Fermi National Accelerator Laboratory

FN-365
1500.000
(Submitted to
Lettere al Nuovo Cimento)

A NEW METHOD FOR POTENTIAL OF A 3-DIMENSIONAL NON-UNIFORM CHARGE DISTRIBUTION

Ken Takayama

May 1982

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

A New Method for Potential of a 3-Dimensional Non-Uniform Charge Distribution

Ken Takayama

Fermi National Accelerator Laboratory , Batavia, Ill 60510, USA

Summary

A new simple method for obtaining the expression of the electrical potential for a 3-dimensional non-uniform charge distribution is given. It is shown how the new method can be used to obtain the expression for a 3-dimensional gaussian distribution.

The traditional method for obtaining the potential of a three-dimensional non-uniform charge distribution has relied on its spatial symmetry(1)(2). In this paper, a simple method is given which is independent of such a spatial symmetry.

We consider the Poisson's equation

$$\nabla^2 \phi(\mathbf{r}) = -4\pi \rho(\mathbf{r}), \tag{1}$$

with the charge distribution $\rho(\mathbf{r})$,
where $\mathbf{r}$ denotes (x,y,z).
The Green's function corresponding to (1) has been well-known in the form

$$G(\mathbf{r}, \boldsymbol{\xi}) = \frac{1}{4\pi |\mathbf{r} - \boldsymbol{\xi}|}, \tag{2}$$

which satisfies the equation

$$\nabla^2 G(\mathbf{r}, \boldsymbol{\xi}) = -\delta(\mathbf{r} - \boldsymbol{\xi}), \tag{3}$$

where $\boldsymbol{\xi}$ denotes (ξ_1, ξ_2, ξ_3). Now we rewrite the Green's function (2) by an integral representation

$$\frac{1}{4\pi |\mathbf{r} - \boldsymbol{\xi}|} = \frac{1}{2\pi^{3/2}} \int_0^\infty e^{-|\mathbf{r} - \boldsymbol{\xi}|^2 \rho^2} d\rho, \tag{4}$$

which is well-known as the integration formula. Note that couplings among three coordinates (x,y,z) appearing in the left-hand side of (4) are separated in the integrand. For later application, we make a change of the integration variable

$$t = \frac{1}{\gamma}, \tag{5}$$

then

$$G(\mathbf{r}, \boldsymbol{\xi}) = \frac{1}{4\pi^{3/2}} \int_0^\infty \frac{e^{-\frac{|\mathbf{r}-\boldsymbol{\xi}|^2}{t}}}{t^{3/2}}\, dt. \tag{6}$$

Using (6), we obtain a formal solution of the Poisson's equation

$$\begin{aligned} \phi(\mathbf{r}) &= 4\pi \iiint_{-\infty}^{\infty} d\boldsymbol{\xi}\; G(\mathbf{r}, \boldsymbol{\xi})\, \rho(\boldsymbol{\xi}) \\ &= \frac{1}{\pi^{1/2}} \int_0^\infty \frac{dt}{t^{3/2}} \iiint_{-\infty}^{\infty} d\boldsymbol{\xi}\; \rho(\boldsymbol{\xi})\, e^{-\frac{|\mathbf{r}-\boldsymbol{\xi}|^2}{t}} \end{aligned} \tag{7}$$

As an example, we consider a three-dimensional gaussian charge distribution

$$\rho(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}\, abc} \exp\left(-\frac{x^2}{2a^2} - \frac{y^2}{2b^2} - \frac{z^2}{2c^2}\right), \tag{8}$$

where a,b,c are the standard deviations.

Substitution of (8) into (7) yields

$$\phi(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}\, abc\, \pi^{1/2}} \int_0^\infty \frac{dt}{t^{3/2}} \iiint_{-\infty}^{\infty} d\boldsymbol{\xi}\, \exp\left(-\frac{|\mathbf{r}-\boldsymbol{\xi}|^2}{t} - \frac{\xi_1^2}{2a^2} - \frac{\xi_2^2}{2b^2} - \frac{\xi_3^2}{2c^3}\right). \tag{9}$$

Integrals of the form

$$(10) \qquad \int_{-\infty}^{\infty} \exp\left[-\frac{(s-\xi)^2}{\tau} - \frac{\xi^2}{2\ell^2}\right] d\xi ,$$

is easily obtained from the integration formula

$$(11) \qquad \int_{-\infty}^{\infty} e^{-\alpha^2\xi^2+\beta\xi}\, d\xi = \frac{\sqrt{\pi}}{\alpha}\, e^{\left(\frac{\beta}{2\alpha}\right)^2} \qquad (\beta>0) .$$

Introducing these results into (9), we therefore have

$$(12) \qquad \phi(\mathbf{r}) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\exp\left(-\frac{x^2}{2a^2+t} - \frac{y^2}{2b^2+t} - \frac{z^2}{2c^2+t}\right)}{\sqrt{(2a^2+t)(2b^2+t)(2c^2+t)}}\, dt .$$

The potential (12) is identical with the expression [3] obtained by application of the traditional method, which makes use of ellipsoidal coordinates.

Separation of variables in the Green's function by introducing the auxiliary parameter q has led to the present success, and its exponential form makes it easy to perform the later integrations. For other charge distributions, choice of a suitable integral representation, as the present case may also help us to obtain analytical expressions of their potentials.

References

(1) O.D.Kellog : Foundations of Potential Theory (Springer-Verlag ,1967), p.184.

(2) E.Weber : Electromagnetic Fields, Theory and Applications Vol 1 (John Wiley & Sons,Inc.,1960), p.414.

(3) B.W.Montague, Fourth-Order Coupling Resonances Excited By Space -Charge Forces In A Synchrotron, CERN/ISR/68-38 (1968).