



**EFFECT OF RANDOM FLUCTUATIONS  
ON SYNCHROTRON PHASE MOTION**

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**PURPOSE**

To determine the effect of random fluctuations of the magnetic field, radio frequency, and cavity voltage on the growth of the phase area associated with synchrotron motion. The synchrotron motion will be represented only in the linear approximation. Application is made to booster.

**EQUATIONS OF MOTION**

The notations and formalism in a previous note on linear synchrotron motion<sup>1</sup> will be employed. In order to introduce fluctuations the reference motion is altered to the following:

$$\dot{\psi}_R = -\frac{V+\Delta V}{2\pi h} \sin \phi_R, \quad (1)$$

and

$$\dot{\phi}_R = h(\omega_R + \Delta\omega_R) - (\omega_{RF} + \Delta\omega_{RF}), \quad (2)$$



where  $\Delta V$ ,  $\Delta\omega_R$ , and  $\Delta\omega_{RF}$  represent the instantaneous fluctuations of the corresponding quantities from the desired values. The change in angular orbital frequency  $\Delta\omega_R$  arises from a change in the magnetic field according to

$$\Delta\omega_R = \frac{\partial\omega_R}{\partial R} \Delta R = -\omega_R \frac{\Delta R}{R} = -\frac{\omega_R}{\gamma_T} \frac{\Delta p}{p} = -\frac{\omega_R}{\gamma_T} \frac{\Delta B}{B}, \quad (3)$$

where  $\gamma_T$  is gamma at the transition energy and B the azimuthally averaged magnetic field. An analysis similar to that previously employed<sup>1</sup> gives

$$\dot{J} = -\frac{V\cos\phi_R}{2\pi h} \eta + \frac{\Delta V\sin\phi_R}{2\pi h} \quad (4)$$

and

$$\dot{\eta} = \frac{h^2\omega_R^2\kappa_R}{E_R} J - h\Delta\omega_R + \Delta\omega_{RF} \quad (5)$$

for the synchrotron motion (J,η). Changing the independent variable to

$$s = \int_0^t \frac{h^2\omega_R^2\kappa_R}{E_R} dt \quad (6)$$

and designating differentiation with respect to s by a prime, Eqs. (4-5) become

$$J' = -K\eta + F \quad (7)$$

and

$$\eta' = J + G, \quad (8)$$

where

$$K = \frac{E_R V \cos \phi_R}{2\pi h^3 \omega_R^2 \kappa_R}, \quad (9)$$

$$F = \frac{E_R \Delta V \sin \phi_R}{2\pi h^3 \omega_R^2 \kappa_R}, \quad (10)$$

$$G = \frac{E_R}{h^2 \omega_R^2 \kappa_R} \left( \frac{\omega_R}{\gamma_T} \frac{\Delta B}{B} + \Delta \omega_{RF} \right). \quad (11)$$

### FORMAL SOLUTION OF EQUATIONS OF MOTION

Two real independent solutions of the homogeneous equations will be used as integrating factors for the inhomogeneous equations. Thus let

$$J_1' = -K\eta_1 \quad \eta_1' = J_1 \quad (12)$$

$$J_2' = -K\eta_2 \quad \eta_2' = J_2 \quad (13)$$

These solutions possess the property that  $\eta_2 J_1 - \eta_1 J_2$  is a constant. Choose the solutions such that

$$\eta_2 J_1 - \eta_1 J_2 = 1. \quad (14)$$

Multiply Eq. (4) by  $\eta_1$ , Eq. (5) by  $J_1$ , integrate each by parts and subtract. Similarly multiply Eq. (4) by  $\eta_2$ , Eq. (5) by  $J_2$ , integrate each by parts and subtract. The results may be put in the form

