national accelerator laboratory

FN-224
0402

BEAM DEBUNCHING*

A. G. Ruggiero

February 4, 1971

SUMMARY

The beam debunching problem is approached directly by writing down the rms bunch length equation in the presence of space charge forces. Besides the usual reactive beam-wall coupling field, the wake fields caused by the resistivity of the wall material and the RF accelerating cavities are also taken into account.

The rms length equation is integrated for the special case of the NAL main ring with special attention to detuning of the cavities. The result is a debunching criterion which can be easily fulfilled for ordinary operating conditions.

THE RMS AMPLITUDE PHASE OSCILLATION EQUATION

We assume a particle beam composed of h bunches all with the same number of particles N. Let $x = \phi - \phi_s$ be the deviation of the phase angle of a particle from the synchronous value, and $f_k(x,t)$ the particle distribution function of the k-th bunch normalized to unity.

---

*This paper is to be presented at the 1971 Particle Accelerator Conference, March 1-3, 1971, Chicago, Illinois.

If we denote by $\bar{x}_k = \int x f_k(x,t)dx$ the phase of the center of mass of the bunch, according to F. J. Sacherer[1] it is possible to derive a differential equation for the rms length $\hat{y}_k$ of the k-th bunch defined by

$$\hat{y}_k^{\,2} = \int (x-\bar{x}_k)^2 \; f_k(x,t)dx \; .$$

For the phase oscillation in presence of space charge forces the equation can be written as

$$\frac{d^2\hat{y}_k}{dt^2} + \Omega^2\hat{y}_k - \left(\frac{\eta h \omega_s^{\,2}}{\beta^2 E}\right)^2 \frac{S^2}{\hat{y}_k^{\,3}} - e\,\frac{\eta h \omega_s^{\,2}}{\beta^2 E}\,\frac{\overline{yF_k}}{\hat{y}_k} = 0 \tag{1}$$

where

Ω = phase oscillation angular frequency,

ω_s = bunch angular velocity,

h = RF harmonic number and number of bunches,

E = particle total energy,

β = ratio of bunch velocity to light velocity, c,

e = particle charge,

$\eta = \gamma_t^{-2} - \gamma^{-2}$, with

$\gamma^{-2} = 1 - \beta^2$,

γ_t = ratio of transition energy to rest energy,

and S is the rms phase space area of a single bunch. In the following S is supposed to be known and constant in time.

Denoting the rms energy spread by $(\Delta E)_k$, we have for a bunch having the shape of a right ellipse

$$S = \hat{y}_k(\Delta E)_k/\omega_s . \tag{2}$$

The quantity

$$\overline{yF_k} = \int yF_k(y)f_k(y,t)dy$$

with $y = x - \bar{x}_k$ in the last term of (1) gives the effect of the space charge.

The space charge function $F_k(y)$ for a particle in the k-th bunch includes interactions with particles both in the same bunch and in other bunches. If $F_k(y)$ is known, Eq. (1) can be integrated for $\hat{y}_k$ which then shows whether the beam will debunch.

THE SPACE CHARGE FORCES

Let θ be the angular coordinate around the beam orbit of radius R and $\psi(\theta-\omega_s t)$ the longitudinal particle distribution function. Integration of ψ over one bunch is normalized to unity so that integration of ψ over the whole beam gives h.

The beam current is

$$\begin{aligned} I &= Ne\omega_s\psi(\theta-\omega_s t) \\ &= \frac{Ne}{2\pi}\,\omega_s \sum_{n=-\infty}^{+\infty} \psi_n\, e^{in(\theta-\omega_s t)}, \end{aligned}$$

where

$$\psi_n = \int_{-\pi}^{+\pi} \psi(z)e^{-inz}dz\ .$$

Let us introduce the beam-wall coupling impedance per turn Z_n. Then, the electric field associated with I is[2]

$$\xi(\theta-\omega_s t) = -\frac{Ne\omega_s}{4\pi^2 R}\sum_n Z_n\psi_n e^{in(\theta-\omega_s t)} ,$$

