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DESIGN OF QUADRUPOLES WITH LINEARLY SEGMENTED PROFILES

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SUMMARY

Danby and Jackson[1] have given an empirical procedure for designing quadrupoles with profiles made up of straight line segments. In this note it is demonstrated that conformal transformations may be employed to give a theoretical basis for designing these quadrupoles. The median plane fields compare favorably with those obtained from LINDA using the profile suggested by the transformation.

METHOD

Following the method outlined previously[2] one may begin by expressing the complex potential W in the λ-plane due to current filaments that are imaged to produce a line of constant flux, U = 0, for $|\lambda| = 1$. Figure 1 indicates the details for one such current filament and its images.

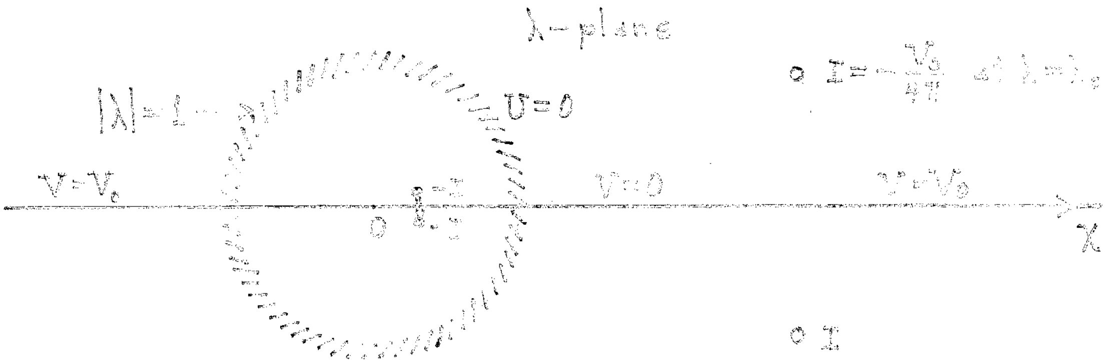

Fig. 1 Magnetostatics for Single Current Filament

If N such current filaments are located at $\lambda = \lambda_{ok}$, then

$$W = \frac{V_o}{2\pi N} \sum_{k=1}^{N} \ell n \left[\frac{(\lambda_{ok}\lambda - 1)(\lambda_{ok}^{*}\lambda - 1)}{(\lambda - \lambda_{ok})(\lambda - \lambda_{ok}^{*})} \right] \qquad (1)$$

The λ-plane is transformed to the s-plane using

$$s = \frac{\tau}{2}\left(\lambda + \frac{1}{\lambda}\right) \qquad (2)$$

and is shown in Fig. 2 where it is considered as the starting plane for a Schwartz-Christoffel transformation.

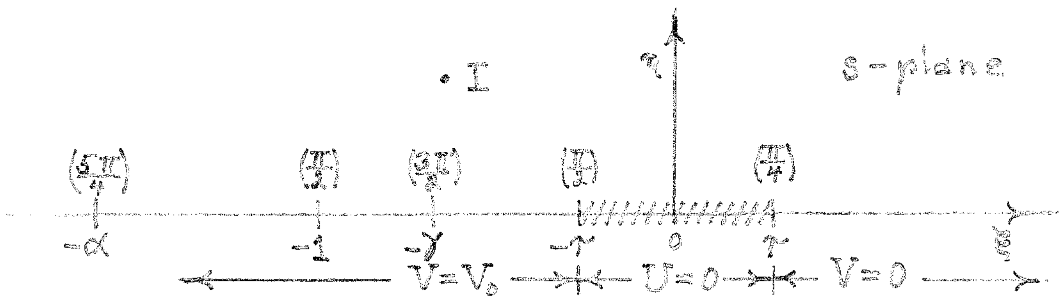

Fig. 2 Geometric Details of Starting s-plane

The real axis of Fig. 2 is bent at the points indicated to produce a polygon with verticies having interior angles shown in parentheses. Figure 3 shows the resulting z-plane. Since the line U = 0 is a line of constant flux, the desired profile of one half of the pole has been formed.