$$\eta_1 J - J_1 \eta = X_1 + H_1 \quad (15)$$

and

$$\eta_2 J - J_2 \eta = X_2 + H_2, \quad (16)$$

where

$$X_1 = \eta_1(0)J(0) - J_1(0)\eta(0), \quad (17)$$

$$X_2 = \eta_2(0)J(0) - J_2(0)\eta(0), \quad (18)$$

$$H_1 = \int_0^S \eta_1 F ds - \int_0^S J_1 G ds, \quad (19)$$

and

$$H_2 = \int_0^S \eta_2 F ds - \int_0^S J_2 G ds. \quad (20)$$

Simultaneous solution of Eqs. (15-16) for  $J$  and  $\eta$  gives

$$J = J_1(X_2 + H_2) - J_2(X_1 + H_1) \quad (21)$$

and

$$\eta = \eta_1(X_2 + H_2) - \eta_2(X_1 + H_1). \quad (22)$$

#### GROWTH IN PHASE AREA DUE TO RANDOM FLUCTUATIONS

The invariant  $W$  associated with the homogeneous equations<sup>1</sup> evaluated using the solutions of the inhomogeneous equations will increase with time because of the perturbations  $F$  and  $G$ . In order to have  $W$  represent motion matched to the small amplitude bucket shape one chooses

$$\beta = \eta_1^2 + \eta_2^2. \quad (23)$$

Then

$$\alpha = -\frac{1}{2}\beta' = -(\eta_1 J_1 + \eta_2 J_2) \quad (24)$$

and

$$\gamma = \frac{1}{\beta}(1+\alpha^2) = J_1^2 + J_2^2. \quad (25)$$

This last equation is evident only after employing Eq. (14).

The unperturbed invariant now becomes

$$\begin{aligned} W = & \frac{1}{2}\beta \left[ J_1(X_2+H_2) - J_2(X_1+H_1) \right]^2 \\ & + \alpha \left[ J_1(X_2+H_2) - J_2(X_1+H_1) \right] \cdot \left[ \eta_1(X_2+H_2) - \eta_2(X_1+H_1) \right] \\ & + \frac{1}{2}\gamma \left[ \eta_1(X_2+H_2) - \eta_2(X_1+H_1) \right]^2. \end{aligned} \quad (26)$$

After considerable algebraic reduction employing Eqs. (23-25)

one has

$$W = \frac{1}{2}(X_1+H_1)^2 + \frac{1}{2}(X_2+H_2)^2 \quad (27)$$

which for  $s = 0$  gives

$$W(0) = \frac{1}{2}(X_1^2 + X_2^2). \quad (28)$$

#### STATISTICAL TREATMENT OF FLUCTUATIONS

Chandrashekar<sup>2</sup> shows that a density distribution  $\rho(W,t)$  in which the variable  $W$  is governed by a random walk process obeys the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial W} \left[ -D_1 \rho + \frac{1}{2} \frac{\partial}{\partial W} (D_2 \rho) \right], \quad (29)$$

where

$$D_1 = \frac{d}{dt} \langle (\Delta W) \rangle_{Av} \quad (30)$$

and

$$D_2 = \frac{d}{dt} \langle (\Delta W)^2 \rangle_{Av}, \quad (31)$$

the brackets representing ensemble averages.

By associating  $\Delta W$  with  $W(s) - W(0)$  one has from Eqs. (27-28)

$$\frac{d}{dt} (\Delta W) = x_1 \dot{H}_1 + x_2 \dot{H}_2 + H_1 \dot{H}_1 + H_2 \dot{H}_2 \quad (32)$$

and

$$\frac{d}{dt} \langle (\Delta W)^2 \rangle = 2x_1^2 H_1 \dot{H}_1 + 2x_1 x_2 (H_1 \dot{H}_2 + H_2 \dot{H}_1) + 2x_2^2 H_2 \dot{H}_2 \quad (33)$$

where higher order terms in Eq. (33) have been dropped. For a random process

$$\langle \dot{H}_1 \rangle_{Av} = 0 \quad \langle \dot{H}_2 \rangle_{Av} = 0. \quad (34)$$