The rate of change of energy of a particle at $\theta = \omega_s t + \zeta(t)$ is

$$\dot{E} = \frac{Ne^2}{4\pi^2}\omega_s^{\,2}\sum_n Z_n\psi_n e^{in\zeta}$$

This gives for a particle in the k-th bunch,

$$F_k(\zeta) = \dot{E}/e\omega_s$$

$$= \frac{Ne}{4\pi^2}\omega_s\sum_n Z_n\psi_n e^{in\zeta} \qquad (3)$$

Equation (3) can be evaluated easily for the following three special cases

1. Perfectly conductive wall

For concentric circular vacuum pipe and beam respectively of radius b and a, we have[2,3]

$$Z_n = 2\pi in\,\frac{1+2\lg(b/a)}{\beta c\gamma^2}$$

and

$$F_k(\zeta) = Ne\,\frac{1+2\lg(b/a)}{R\gamma^2}\,\frac{d}{d\zeta}\,\psi(\zeta)$$

2. Resistive wall

For the same concentric circular geometry and denoting by σ the conductivity of the wall material we have[2,3]

$$Z_n = (1-i)\,\frac{R\sqrt{n}}{cb}\sqrt{\frac{2\pi\omega_s}{\sigma}}$$

and

$$F_k(\zeta) = -\frac{Ne}{\pi}\frac{\beta}{b}\sqrt{\frac{\omega_s}{\sigma}}\frac{d}{d\zeta}\int_{\zeta}^{\infty}\frac{\psi(z)dz}{\sqrt{z-\zeta}} \qquad (4)$$

3. Lumped resonator

For a resonator

$$Z_n = \frac{in\omega_s/C}{n^2\omega_s^{\ 2} + in\omega_s/\tau - \omega_o^{\ 2}}$$

where C is the capacitance, τ the damping time and ω_o the resonating angular frequency of the resonator.

If the quality $Q = \tau\,\omega_o$ of the resonator is not too small, we get

$$F_k(\zeta) = \frac{Ne}{2\pi C}\int_{\zeta}^{\infty} e^{-\Gamma h(z-\zeta)}\cos\left[\alpha h(z-\zeta)\right]\psi(z)\,dz \qquad (5)$$

where

$$\Gamma = \alpha/2Q, \qquad \alpha = \omega_o/\omega_s h\ .$$

For a particle in the k-th bunch

$$x = h\zeta,\ \bar{x}_k = h\,\bar{\zeta}_k \text{ and } y = h(\zeta-\bar{\zeta}_k)$$

The function $\psi(\zeta)$ in Equation (5) is the sum of the distributions of the h particle bunches, i.e.

$$\psi(\zeta) = \sum_l \psi_l(\zeta) = h\sum_l f_l(y)$$

Let us calculate $\overline{yF_k}$ for these three special cases keeping the contributions from the k-th bunch and the other bunches separate.

A. Contribution from the k-th bunch

1. Perfectly conductive wall

This is the only contribution for this case.

$$F_k(y) = Ne \frac{1+2\lg(b/a)}{R\gamma^2} h^2 \frac{df_k(y)}{dy}$$

$$\overline{yF_k} = -Ne \frac{1+2\lg(b/a)}{2R\gamma^2} h^2 \frac{\varepsilon_c}{\hat{y}_k} \qquad (6)$$

where we have assumed an $f_k(y)$ symmetric around $y = 0$. The quantity ε_c is a coefficient describing the shape of the distribution. For rectangular distribution: $\varepsilon_c = 1/2\sqrt{3}$.

2. Resistive wall

For this case,

$$F_k(y) = -\frac{Ne}{\pi} \frac{\beta}{b} \sqrt{\frac{\omega_s}{\sigma}}\, h^{3/2} \frac{d}{dy} \int_y \frac{f_k(y')dy'}{\sqrt{y'-y}}$$

which gives, again assuming a symmetric distribution

$$\overline{yF_k} = \frac{Ne}{\pi} \frac{\beta}{b} \sqrt{\frac{\omega_s}{\sigma}}\, h^{3/2} \frac{\varepsilon_r}{\hat{y}_k^{\,1/2}} \qquad (7)$$

where $\varepsilon_r = \sqrt{6\sqrt{3}}/18$ for a rectangular distribution.

