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ν-SPREAD IN A PROTON BEAM DUE TO TRANSVERSE SPACE CHARGE FIELDS

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SUMMARY

For a circular beam with uniform density the transverse space charge forces are linear.[1,2] The effect of the linear space charge is just a reduction $\Delta\nu$ of the transverse oscillation wave number. With nonuniform density distribution in addition to a reduction $\Delta\nu$ of the average ν-value, one expects the nonlinear space charge forces to produce a ν-spread $\delta\nu$. In many cases[3,4] a ν-spread is desirable for supplying Landau damping to transverse instabilities.

The ν-spread can be enhanced by the insertion of a thin beam of particles of the opposite charge in the middle of the original beam.[5] Because of the dependence of the space charge force on the inverse power of γ we can expect a low-γ thin beam of opposite charge to produce a large ν-spread. The ν-spreads in a nonuniform beam and its enhancement by the beam of opposite charge are computed in this report.

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1. NONLINEAR SPACE CHARGE FORCES IN PROTON BEAM

Let us consider an infinitely long straight beam of protons with uniform distribution in the longitudinal direction and gaussian distribution in the transverse plane. We derive the potential function from Buffet and Potaux[6] for the special case that the beam has circular geometry. We have

$$V(x,y) = V_0 + \lambda_p e \sum_{k=1}^{\infty} \left(\frac{x^2+y^2}{\sigma_p^{\ 2}} \right)^k \frac{(-1)^k}{k \cdot k!} \tag{1}$$

with

- (x,y) = transverse cartesian orthogonal coordinates with the origin on the proton beam axis
- V_0 = the potential at the origin, $x = y = 0$
- e = particle charge
- λ_p = protons per unit length
- σ_p = gaussian standard deviation of the proton distribution in the transverse plane.

Neglecting the edges effect, we can take (1) valid also for a beam bunch. In this case λ_p can be the ratio of the total number of particles to the total length of the bunch.

The vector potential function, $\vec{A}$, is directed as the beam velocity $\vec{v}$ which is supposed to be totally longitudinal. Thus it is

$$A = \frac{v}{c} V = \beta V, \tag{2}$$

and c is the light velocity.

From (1) and (2) we can calculate the electric, $\vec{E}$, and magnetic, $\vec{H}$, fields and, then, the total space charge force $\vec{F}$ per particle,

$$E = -\text{grad } V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \qquad \vec{H} = \text{rot } \vec{A}$$

$$\vec{F} = e\vec{E} + e\frac{\vec{v}}{c} \times \vec{H} .$$

We have for the transverse components

$$F_x = -e(1-\beta^2)\frac{\partial V}{\partial x}$$

$$F_y = -e(1-\beta^2)\frac{\partial V}{\partial y}$$

which take the following form on the midplanes y = 0 and x = 0, respectively

$$F_x\Big|_{y=0} = 2\frac{\lambda_p e^2}{\sigma_p \gamma^2} f\left(\frac{x}{\sigma_p}\right)$$

$$F_y\Big|_{x=0} = 2\frac{\lambda_p e^2}{\sigma_p \gamma^2} f\left(\frac{y}{\sigma_p}\right) \tag{3}$$

where $\gamma^{-2} = (1-\beta^2)$, and the function

$$f(u) = \frac{1-e^{-u^2}}{u} \tag{4}$$

has been plotted in Fig. 2 against positive u. f(u) is an odd function of u.

The equation of transverse motion of a single proton in the presence of space charge forces of the proton beam is, making use of (3),

$$z'' + \nu_0^2\, z - \frac{2\lambda_p e^2}{m_0 \gamma^3 \sigma_p \omega_0^2}\, f\left(\frac{z}{\sigma_p}\right) = 0 \tag{5}$$

where a dash denotes derivative of z with respect to the angular longitudinal coordinate, θ

z = it can be either x or y

ω_0 = beam angular velocity

ν_0 = betatron oscillation number per revolution due to the guiding external forces

m_0 = proton mass at rest.

Observe that (5) has the equilibrium solution $z = 0$ around which all the other solutions oscillate. We assume that the space charge term has a small contribution compared to the term in ν_0^2. Thus let us search for a general solution of (5) having the form

$$z(\theta) = a \sin \psi(\theta). \tag{6}$$

Actually, also, the amplitude a is a (weak) function of θ. But, for our purposes, as we are interested in the betatron oscillation frequency shift, and assuming a and ψ uncoupled, we take a constant.

