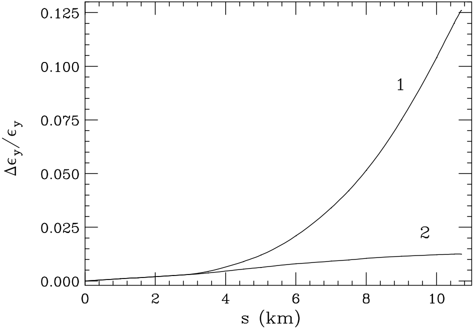
  For FODO lattice of phase advance μ, natural chromaticity is $\xi_N=-\dfrac{2}{\pi}\tan\dfrac{\mu}{2}$
- If we take $\xi_N=-1$, we get required energy spread of 2.7%

- Simulation has been performed by Stupakov, assuming initial bunch offset of 1 μm. [41]
- The vertical emittance of the last or 95th bunch was monitored.

Increase of vertical emittance of 95th bunch:

curve 1: w/o energy variation

curve 2: 0.8% energy variation



- At 0.8% energy variation, vertical emittance of 95th bunch increases by only 1.2%.
- Analytic treatment of multi-bunch bbu has been performed by Bohn, Delayen, and Ng, with good agreement with simulations. [42, 43, 44]

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