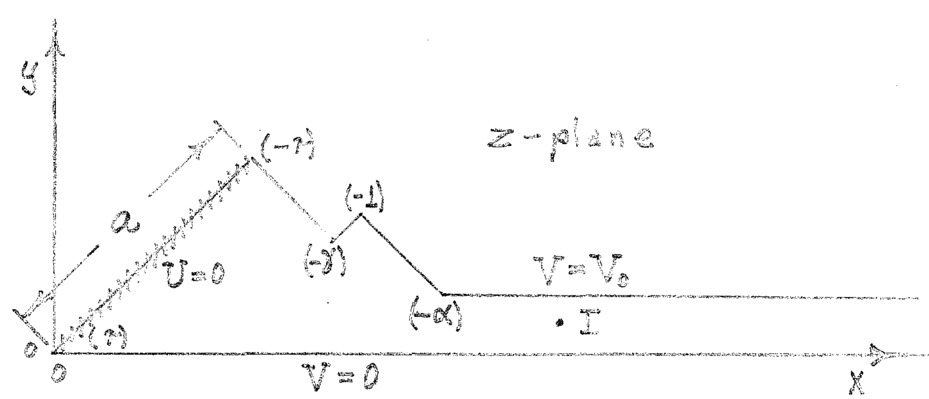

Fig. 3 Geometric Details of Final z-plane

The transformation is

$$z = \frac{aA_1 e^{-i\frac{\pi}{4}}}{C_1} \int_{\tau}^{s} \frac{1}{G(s)} \cdot \frac{ds}{(s-\tau)^{3/4}(s+\tau)^{1/2}} , \tag{3}$$

where the form of the arbitrary constant has been chosen for later convenience and

$$G(s) = \frac{(1+s)^{1/2}}{\left(1+\frac{s}{\alpha}\right)^{1/4}\left(1+\frac{s}{\gamma}\right)^{1/2}} . \tag{4}$$

If a w-plane is formed using

$$w = A_1 \int_{\tau}^{s} \frac{ds}{(s-\tau)^{3/4}(s+\tau)^{1/2}} \tag{5}$$

Eq. (3) may be written as

$$z = \frac{ae^{-i\frac{\pi}{4}}}{C_1} \int_{0}^{w} \frac{dw}{G(s)} \tag{6}$$

where a is considered as the half aperture as shown in Fig. 3.

To evaluate Eq. (5) let

$$s = \tau - 2\tau\ t^4. \tag{7}$$

Thus

$$w = \frac{4A_1 e^{i\frac{\pi}{4}}}{\sqrt[4]{2\tau}} \int_{0}^{t} \frac{dt}{\sqrt{1-t^4}} \tag{8}$$

or

$$w = \frac{4A_1 e^{i\frac{\pi}{4}}}{\sqrt{2}\ \sqrt[4]{2\tau}} \left[K\left(\frac{1}{\sqrt{2}}\right) - \mathrm{cn}^{-1}\left(t, \frac{1}{\sqrt{2}}\right)\right] \tag{9}$$

where K is the complete elliptic integral[3] and cn is the Jacobi elliptic function.[3] For convenience choose $w = i\pi$ for $s = -\tau$ or $t = 1$. Thus

$$A_1 = \frac{\pi\ \sqrt[4]{2\tau}}{2\ \sqrt{2}\ K\left(\frac{1}{\sqrt{2}}\right)}\, e^{i\frac{\pi}{4}} \tag{10}$$

and Eqs. (7), (9), and (10) give

$$s = \tau - 2\tau\ \mathrm{cn}^4\left[K\left(\frac{1}{\sqrt{2}}\right)\left(1 + i\frac{w}{\pi}\right),\ \frac{1}{\sqrt{2}}\right] \tag{11}$$

Figure 4 indicates the relations in the w-plane.