In the absence of space charge forces a solution for $\psi(\theta)$ is

$$\psi(\theta) = \nu_0\theta + \text{constant}$$

where the constant can be brought to zero with a proper shift of the origin $\theta = 0$. With space charge forces we can write

$$\psi(\theta) = \int \nu(\theta) d\theta$$

$$\nu(\theta) = \nu_0 + \delta_p(a,\theta) \text{ with } |\delta_p(a,\theta)| << \nu_0.$$

Introducing (6) in (5) and neglecting the terms in ψ'' and δ_p^2 give

$$\delta_p(a,\theta) = -\frac{\lambda_p e^2}{m_0\gamma^3\sigma_p^2\omega_0^2\nu_0}\,\frac{f\left(\dfrac{a\sin\nu_0\theta}{\sigma_p}\right)}{\dfrac{a\sin\nu_0\theta}{\sigma_p}}.$$

We define the average of $\delta_p(a,\theta)$ over one turn as

$$\delta_p(a) = \frac{1}{2\pi}\int_0^{2\pi} \delta_p(a,\theta)\,d\theta$$

$$= -\eta_p\, I\left(\frac{a}{\sigma_p},0\right) \qquad (7)$$

with

$$\eta_p = \frac{\lambda_p r_0 R^2}{2\pi\gamma^3\sigma_p{}^2\beta^2\nu_0{}^2} \qquad (8)$$

$$I(w,p) = \int_0^{2\pi\nu_0} \frac{f(w\sin\theta+p) - f(p)}{w\sin\theta}\,d\theta \qquad (9)$$

$r_0 = \frac{e^2}{m_0c^2}$ = classical proton radius = 1.53×10^{-16} cm

R = closed orbit radius.

The function I(w,p) has been plotted in **Fig.** 3 for $\nu_0 = 20.25$ which corresponds to the NAL Main Ring. I(w,p) is an even function of p if ν_0 is an integer number. For large values of ν_0, it is approximately an even function of p and has only a weak dependence on the decimal part of ν_0.

We can define the full ν-spread in the proton beam due to the space charge of the beam itself in the following way

$$\Delta_p = \eta_p[I(0,0) - I(1,0)] \qquad (10)$$

where after inspecting Fig. 3

$$I(0,0) - (1,0) = 25.$$

Let us apply this result to the NAL Main Ring. We have at the injection and for full intensity

$$\lambda_p = 7 \times 10^8 \text{ cm}^{-1}, \qquad \beta = 1.0$$

$$R = 10^5 \text{ cm}, \qquad \gamma = 10$$

$$\nu_0 = 20.25, \qquad \sigma_p = 0.5 \text{ cm}$$

and

$$\eta = 1.7 \times 10^{-3}, \qquad \Delta_p = 0.04.$$

Observe that the ν-shift is negative. It is -0.21 for $a/\sigma_p = 0$ and -0.25 for $a/\sigma_p = 1$.

The above ν-spread is only 40% of the minimum required for beam stabilization against coherent oscillations.[3]

2. EFFECT OF THE NONLINEAR SPACE CHARGE FORCES INDUCED BY AN ELECTRON BEAM

We can expect also a ν-spread in the proton beam due to the space charge forces of an electron beam circulating along the protons for a length ℓ of the accelerator circumference. Assuming the electron beam centered to the proton beam and having a gaussian distribution of standard deviation σ_e, we have now the following ν-shift as a function of the amplitude oscillation a

$$\delta_e(a) = \eta_e I\left(\frac{a}{\sigma_e}, 0\right) \tag{11}$$

with

$$\eta_e = \frac{\lambda_e r_0 R^2}{2\pi\gamma\gamma_e^2\sigma_e^2\beta^2\nu_0^2}\,\frac{\ell}{2\pi R}$$

$$= \frac{\ell}{2\pi R}\,\frac{\lambda e}{\lambda p}\left(\frac{\gamma}{\gamma_e}\right)^2\left(\frac{\sigma_p}{\sigma_e}\right)^2 \eta_p = \alpha\eta_p \tag{12}$$

λ_e = electrons per unit length

γ_e = relativistic energy factor for electrons.