By the ergodic theorem ensemble averages and time averages are identical. Hence

$$\frac{d}{dt} \langle \Delta W \rangle_{Av} = \frac{1}{2t} (H_1^2 + H_2^2) \quad (35)$$

and

$$\frac{d}{dt} \langle (\Delta W)^2 \rangle_{Av} = \frac{1}{t} (x_1 H_1 + x_2 H_2)^2. \quad (36)$$

This last equation after employing Eqs. (28) and (35) may be written as

$$\frac{d}{dt} \langle (\Delta W)^2 \rangle_{Av} = 2W(0) \frac{d}{dt} \langle \Delta W \rangle_{Av} + \frac{1}{2} (X_1^2 - X_2^2) (H_1^2 - H_2^2) + 2X_1 X_2 H_1 H_2. \quad (37)$$

In Appendix A it is shown that the terms  $H_1^2 - H_2^2$  and  $H_1 H_2$  have zero mean values. Hence Eq. (37) becomes

$$\frac{d}{dt} \langle (\Delta W)^2 \rangle_{Av} = 2W \frac{d}{dt} \langle \Delta W \rangle_{Av} \quad (38)$$

where the evaluation of  $W$  at  $t = 0$  is considered to be the value of  $W$  at  $t$  on a time scale in which appreciable diffusion occurs.

The Fokker-Planck equation, after employing Eq. (38) becomes

$$\frac{\partial \rho}{\partial t} = \left[ \frac{d}{dt} \langle \Delta W \rangle_{Av} \right] \frac{\partial}{\partial W} \left( W \frac{\partial \rho}{\partial W} \right). \quad (39)$$

Clearly it is useful to introduce a new time variable  $w$  such that

$$w = \int_0^t \left[ \frac{d}{dt} \langle \Delta W \rangle_{Av} \right] dt. \quad (40)$$

Then Eq. (39) becomes

$$\frac{\partial \rho}{\partial w} = \frac{\partial}{\partial W} \left( W \frac{\partial \rho}{\partial W} \right). \quad (41)$$

which has a fundamental solution.

$$\rho = \frac{1}{w} e^{-\frac{W}{w}} \quad (42)$$

with the properties

$$\int_0^{\infty} \rho dW = 1 \quad (43)$$

and

$$\int_0^{\infty} W \rho dW = w. \quad (44)$$

Since the time variation of  $W$  averaged over the distribution, here taken initially as a  $\delta$ -function, is a measure of the growth in phase space associated with the particles, a determination of  $w$  is the significant quantity to be found. To this end, Eqs. (35) and (40) give

$$w = \frac{1}{2} \int_0^t \frac{1}{t} (H_1^2 + H_2^2) dt \quad (45)$$

thereby reducing the problem to a determination of  $H_1^2 + H_2^2$ .

If the individual contributions to the fluctuations are independent, one may consider each one in turn and add the results. Of course, if feedback is employed to correlate  $\Delta\omega_R$  with  $\Delta\omega_{RF}$ , the problem is more complex and is not considered here. For convenience, let

$$F \dot{s} \equiv f \Delta V \quad \text{or} \quad f = \frac{\sin \phi_R}{2\pi h}, \quad (46)$$

$$G_B \dot{s} \equiv g_B \Delta B \quad \text{or} \quad g_B = \frac{h\omega_R}{\gamma_T^2 B} \quad (47)$$

and

$$G_{RF} \dot{s} \equiv g_{RF} \Delta\omega_{RF} \quad \text{or} \quad g_{RF} = 1 \quad (48)$$



where  $\dot{s}$  is given by Eq. (6)

Appendix A gives for the contribution to  $H_1^2 + H_2^2$  due to cavity voltage fluctuations

$$\left(H_1^2 + H_2^2\right)_{\text{cav}} = \pi \int_0^t \beta(t) f^2(t) J_{\text{cav}}(\Omega) dt \quad (49)$$

where  $J_{\text{cav}}(\Omega)$  is the spectral density of the fluctuation volts per turn (power spectrum). Similarly

$$\left(H_1^2 + H_2^2\right)_{\text{Mag}} = \pi \int_0^t \gamma(t) g_B^2(t) J_{\text{Mag}}(\Omega) dt \quad (50)$$

and

$$\left(H_1^2 + H_2^2\right)_{\text{RF}} = \pi \int_0^t \gamma(t) J_{\text{RF}}(\Omega) dt. \quad (51)$$