3. Resonator

For this case $\overline{yF_k}$ is rather complex even for a rectangular distribution. Without loss of important physical information, we can assume for $f_k(y)$ two delta-functions spaced by $\hat{y}_k$. This gives

$$F_k(y) = \frac{Ne}{2\pi C} \int_y e^{-\Gamma(y'-y)} \cos\left[\alpha(y'-y)\right] f_k(y')dy'$$

and

$$\overline{yF_k} = -\frac{Ne}{8\pi C}\, \hat{y}_k e^{-2\Gamma\hat{y}_k} \cos(2\alpha y_k) \qquad (8)$$

B. Contribution from all other bunches

We shall take delta-functions for the particle distributions for all other bunches on the right-hand side of (4) and (5).

1. Perfectly conductive wall

Because of the absence of wake field there is no contribution of this type in this case.

2. Resistive wall

If $y \ll 2\pi m$, $m = 1,2,\ldots.$

$$F_k(y) = -\frac{Ne}{2\pi}\frac{\beta}{b}\sqrt{\frac{\omega_s}{\sigma}}\, h^{3/2} \sum_{m=1}^{\infty} \frac{1+\frac{3}{2}\frac{y}{2\pi m}}{(2\pi m)^{3/2}}$$

and

$$\overline{yF_k} = -\frac{3}{4}\frac{Ne}{\pi}\frac{\beta}{b}\sqrt{\frac{\omega_s}{\sigma}}\, h^{3/2}\hat{y}_k^{\,2} \sum_{m=1}^{\infty}(2\pi m)^{-5/2} \tag{9}$$

where

$$\sum_{m=1}^{\infty}(2\pi m)^{-5/2} = 0.0136$$

3. Resonating circuit

$$F_k(y) = \frac{Ne}{2\pi C}\sum_{m=1}^{\infty} e^{-\Gamma(2\pi m-y)}\cos\left[\alpha(2\pi m-y)\right]$$

and

$$\overline{yF_k} = \frac{Ne}{2\pi C}\,\hat{y}_k \left\{A(\Gamma,\alpha)\,\text{Sinh}(\Gamma\hat{y}_k)\cos(\alpha\hat{y}_k) + B(\Gamma,\alpha)\,\text{Cosh}(\Gamma\hat{y}_k)\sin(\alpha\hat{y}_k)\right\} \tag{10}$$

where

$$A(\Gamma,\alpha) = \sum_{m=1}^{\infty} e^{-2\pi m\Gamma}\cos(2\pi m\alpha)$$

$$B(\Gamma,\alpha) = \sum_{m=1}^{\infty} e^{-2\pi m\Gamma}\sin(2\pi m\alpha)$$

Equations (8) and (10) can be applied to the case of the RF accelerating cavity for which $\alpha \sim 1$, and we introduce the detuning factor $q = \alpha-1$. Generally, $\Gamma\hat{y}_k$ is very small for all values $\hat{y}_k$ of interest, so (8) becomes

$$\overline{yF_k} = -\frac{Ne}{8\pi C}\,\hat{y}_k\cos(2\hat{y}_k), \tag{11}$$

and, since for $|q|>>1/2Q^2$ or $q = 0$

$$A = \frac{Q/\pi}{1+4q^2Q^2}\,, \qquad B = \frac{2qQ^2/\pi}{1+4q^2Q^2}\,,$$

(10) becomes

$$\overline{yF_k} = \frac{Ne}{4\pi^2 C}\,\hat{y}_k\,\frac{\hat{y}_k\cos\hat{y}_k + 4qQ^2\sin\hat{y}_k}{1+4q^2Q^2} \tag{12}$$