Eq. (11) has been derived taking only the effect of the electron beam averaged over one turn.

The ν-spread in the proton beam now is

$$\begin{aligned}\Delta_e &= \eta_e \left[I(0,0) - I\left(\frac{\sigma_p}{\sigma_e}, 0\right)\right] \\ &= \alpha K \left(\frac{\sigma_p}{\sigma_e}\right) \Delta_p\end{aligned} \tag{13}$$

with

$$K(x) = \frac{I(0,0) - I(x,0)}{I(0,0) - I(1,0)}. \tag{14}$$

The total ν-spread Δ_t in the proton beam is given by the algebraic sum of the partial spreads due to the electrons and the protons, namely, if $\Delta_e > \Delta_p$,

$$\begin{aligned}\Delta_t &= \Delta_e - \Delta_p \\ &= (\alpha K - 1)\Delta_p\end{aligned}$$

from which

$$\alpha K = 1 + \frac{\Delta_t}{\Delta_p}. \tag{15}$$

Let us make application of (15) to the NAL Main Ring. We have $\Delta_p = 0.04$ and we want $\Delta_t = 0.10$, so that it must be

$$\alpha K = 3.5.$$

Let us take $\sigma_p/\sigma_e = 10^{(*)}$. After inspection of Fig. 3 we have

$$K = 4.25 \qquad \text{and} \qquad \alpha = 0.77.$$

Taking an electron beam with kinetic energy E_{kin} very much smaller than rest energy E_0, and $\ell/2\pi R = 10^{-3}$ we obtain

$$\lambda_e = 0.077\ \lambda_p$$
$$= 5.4\ 10^7\ \text{cm}^{-1}$$

which corresponds to an electron current i_e given by

$$i_e = 370 \sqrt{\frac{E_{kin}}{E_0}}\ \text{mA},\ E_{kin} \ll E_0.$$

In the following we shall make more exact calculations considering also the case that the electron beam is out of the center of the proton beam. The space charge forces of both the beams will be taken into account.

3. EXACT CALCULATIONS WITH SPACE CHARGE FORCES DUE TO PROTON BEAMS AND UNCENTERED ELECTRON BEAM

The space charge forces of the proton beam have already been calculated in Section 1 and are given by Eq. (3). Let the electron beam center be sitting at the point of coordinates (x_0, y_0) in the (x,y) proton beam reference frame (see

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*But this number can be even larger if the electron beam is circulating along the proton beam in high β-(Twiss parameter) section where the proton beam size is also larger.

Fig. 2). Let also (x_e, y_e) be the cartesian orthogonal coordinate frame with the origin at the electron beam center. We noticed in the foregoing section we need an electron beam with very small kinetic energy $(\gamma_e \sim 1)$, thus we neglect in the following the electron motion and the associated magnetic field. The scalar potential function is[6]

$$V(x_e, y_e) = V_0 - \lambda_e e \sum_{k=1}^{\infty} \left(\frac{x_e^2 + y_e^2}{\sigma_e^2} \right)^k \frac{(-1)^k}{kk!}$$

and

$$\vec{F} = -e \text{ grad } V.$$

Let us consider only the cases where the electron beam center is lying on either the x or y-plane (respectively, either $y_0=0$ or $x_0=0$). All the other cases can be reduced to these two with a proper rotation of the system for the supposed cylindrical symmetry.

The transverse components of $\vec{F}$ are

$$F_x\Big|_{y=0} = -2\frac{\lambda_e e^2}{\sigma_e} f\left(\frac{x-x_0}{\sigma_e}\right), \text{if } y_0 = 0$$

$$F_y\Big|_{x=0} = -2\frac{\lambda_e e^2}{\sigma_e} f\left(\frac{y-y_0}{\sigma_e}\right), \text{if } x_0 = 0. \tag{16}$$

The equation of motion on the transverse plane of a single proton in the presence of the total space charge forces given by (3) and (16), taking only the average over one turn for the contribution of the electron beam, is

$$z'' + \nu_0^2 z - \frac{2\lambda_p e^2}{m_0 \gamma^3 \sigma_p \omega_0^2} f\left(\frac{z}{\sigma_p}\right) + \frac{\ell}{2\pi R} \frac{2\lambda_e e^2}{m_0 \gamma \sigma_e \omega_0^2} f\left(\frac{z-z_0}{\sigma_e}\right) = 0 \quad (17)$$

where z can be either x or y and z_0, hence, respectively, either x_0 or y_0, while, respectively, either $y_0=0$ or $x_0=0$.