In Eqs. (49-51),  $\beta$  and  $\gamma$  are given by Eqs. (23) and (25). The spectral densities  $J_{\text{cav}}(\Omega)$ ,  $J_{\text{Mag}}(\Omega)$ , and  $J_{\text{RF}}(\Omega)$  are given in the Appendix. The frequency  $\Omega$  is the synchrotron frequency

$$\Omega = \frac{\dot{s}}{\beta}. \quad (52)$$

#### APPLICATION TO BOOSTER

Only the random fluctuation  $\Delta\omega_{\text{RF}}$  is significant in producing a growth in longitudinal phase space area associated with the beam. The function  $\gamma(t)$  characterizing the beam bunch shape assuming a constant bucket area regime<sup>3</sup> is shown in Fig. 1.

For an estimate of the growth let the captured beam have zero phase space area. Then Eq. (44) gives the average value of  $W = E/2\pi$  to be expected at a later time due to random fluctuations in the frequency correcting circuit. Combining Eq. (45) with Eq. (51) and using Eq. (A-23) for the spectral density gives

$$E = 2\pi kT \int_0^t \frac{dt}{t} \int_0^t \gamma \cdot \left| T_{RF}(\Omega) \right|^2 \cdot \text{Real} \{ Z_{RF}(\Omega) \} dt \quad (53)$$

The transfer function depends on the synchrotron frequency  $\Omega$ , but, since  $\gamma$  is peaked at transition, use  $T_{RF}(0)$ . The impedance  $Z_{RF}(\Omega)$  may be considered independent of frequency, hence also use  $Z_{RF}(0)$ . Thus<sup>4</sup> put

$$kT = 5 \times 10^{-21} \text{ J}$$

$$T_{RF}(0) = 2\pi \times 7.5 \text{ MHz/V}$$

$$Z_{RF}(0) = 1 \text{ M}\Omega$$

Appropriate integration using curve in Fig. 1 gives

$$\int_0^{T_{trans}} \frac{dt}{t} \int_0^t \gamma(t) dt = 87 \times 10^{-6} \text{ eV-sec}^2.$$

Hence at transition the beam area has grown by

$$E = .006 \text{ eV-sec.} \quad (\text{one bunch})$$

This is to be compared with an initial beam area of

$$E_{beam} = .02 \text{ eV-sec.} \quad (\text{one bunch})$$

Hence the growth is a significant fraction of the initial beam since numerical studies<sup>5</sup> show that the bucket area is just sufficient to contain the beam.

The missing bunch<sup>6</sup> phenomenon could possibly be explained by postulating randomly distributed central holes in the trapped beam due to the microbunches from the linac beam. Since there are approximately 6 microbunches per booster bunch and the outermost microbunches are expected to be mixed by nonlinear forces, it is possible that randomly distributed holes could be present in the linear region. Subsequent growth of the beam area could leave only the holes for some bunches. This explanation, of course, assumes that all bunches are subject to some loss.

# APPENDIX A. Various Integrals and Nyquist's Theorem

Using Eqs. (19-20) to define  $H_1$  and  $H_2$  one has

$$\begin{aligned} H_1^2 + H_2^2 = & \int_0^S \int_0^S \left[ \eta_1(s_1) \eta_1(s_2) + \eta_2(s_1) \eta_2(s_2) \right] F(s_1) F(s_2) ds_1 ds_2 \\ & - \int_0^S \int_0^S \left[ \eta_1(s_1) J_1(s_2) + \eta_2(s_1) J_2(s_2) \right] F(s_1) G(s_2) ds_1 ds_2 \\ & - \int_0^S \int_0^S \left[ \eta_1(s_2) J_1(s_1) + \eta_2(s_2) J_2(s_1) \right] F(s_2) G(s_1) ds_1 ds_2 \\ & + \int_0^S \int_0^S \left[ J_1(s_1) J_1(s_2) + J_2(s_1) J_2(s_2) \right] G(s_1) G(s_2) ds_1 ds_2. \end{aligned} \quad (A-1)$$