4. <u>Beam Debunching Equations</u>

We assume that at time $t = 0$ the RF cavity is turned off for the beam to debunch. Hence, for $t \geq 0$, $\Omega = 0$. Equation (1) can be written as two first order equations (dropping the index k)

$$\begin{cases} \dfrac{d\hat{y}}{d\tau} = \hat{p} \\[2ex] \dfrac{d\hat{p}}{d\tau} = \dfrac{1}{\hat{y}^3} + Kg(\hat{y}) \end{cases} \tag{13}$$

where τ is the scaled time

$$\tau = \frac{\eta h\omega_s{}^2}{\beta^2 E}\,St, \tag{14}$$

$$K = \frac{Ne^2\beta^2 E}{\eta h\omega_s{}^2 S^2} \tag{15}$$

and g is the function of $\hat{y}$ obtained by summing (6), (7), (9), (11) and (12), namely

$$g(\hat{y}) = -\frac{a_1}{\hat{y}^2} + \frac{a_2}{y^{3/2}} - a_3\hat{y} - a_4\cos(2\hat{y}) +$$

$$+ a_5 \frac{\hat{y}\cos\hat{y} + 4qQ^2\sin\hat{y}}{1 + 4q^2Q^2}$$

where

$$a_1 = h^2 \frac{1 + 2\lg(b/a)}{R\gamma^2} \frac{\varepsilon_c}{2}$$

$$a_2 = h^{3/2} \frac{\beta}{b} \sqrt{\frac{\omega_s}{\sigma}} \frac{\varepsilon_r}{\pi}$$

$$a_3 = \frac{3}{4} \frac{0.0136}{\pi} h^{3/2} \frac{\beta}{b} \sqrt{\frac{\omega_s}{\sigma}}$$

$$a_4 = \frac{1}{8\pi C}, \qquad a_5 = \frac{1}{4\pi^2 C}$$

We assume that at time $t = 0$ the bunches have the shape of right ellipses, so that (2) applies. Denoting the bunch rms length by $\hat{y}_o$ and the rms energy spread by ΔE_o, at $t = 0$, we have for the constant value of S

$$S = \hat{y}_o \Delta E_o / \omega_s,$$

and the quantity $\hat{p}$, which is a measure of the obliquity of the bunch, is taken to be $\hat{p} = 0$ at $t = 0$.

5. Application to the NAL Main Ring

For the NAL Main Ring

$1 + 2\lg(b/a) = 4.5$ $\qquad h = 1113$

$R = 10^5$ cm $\qquad \gamma = 214$

$b = 4$ cm $\qquad \beta = 1.000$

$\sigma = 5.10^{16}$ sec^{-1} $\qquad \omega_s = 3.10^5$ sec^{-1}

$\hat{y}_o = 0.1$

and

C = 16 cm, for a total of 16 cavities and taking 13 ohm for the characteristic impedance of each cavity.

Substituting these parameters we get

$a_1 = 1.8 \cdot 10^{-4}\ \mathrm{cm}^{-1}$

$a_2 = 1.3 \cdot 10^{-3}\ \mathrm{cm}^{-1}$, $a_3 = 7.4 \cdot 10^{-5}\ \mathrm{cm}^{-1}$,

$a_4 = 2.5 \cdot 10^{-3}\ \mathrm{cm}^{-1}$, $a_5 = 1.6 \cdot 10^{-3}\ \mathrm{cm}^{-1}$.

Two possible modes of operation were studied: (a) Leave the RF cavities tuned to the ejection frequency ($q = 0$). (b) Detune the RF cavities back down to the injection frequency ($q = -0.37\ \%$).

The results for mode (b) which turns out to be the more limiting of the two are shown in Figure 1. The rms amplitude $\hat{y}$ is plotted against $\tau/\hat{y}_o$ for several values of K. If we take as debunching criterion that $\hat{y} > \pi$, the results yield the condition

$$K < 58\ \mathrm{cm}$$

or from (15)

$$\frac{N}{(\Delta p/p)^2} < 2.3 \cdot 10^{18}$$

In normal operation the number of particles per bunch is $N = 4.5 \cdot 10^{10}$, and the rms momentum spread $\Delta p/p$ is expected to be $\sim 8 \cdot 10^{-4}$. This gives

$$\frac{N}{(\Delta p/p)^2} = 0.07 \cdot 10^{18}$$

so that there is a very large margin of safety for the beam debunching.