Eq. (17) has the equilibrium solution $z = \bar{z}$ = constant around which all the other solutions oscillate. Observe that $\bar{z}=0$ only when $\lambda_e=0$ or when $\lambda_e \neq 0$ and $z_0=0$. More generally, it is, from (8) and (12),

$$\frac{\bar{z}}{\sigma_e} = 4\pi\eta_p \left\{ \frac{\sigma_p}{\sigma_e} f\left(\frac{\bar{z}/\sigma_e}{\sigma p/\sigma_e}\right) - \alpha f\left(\frac{\bar{z}-z_0}{\sigma_e}\right) \right\} . \quad (18)$$

Thus let us introduce the new variable

$$w = z - \bar{z},$$

which satisfies the following equation

$$w'' + \nu_0^2 w - 4\pi\nu_0^2 \sigma_e \eta_p \left\{ \frac{\sigma_p}{\sigma_e} \left[f\left(\frac{w+\bar{z}}{\sigma_p}\right) - f\left(\frac{\bar{z}}{\sigma_p}\right) \right] + - \alpha \left[f\left(\frac{w+\bar{z}-z_0}{\sigma_e}\right) - f\left(\frac{\bar{z}-z_0}{\sigma_e}\right) \right] \right\} = 0.$$

Let us search for a solution of the above equation with the form

$$w = a \sin \left(\int [\nu_0 + \delta_t(a,\theta)] d\theta \right)$$

with a constant.

Proceeding as in the foregoing section, i.e., neglecting the terms in δ_t' and in δ_t^2, and performing the average of $\delta(a,\theta)$ over one revolution, we obtain

$$\delta_t(a) = -\eta_p \left\{ I\left(\frac{a}{\sigma_p}, \frac{\bar{z}}{\sigma_p}\right) - \alpha I \left(\frac{a}{\sigma_e}, \frac{\bar{z}-z_0}{\sigma_e}\right) \right\} \tag{19}$$

It is reasonable to assume that the closed orbit shift $\bar{z}$ caused by the mutual interaction between the protons and electron beams is only a small fraction of σ_e for not too large values of η_p and α. Thus in the following we shall take z=0 confiding that it does change the result significantly. Then Eq. (19) becomes

$$\delta_t(a) = -\eta_p \left\{ I\left(\frac{a}{\sigma_p}, 0\right) - \alpha I\left(\frac{a}{\sigma_e}, -\frac{z_0}{\sigma_e}\right) \right\} \tag{20}$$

We can immediately see a practical application of (20). For some values of α, σ_p/σ_e and a it is possible to have $\delta_t(a) = 0$.

To see the effect on the total ν-spread in the proton beam due to an out of center electron beam we shall neglect the contribution of the protons at the right hand side of Eq. (20).

Let us again refer to the NAL Main Ring where a minimum ν-spread of 0.1 is required and let us still take $\sigma_p/\sigma_e = 10$.

The minimum α required has been plotted in Fig. 4 against the lateral displacements z_0/σ_e. Taking the same number of the foregoing section and from (12) we have

$$\lambda_e = 0.1\ \alpha\lambda_p = 7.0 \times 10^7\ \alpha\ \text{cm}^{-1},$$

or in terms of electron current i_e

$$i_e = 480\ \alpha \sqrt{\frac{E_{kin}}{E_0}}\ \text{mA},\ E_{kin} \ll E_0.$$

Using this scaling we have the required i_e to achieve a ν-spread of 0.1 with an electron beam of ~5 keV at the right-hand side of Fig. 4.