$$\begin{aligned} H_1^2 - H_2^2 = & \int_0^S \int_0^S \left[ \eta_1(s_1) \eta_1(s_2) - \eta_2(s_1) \eta_2(s_2) \right] F(s_1) F(s_2) ds_1 ds_2 \\ & - \int_0^S \int_0^S \left[ \eta_1(s_1) J_1(s_2) - \eta_2(s_1) J_2(s_2) \right] F(s_1) G(s_2) ds_1 ds_2 \\ & - \int_0^S \int_0^S \left[ \eta_1(s_2) J_1(s_1) - \eta_2(s_2) J_2(s_1) \right] F(s_2) G(s_1) ds_1 ds_2 \\ & + \int_0^S \int_0^S \left[ J_1(s_1) J_1(s_2) - J_2(s_1) J_2(s_2) \right] G(s_1) G(s_2) ds_1 ds_2. \end{aligned} \quad (A-2)$$

$$\begin{aligned} 2H_1 H_2 = & \int_0^S \int_0^S \left[ \eta_1(s_1) \eta_2(s_2) + \eta_1(s_2) \eta_2(s_1) \right] F(s_1) F(s_2) ds_1 ds_2 \\ & - \int_0^S \int_0^S \left[ \eta_1(s_2) J_2(s_1) + \eta_2(s_2) J_1(s_1) \right] F(s_1) G(s_2) ds_1 ds_2 \\ & - \int_0^S \int_0^S \left[ \eta_1(s_1) J_2(s_2) + \eta_2(s_1) J_1(s_2) \right] F(s_2) G(s_1) ds_1 ds_2 \\ & + \int_0^S \int_0^S \left[ J_1(s_1) J_2(s_2) + J_1(s_2) J_2(s_1) \right] G(s_1) G(s_2) ds_1 ds_2. \end{aligned} \quad (A-3)$$

Solutions of Eqs. (12-13) consistent with Eq. (14) may be taken as

$$\eta_1 = \sqrt{\beta} \sin \int_0^s \frac{ds}{\beta}, \quad J_1 = \frac{1}{\sqrt{\beta}} \left[ \cos \int_0^s \frac{ds}{\beta} - \alpha \sin \int_0^s \frac{ds}{\beta} \right] \quad (A-4)$$

and

$$\eta_2 = \sqrt{\beta} \cos \int_0^s \frac{ds}{\beta}, \quad J_2 = -\frac{1}{\sqrt{\beta}} \left[ \sin \int_0^s \frac{ds}{\beta} + \alpha \cos \int_0^s \frac{ds}{\beta} \right]. \quad (A-5)$$

Since F and G are considered to be random variables having no correlation, all the integrals involving products of F and G average to zero. The remainder, after using Eqs. (A-4) and (A-5) become

$$\begin{aligned} H_1^2 + H_2^2 = & \int_0^s \int_0^s \sqrt{\beta(s_1)\beta(s_2)} F(s_1)F(s_2) \cos \int_{s_1}^{s_2} \frac{ds}{\beta} \cdot ds_1 ds_2 \\ & + \int_0^s \int_0^s \frac{G(s_1)G(s_2)}{\sqrt{\beta(s_1)\beta(s_2)}} \left\{ \left[ 1 + \alpha(s_1)\alpha(s_2) \right] \cos \int_{s_1}^{s_2} \frac{ds}{\beta} \right. \\ & \left. + \left[ \alpha(s_1) - \alpha(s_2) \right] \sin \int_{s_1}^{s_2} \frac{ds}{\beta} \right\} ds_1 ds_2, \end{aligned} \quad (A-6)$$