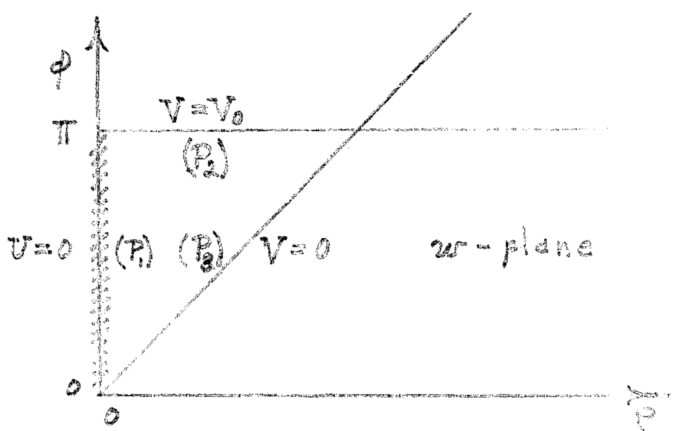

Fig. 4 Geometric Details of w-plane

For $w = i\phi$ (Path P_1), s varies from τ to $-\tau$.

For $w = i\pi + \rho$ (Path P_2), s is real and negative.

For $\phi = \rho$ (Path P_3), s is real and positive.

If in Eq. (6) one integrates along path P_1 from 0 to $i\pi$ and associates $i\pi$ with $z = ae^{i\frac{\pi}{4}}$ one finds

$$C_1 = \int_0^\pi \frac{d\phi}{G(\xi)} \tag{12}$$

where

$$\xi = \tau - 2\tau \, cn^4\left[K\left(\frac{1}{\sqrt{2}}\right)\left(1 - \frac{\phi}{\pi}\right), \frac{1}{\sqrt{2}}\right]. \tag{13}$$

The pole profile is given by Eq. (6) using an integration first along path P_1 and subsequently along path P_2. Thus the contour is

$$z = a\left[e^{i\frac{\pi}{4}} + \frac{e^{-i\frac{\pi}{4}}}{C_1}\int_0^\rho \frac{d\rho}{G(s)}\right], \tag{14}$$

where s is given by Eq. (11) using $w = i\pi + \rho$. On the median plane of Fig. 3 the relation between x and ρ in the w-plane is given by integrating Eq. (6) along path P_3. Thus

$$x = \frac{a\sqrt{2}}{C_1}\int_0^\rho \frac{d\rho}{G(s)}, \tag{15}$$

where s is determined from Eq. (11) using w for which $\phi = \rho$. Thus

$$s = \tau - 2\tau \, cn^4\left[K\left(\frac{1}{\sqrt{2}}\right)\left(1 - (1-i)\frac{\rho}{\pi}\right), \frac{1}{\sqrt{2}}\right] \tag{16}$$

which by properties of the Jacobi elliptic function is real.

MAGNETIC FIELD

Since the imaginary part of the complex potential W has been chosen as the magnetostatic potential, if one lets $H = H_x + iH_y$ then

$$H^* = i\frac{dW}{dz} = i\ \frac{dW}{d\lambda} \cdot \frac{d\lambda}{ds} \cdot \frac{ds}{dw} \cdot \frac{dw}{dz}\ . \tag{17}$$

After obtaining the various derivatives from Eqs. (1), (2), (5), and (6) one finds

$$H^* = i\frac{V_o}{\pi} \cdot \frac{2\sqrt{2}K\left(\frac{1}{\sqrt{2}}\right)C_1}{\pi a} \cdot \sqrt[4]{\frac{s-\tau}{2\tau}}\ G(s)\left[1 + C(\lambda)\right], \tag{18}$$

where

$$C(\lambda) = \frac{\lambda}{2N}\sum_{k=1}^{N}\left(\frac{1}{\lambda - \frac{1}{\lambda_{ok}}} + \frac{1}{\lambda - \frac{1}{\lambda^*_{ok}}} - \frac{1}{\lambda - \lambda_{ok}} - \frac{1}{\lambda - \lambda^*_{ok}}\right) - 1. \tag{19}$$

The gradient of the magnetic field may be found from

$$\frac{dH^*}{dz} = \frac{dH^*}{ds} \cdot \frac{ds}{dw} \cdot \frac{dw}{dz}\ . \tag{20}$$