DISCUSSION

In section 1 we calculated the ν-shift due to the self-fields of a proton with oscillation amplitude a. This is given by Eq. (7), (8) and (9). For a = 0 we have

$$I(0,0) = 2\pi\nu_0$$

and

$$\delta_p(0) = -\frac{\lambda_p r_0 R^2}{\gamma^3 \sigma_p^{\ 2} \beta^2 \nu_0},$$

which is exactly the same as that Kerst and Laslett obtained in free space and for uniform distribution. Thus we can presumably infer that the ν-shift induced by the self-fields does not depend on the particle distribution. But that is not evidently true for the ν-spread. Although we think that a gaussian distribution describes almost all the practice cases, nevertheless, some extra work should be done to inspect how much more important is the particle distribution.

We considered in this paper the ν-shift of those particles executing only perfectly radial oscillations. The ν-shift might be different for the other particles with an ellipse for transverse trajectory and that could yield a different value of the ν-spread in the beam. For the same reason, it is our opinion that the current of the electron beam we calculated in Section 3 for the case of out-of-center electrons might be pessimistic. In fact, we investigated only the contribution of all the protons oscillating in the same direction through the electron beam. If we could take into account also the protons crossing the electron beam from different directions the current to achieve a given ν-spread in the proton beam should be considerably less.

The ideal approach of the problem is to write the exact expression of the space charge forces in any point of the space and to solve at the same time the two equations of motion of a single proton in the x and y directions. But in this way we discover the possibility of coupling between the two modes of oscillation induced by the space charge forces either of the proton beam itself or of an extra electron beam.

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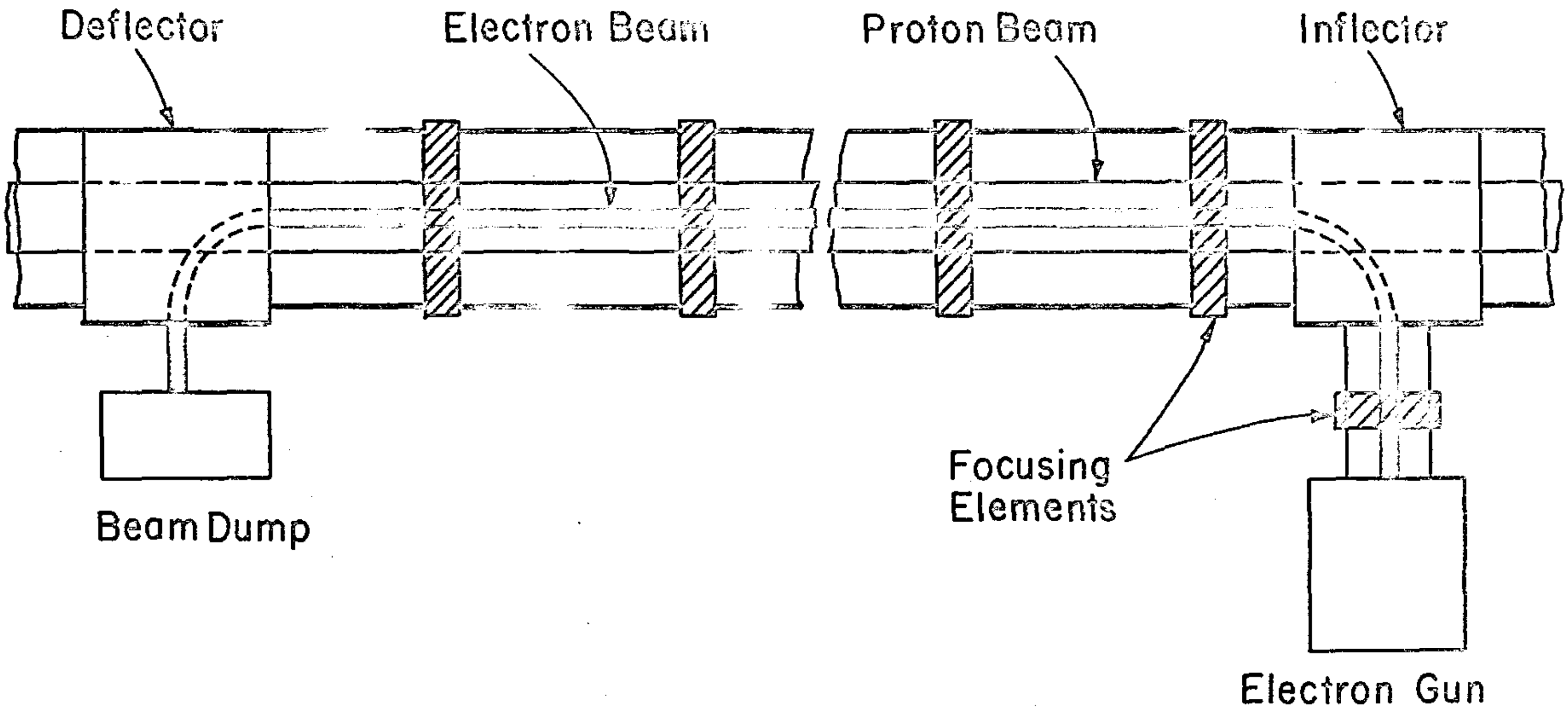
Deflector
Electron Beam
Proton Beam
Inflector
Beam Dump
Focusing
Elements
Electron Gun