$$\begin{aligned} H_1^2 - H_2^2 = & - \int_0^s \int_0^s \sqrt{\beta(s_1)\beta(s_2)} F(s_1)F(s_2) \cos \left( \int_0^{s_1} \frac{ds}{\beta} + \int_0^{s_2} \frac{ds}{\beta} \right) ds_1 ds_2 \\ & + \int_0^s \int_0^s \frac{G(s_1)G(s_2)}{\sqrt{\beta(s_1)\beta(s_2)}} \left\{ \left[ 1 - \alpha(s_1)\alpha(s_2) \right] \cos \left[ \int_0^{s_1} \frac{ds}{\beta} + \int_0^{s_2} \frac{ds}{\beta} \right] \right. \\ & \left. - \left[ \alpha(s_1) + \alpha(s_2) \right] \sin \left[ \int_0^{s_1} \frac{ds}{\beta} + \int_0^{s_2} \frac{ds}{\beta} \right] \right\} ds_1 ds_2, \end{aligned} \quad (A-7)$$

and

$$\begin{aligned}
 2H_1H_2 = & \int_0^s \int_0^s \sqrt{\beta(s_1)\beta(s_2)} F(s_1)F(s_2) \sin \left[ \int_0^{s_1} \frac{ds}{\beta} + \int_0^{s_2} \frac{ds}{\beta} \right] ds_1 ds_2 \\
 & - \int_0^s \int_0^s \frac{G(s_1)G(s_2)}{\sqrt{\beta(s_1)\beta(s_2)}} \left\{ \left[ 1 - \alpha(s_1)\alpha(s_2) \right] \sin \left[ \int_0^{s_1} \frac{ds}{\beta} + \int_0^{s_2} \frac{ds}{\beta} \right] \right. \\
 & \quad \left. + \left[ \alpha(s_1) + \alpha(s_2) \right] \cos \left[ \int_0^{s_1} \frac{ds}{\beta} + \int_0^{s_2} \frac{ds}{\beta} \right] \right\} ds_1 ds_2. \quad (A-8)
 \end{aligned}$$

It is expected that  $F$  and  $G$  are each autocorrelated only for  $s_1 \approx s_2$ . For  $s_1$  near  $s_2$  Eq. (A-6) gives a finite result. Notice, however, that Eqs. (A-7) and (A-8) contain rapidly varying trigonometric terms for  $s_1$  and  $s_2$  and, because of this, have zero means. Hence,

$$H_1^2 - H_2^2 = 0, \quad (A-9)$$

and

$$H_1H_2 = 0. \quad (A-10)$$

The contribution to  $H_1^2 + H_2^2$  from  $F$  becomes, after setting  $t_2 = t_1 + \tau$  and letting the limits on  $\tau$  be  $\pm\infty$  since only the contributions near  $\tau = 0$  are significant,

$$\begin{aligned}
 (H_1^2 + H_2^2)_{\text{cav}} = & \int_{-\infty}^{\infty} d\tau \cos \Omega\tau \int_0^t \sqrt{\beta(t_1)\beta(t_2)} f(t_1)f(t_1+\tau) \cdot \\
 & \Delta V(t_1)\Delta V(t_1+\tau) dt_1. \quad (A-11)
 \end{aligned}$$

Note the change in independent variable from  $s$  back to  $t$  and the use of Eq. (47) in expressing  $F$ . Also note that the synchrotron frequency  $\Omega$

$$\Omega = \frac{\dot{s}}{\beta} \quad (\text{A-12})$$

has been introduced and that because  $t_2 \approx t_1$

$$\int_{t_1}^{t_2} \Omega dt \approx \Omega \tau. \quad (\text{A-13})$$

Since the inner integral in Eq. (A-11) is the product of a smooth function of time, say  $R(t)$ , with a random function  $\Delta V(t)\Delta V(t+\tau)$ , it may be evaluated as follows. Designate the integral by  $I$ . Then, breaking up the integral into time slots  $\Delta t$  that are large compared with  $\tau$  but small compared with the total excursion of  $t$ , one may write

$$I = \int R(t_i) \Delta t \cdot \frac{1}{\Delta t} \int_{t_i}^{t_i+\Delta t} \Delta V(t) \Delta V(t+\tau) dt. \quad (\text{A-14})$$