Taking the imaginary part on the median plane (y = 0) gives

$$H_y'(x) = -\frac{V_o}{\pi}\left[\frac{2\sqrt{2}K\left(\frac{1}{\sqrt{2}}\right)C_1}{\pi a}\right]^2 \cdot \left\{\left[\frac{1}{4}+(\xi-\tau)\cdot\left(\frac{1}{2}\cdot\frac{1}{1+\xi}-\frac{1}{4}\cdot\frac{1}{\alpha+\xi}-\frac{1}{2}\cdot\frac{1}{\gamma+\xi}\right)\right] \cdot \right.$$

$$\left. \sqrt{\frac{\xi+\tau}{2\tau}} \cdot \Big(1+C(\chi)\Big) + \sqrt{\frac{\xi-\tau}{2\tau}}\ \Big(C(\chi)-D(\chi)\Big)\right\} G^2(\xi), \tag{21}$$

where χ is the real part of λ and

$$D(\chi) = \frac{\chi^2}{2\pi N}\sum_{k=1}^{N}\left\{\frac{1}{\left(\chi-\frac{1}{\lambda_{ok}}\right)^2} + \frac{1}{\left(\chi-\frac{1}{\lambda^*_{ok}}\right)^2} - \frac{1}{\left(\chi-\lambda_{ok}\right)^2} - \frac{1}{\left(\chi-\lambda^*_{ok}\right)^2}\right\} - 1 \qquad (22)$$

In designing a quadrupole it is customary to specify the central gradient $H_y'(0)$ instead of the excitation V_o. From Eq. (21) for $\xi = \tau$ and $\chi = 1$

$$H_y'(0) = -\frac{V_o}{\pi}\left[\frac{2\sqrt{2}K\left(\frac{1}{\sqrt{2}}\right)C_1}{\pi a}\right]^2 \cdot \frac{1}{4}G^2(\tau)\cdot\left[1+C(1)\right] \equiv -B_o'. \qquad (23)$$

Thus

$$\frac{H_y'(x)}{H_y'(0)} = \left\{\left[1 + 4(\xi-\tau)\left(\frac{1}{2}\frac{1}{1+\xi} - \frac{1}{4}\frac{1}{\alpha+\xi} - \frac{1}{2}\frac{1}{\gamma+\xi}\right)\right]\sqrt{\frac{\xi+\tau}{2\tau}}\left[1 + C(\chi)\right] + 4\sqrt{\frac{\xi-\tau}{2\tau}}\left[C(\chi)-D(\chi)\right]\right\}\cdot\frac{G^2(\xi)}{G^2(\tau)}\cdot\frac{1}{1+C(1)} \qquad (24)$$

PARAMETER ADJUSTMENT

The profile design is accomplished by finding the constants ρ_o, ρ_e, α, γ, τ and the set w_{01}, w_{02}, ... w_{0N} where ρ_o and ρ_e are the values of ρ in the w-plane corresponding to the maximum field gradient and the gradient at the "good field limit," and w_{ok} is the location of the k^{th} current filament in the w-plane. The design data is considered to be B_o', B_{max}', B_{end}', a, x_{end}, and the set z_{01}, z_{02}, ...z_{0N} where z_{ok} is the location of the k^{th} current filament in the z-plane.

Since the design is independent of the level of excitation, there are five dimensionless constants plus the current filament locations w_{ok} to be determined from four dimensionless input numbers plus the desired coil locations z_{ok}, one other condition may be imposed on the unknowns. A condition of the type imposed previously in connection with a continuously curved profile[2] is not possible for the present contour. Instead of removing the duodecipole term, one may find a γ that yields a given slot width. The remaining parameters are adjusted by Newton's method as explained previously.

NUMERICAL RESULTS

The relations developed here have been coded for the CDC 6600. The parameters are adjusted by alternating between adjustment of ρ_o, ρ_e, α, τ in the main routine and adjustment of the current filament locations w_{ok} in a subroutine. After an adjustment has been found another subroutine calculates the contour, the field along the contour, the accumulated flux along the contour, and the gradient along the median plane. For confirmation, the contour determined by the complex variable transformation was inserted into an iterative magnetostatic program LINDA. The median plane gradients agree within .1 percent.

REFERENCES

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3. M. Abramowitz and I. A. Stegun, Eds., Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series 55, Sections 16 and 17 (1964). Use formula 17.4.52.