Figure 1

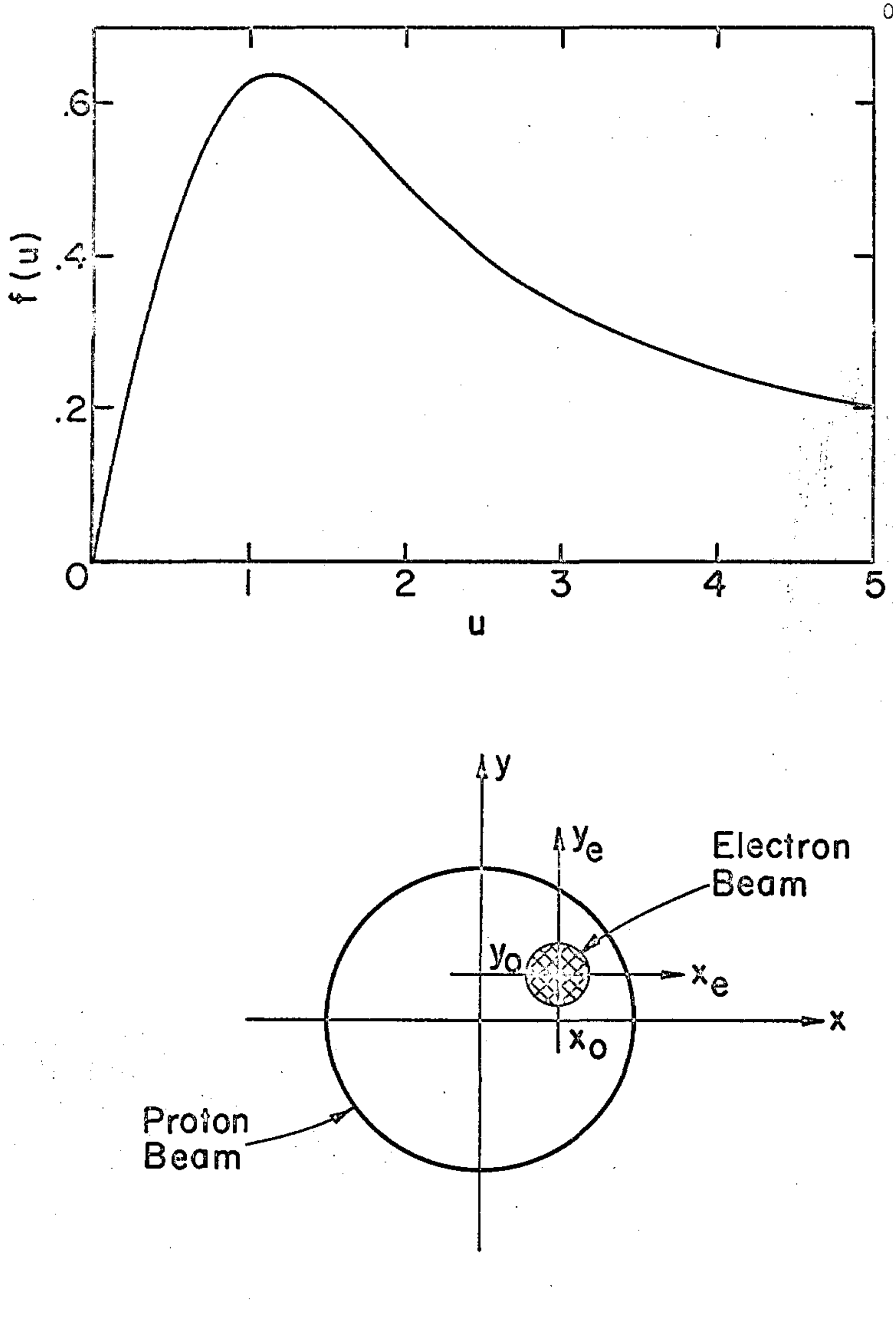