However, the correlation function of the fluctuating voltage is

$$C(\tau) = \frac{1}{\Delta t} \int_{t_i}^{t_i+\Delta t} \Delta V(t) \Delta V(t+\tau) dt \quad (\text{A-15})$$

independent of  $t_i$  and  $\Delta t$  by virtue of attributes of the random process. Thus, Eq. (A-14) becomes

$$I = C(\tau) \int_0^t R(t) dt. \quad (\text{A-16})$$

Accordingly,

$$\left( H_1^2 + H_2^2 \right)_{\text{cav}} = \pi \int_0^t \beta(t) f^2(t) J_{\text{cav}}(\Omega) dt, \quad (\text{A-17})$$

where

$$J_{\text{cav}} = \frac{1}{\pi} \int_{-\infty}^{\infty} C(\tau) \cos \Omega \tau d\tau \quad (\text{A-18})$$

is the power spectrum<sup>5</sup> of the fluctuating voltage.

Nyquist's theorem<sup>7</sup> states that the power spectrum is equal to

$$J_{\text{cav}}(\Omega) = \frac{2}{\pi} kT R_{\text{cav}}(\Omega) \quad (\text{A-19})$$

where  $R_{\text{cav}}(\Omega)$  is the real part of the impedance looking back into the sum of all the cavity voltages,  $k$  is Boltzmann's constant, and  $T$  is the absolute temperature.

The contribution to  $H_1^2 + H_2^2$  from that part of  $G$  due to magnetic field fluctuations  $\Delta B$  evaluated in a manner similar to that of Eq. (A-11) is

$$\left( H_1^2 + H_2^2 \right)_{\text{Mag}} = \pi \int_0^t \gamma(t) g_B^2(t) J_{\text{Mag}}(\Omega) dt, \quad (\text{A-20})$$

where

$$J_{\text{Mag}}(\Omega) = \frac{2}{\pi} kT \cdot \left| T_B(\Omega) \right|^2 \text{Real} \{ Z(\Omega) \} \quad (\text{A-21})$$

is the spectral density of the fluctuating magnetic field. Here  $T_B(\Omega)$  is the transfer function<sup>8</sup> between the azimuthally averaged



magnetic field and the excitation voltage and  $Z(\Omega)$  is the impedance<sup>6</sup> of the entire ring of magnets.

Finally, the contribution to  $H_1^2 + H_2^2$  from that part of  $G$  due to low level RF frequency  $\Delta\omega_{RF}$  is

$$\left(H_1^2 + H_2^2\right)_{RF} = \pi \int_0^t \gamma(t) g_{RF}^2(t) J_{RF}(\Omega) dt \quad (A-22)$$

where

$$J_{RF}(\Omega) = \frac{2}{\pi} kT \cdot \left| T_{RF}(\Omega) \right|^2 \cdot \text{Real} \left\{ Z_{RF}(\Omega) \right\} \quad (A-23)$$

is the spectral density of the fluctuating RF angular frequency. The transducer frequency response varies<sup>4</sup> from  $2\pi \times 10$  MHz/V at injection to  $2\pi \times 3$  MHz/V at full energy. The impedance  $Z_{RF}(\Omega)$  is about 1 M $\Omega$  and is obtained by observing<sup>4</sup> an rms noise voltage of 40  $\mu$ V in the 75 kHz bandwidth circuit.