As the actual value of K is much smaller than 58 cm, the debunching time, T, can be calculated using the K = 0 curve. We have from (14) and $\tau/\hat{y}_o = \pi$

$$T = \pi(\eta h\omega_s \Delta p/p)^{-1} = 4.5 \text{ msec}$$

This time can be reduced if the cavities are detuned to a frequency above the ejection frequency to give q = + 1/2Q. But numerical calculation shows that this reduction of T is only ∿ 10%.

REFERENCES

1. F. J. Sacherer, "RMS Envelope Equations with Space Charge," CERN/SI/Int. DL/70-12 (Nov. 1970).

2. A. G. Ruggiero, "Longitudinal Space Charge Forces within Bunched Beams," FN-219-0402, NAL (Dec. 1970).

3. V. K. Neil, A. M. Sessler, Rev. Scient. Instr. Vol. 36, No. 4, p. 429 (April 1965).

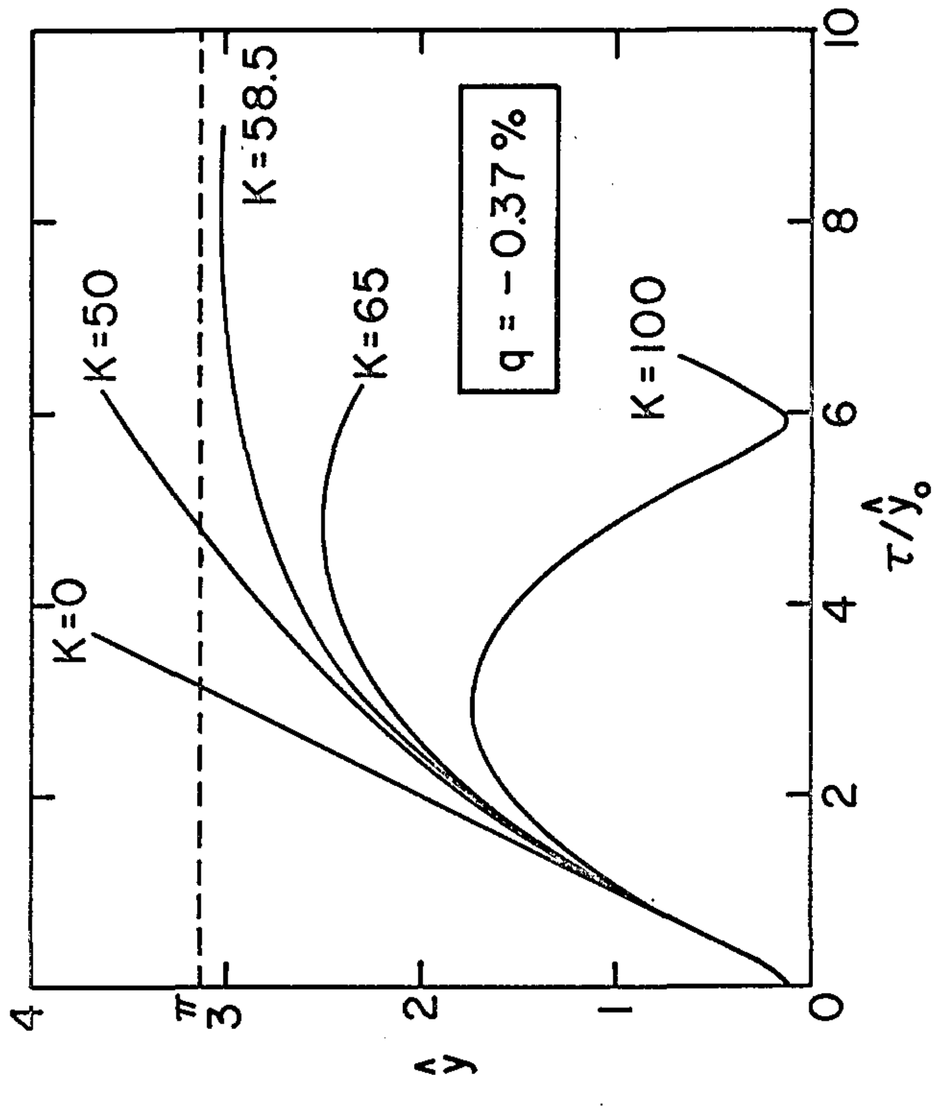

Figure 1