Figure 2

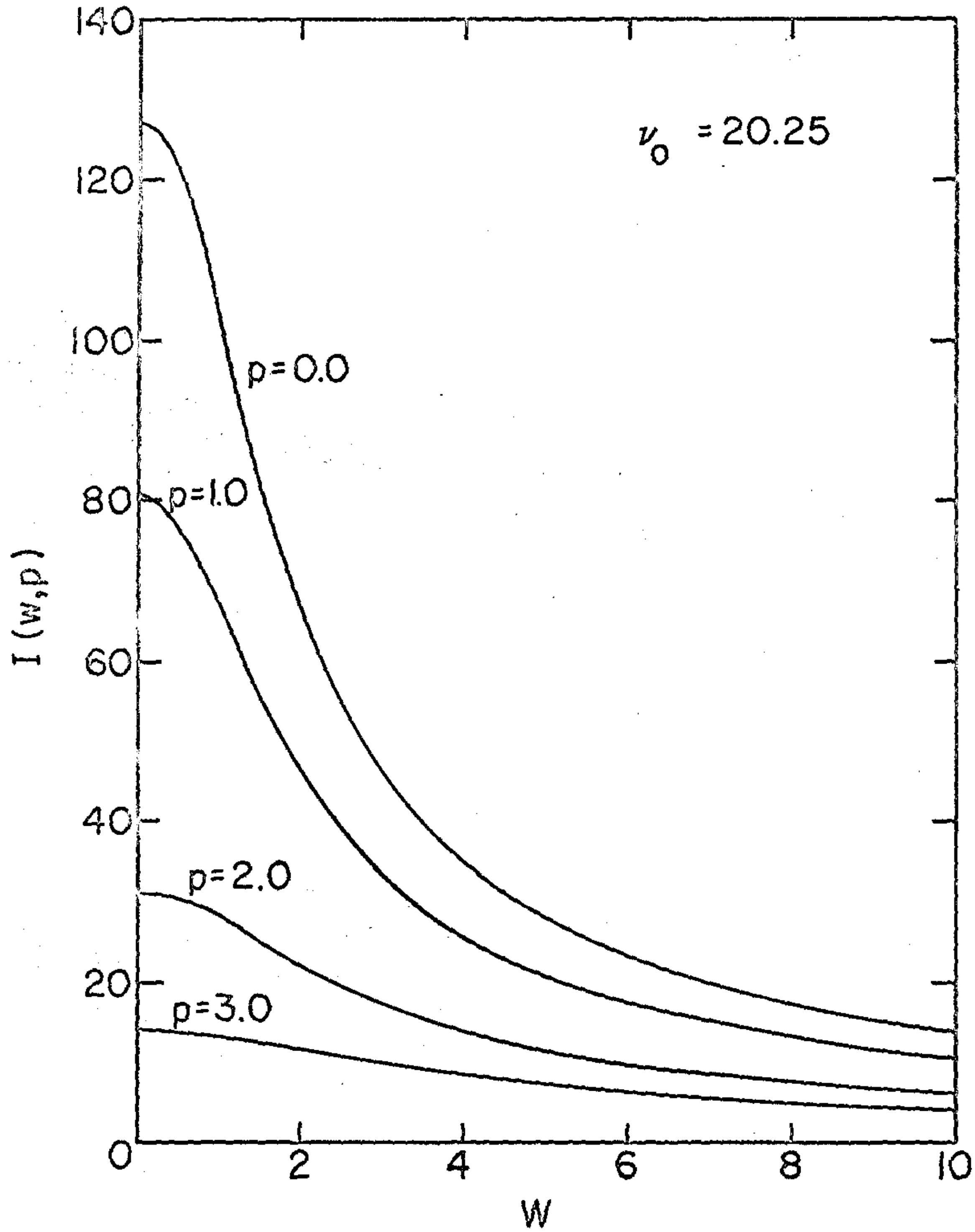

Figure 3