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6. A.G. Ruggiero, Missing Bunches in the Booster, EXP-6, March 10, 1972.
7. F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw Hill Book Co., New York, 1965, p. 587.
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# BOOSTER LONGITUDINAL PHASE SPACE PARAMETER

$T_{MAX} = 8 \text{ GeV}$

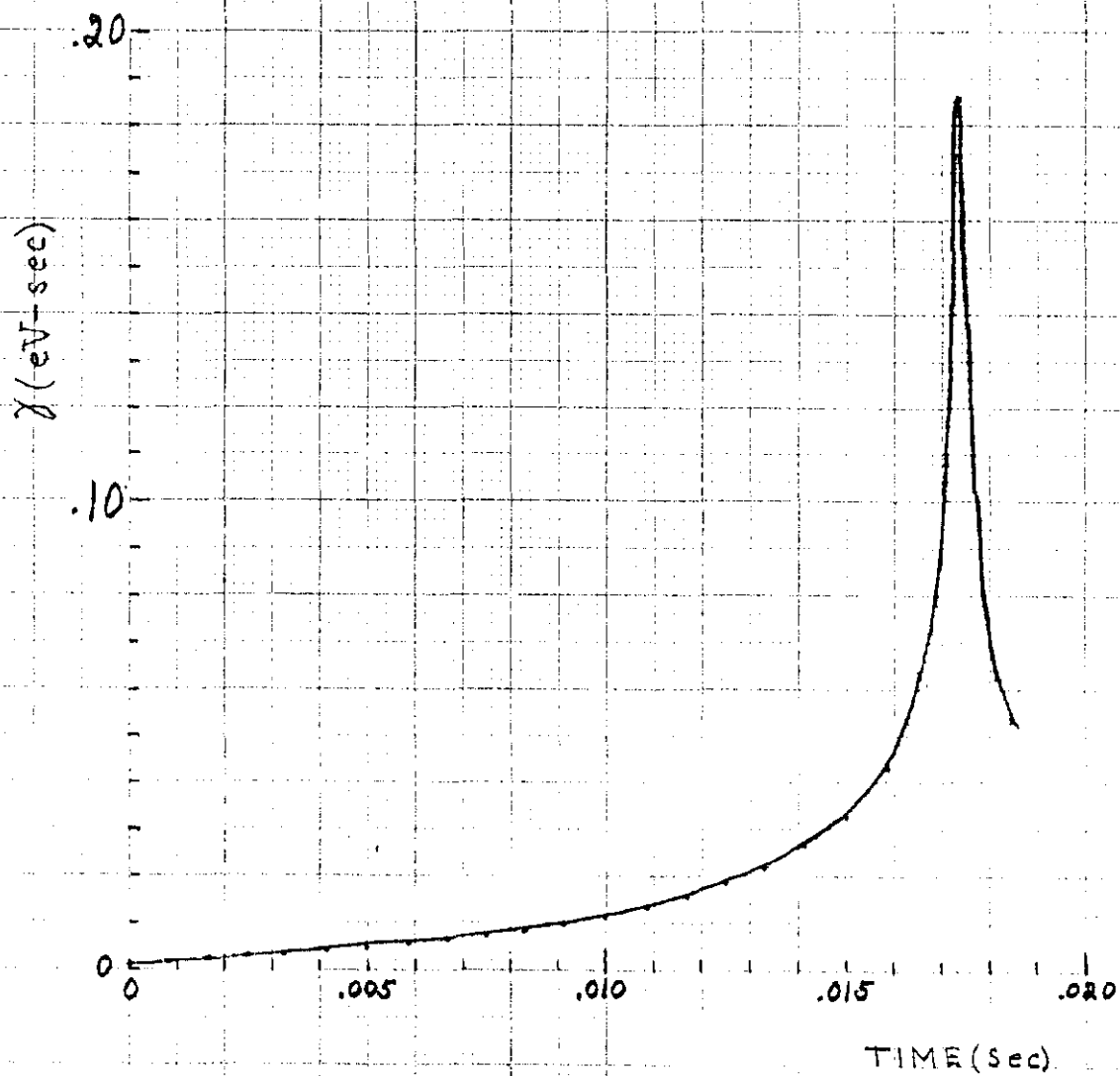


FIG. 1