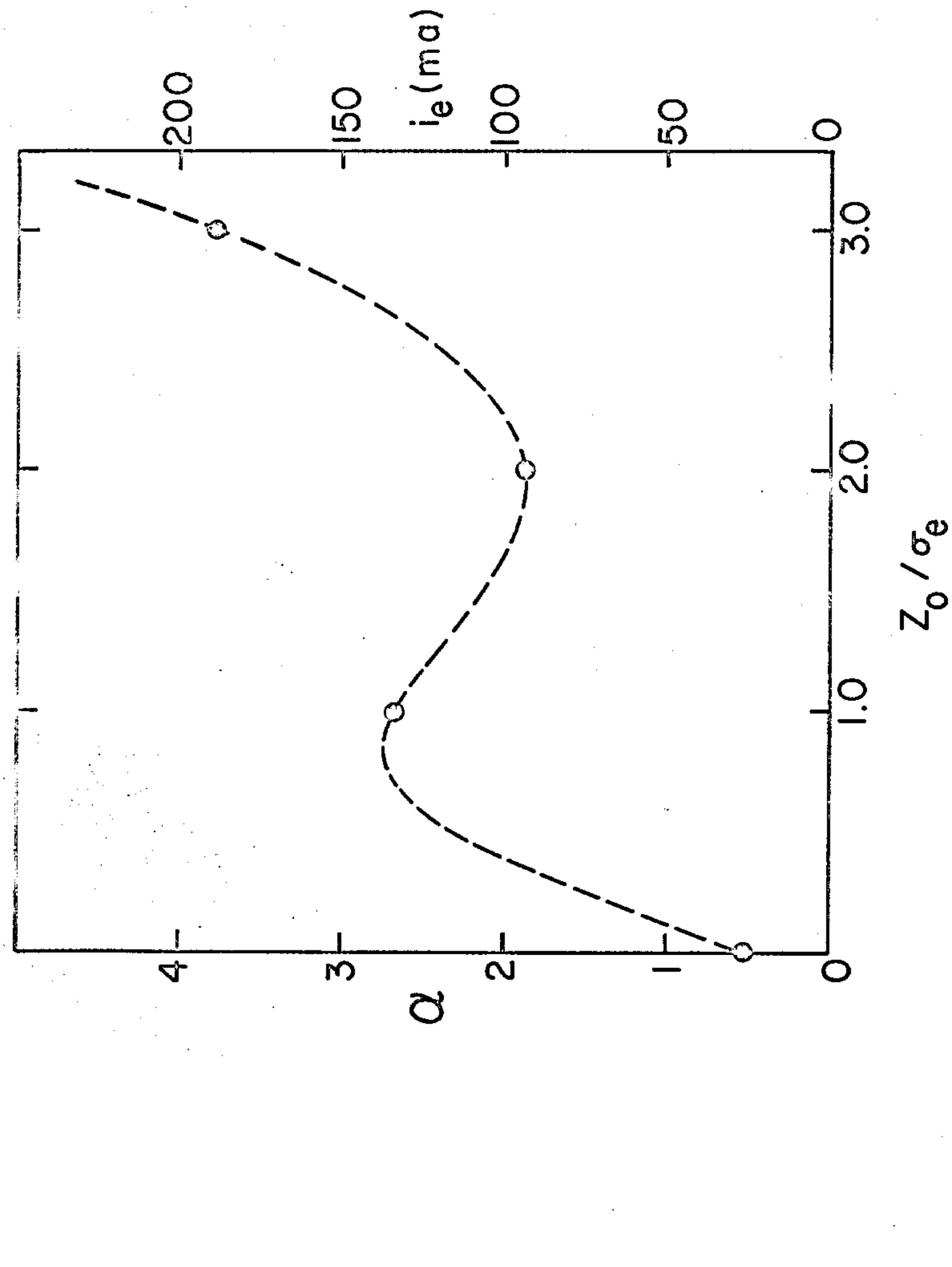

Figure 4