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LONGITUDINAL SPACE CHARGE FORCES WITHIN BUNCHED BEAMS

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SUMMARY

Longitudinal space charge forces within bunched beams are exactly computed in the presence of vacuum chamber wall material with general electric and magnetic properties.

The forces can be split in two terms. The first one does not depend on the wall material properties and corresponds to the usual forces in the presence of a perfectly conductive wall. These forces suffer the magnetic cancellation and hold the γ^{-2} factor. The second term is the effect of the induced current at the wall and of the potential distribution they produce by crossing the equivalent surface characteristic impedance of the material. These kind of forces do not hold any γ-factor, so that their contribution can be large at a very high beam energy.

Application is done to the special case of resistive vacuum chamber wall.

It is shown that the resistive forces are negligible at any frequency and at any energy in the NAL Booster.

It is shown, also, that they can be neglected at the injection and at the transition crossing in the NAL Main Ring, but that they are predominant at the top energy of 200 GeV.

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1. NOTATION

Let us consider the case of one bunch of N particles circulating in a cyclic accelerating machine. We shall denote by R the radius of the bunch orbit.

The beam and the vacuum chamber wall are approximated by infinitely long straight cylinders of radii a and b, respectively.

We shall make use of the cylindric system of coordinates r, ϕ, z, with z along the pipe and beam axis.

The charge density of the beam bunch is written as

$$\rho = \frac{Ne}{\pi a^2} f\ (z-vt)\ Q\ (a-r) \tag{1}$$

where

$$Q\ (x) = 0 \quad \text{for} \quad x < 0$$
$$= 1 \qquad\quad x > 0,$$

e is the particle charge and v the bunch velocity. t is the time.

f(x) is the longitudinal particle distribution function. It is normalized to unity.

We take the vacuum chamber wall with very large thickness so that only the boundary conditions at the inner pipe surface are required. Besides, we take homogeneous wall material with the most general electric properties.

We are interested only in fields with cylindrical symmetry (independent of ϕ). These are E_r, E_z and H_ϕ; whereas $H_z = H_r = E_\phi = 0$. The fields can therefore be expressed in terms of a vector potential which has only a z component. We write

$$A_z\ (r,z,t) = \beta\ V\ (r,z,t)$$
$$A_\phi = A_r = 0 \qquad (2)$$

where β is the ratio of the bunch velocity $\underline{v}$ to the light velocity $\underline{c}$.

In terms of the scalar potential V we have

$$E_z = -\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right) V$$
$$E_r = -\frac{\partial V}{\partial r}$$
$$H_\phi = -\beta\frac{\partial V}{\partial r}\ . \qquad (3)$$

It is advantageous to use the Fourier transform, $\tilde{f}_n$, of the distribtuion function f(x). We have

$$f\ (z-vt) = \frac{1}{2\pi R}\sum_{n=-\infty}^{+\infty}\tilde{f}_n\ e^{i\ (kz-\omega t)}$$

where

$$\tilde{f}_n = \int_{-\pi R}^{\pi R} f(x)\ e^{-in\frac{x}{R}}\ dx$$

and

$$k = n/R,\ \omega = n\ \Omega_0$$

with $\Omega_0 = v/R$ the bunch angular velocity.

In the following we shall denote by the tilde and the index n the Fourier transform of any function in $x = z-vt$.

In particular, we have from Eq. (3)

$$\tilde{E}_{z_n} = -ik\gamma^{-2}\,\tilde{V}_n$$

$$\tilde{E}_{r_n} = -\frac{d\,\tilde{V}_n}{dr} \tag{4}$$

$$\tilde{H}_{\phi_n} = -\beta\,\frac{d\,\tilde{V}_n}{dr}$$

with $\gamma = (1-\beta^2)^{-1/2}$.

2. THE WAVE EQUATION INTEGRATION FOR THE TRANSFORMED POTENTIAL

The wave equation for $\tilde{V}_n$ is

$$\frac{d^2\tilde{V}_n}{dr^2} + \frac{1}{r}\,\frac{d\tilde{V}_n}{dr} - \frac{k^2}{\gamma^2}\,\tilde{V}_n = -\,4\pi\tilde{\rho}_n\ , \tag{5}$$

with

$$4\pi\,\tilde{\rho}_n = \frac{2Ne\tilde{f}_n}{\pi a^2 R}\ \ Q\,(a-r)$$

The solution of (5) which is bounded at $r = 0$ and satisfies the continuity of $\tilde{V}_n$ and $\frac{d\tilde{V}_n}{dr}$ at $r = a$, is

$$\tilde{V}_n = P_n\,I_0\left(\frac{nr}{\gamma R}\right) + \alpha_n\ \ ,\ (r<a)$$

$$\tilde{V}_n = P_n\,I_0\left(\frac{nr}{\gamma R}\right) + \alpha_n\,S_a\left(\frac{nr}{\gamma R}\right),\ (r>a) \tag{6}$$

with

$$\alpha_n = \frac{2Ne}{\pi R} \left(\frac{\gamma R}{na}\right)^2 \tilde{f}_n \tag{7}$$

$$S_a \left(\frac{nr}{\gamma R}\right) = \frac{na}{\gamma R} \left[K_1 \left(\frac{na}{\gamma R}\right) I_0 \left(\frac{nr}{\gamma R}\right) + I_1 \left(\frac{na}{\gamma R}\right) K_0 \left(\frac{nr}{\gamma R}\right)\right] \tag{8}$$

and I_m, K_m are the modified Bessel function of order m and of the first and second kind, respectively.

The constant P_n is to be determined by the boundary condition at the inner vacuum chamber wall, r = b.

We postulate the general electric properties of the wall material can be described by an equivalent surface characteristic impedance ζ_n, at the angular frequency $\omega = n\ \Omega_0$, such that the following fields normalization holds

$$\tilde{E}_{z_n} = - \zeta_n \tilde{H}_{\phi_n}$$

or in terms of $\tilde{V}_n$ from Eq. (4)

$$i \frac{k}{\gamma^2} \tilde{V}_n = - \zeta_n \beta \frac{d\tilde{V}_n}{dr} , \text{ (at } r=b)$$

which gives for P_n

$$P_n = - \alpha_n (\pi_1 - \pi_2) \tag{9}$$

with

$$\pi_1 = S_a \left(\frac{nb}{\gamma R}\right) / I_0 \left(\frac{nb}{\gamma R}\right) \tag{10}$$

$$\pi_2 = T_a \left(\frac{nb}{\gamma R}\right) \pi_1 \tag{11}$$

$$T_a\left(\frac{nb}{\gamma R}\right) = -\,i\zeta_n\beta\gamma\,\frac{\dfrac{I_1\left(\frac{nb}{\gamma R}\right)}{I_0\left(\frac{nb}{\gamma R}\right)} - \dfrac{S_a{}'\left(\frac{nb}{\gamma R}\right)}{S_a\left(\frac{nb}{\gamma R}\right)}}{1-i\zeta_n\beta\gamma\,\dfrac{I_1\left(\frac{nb}{\gamma R}\right)}{I_0\left(\frac{nb}{\gamma R}\right)}} \tag{12}$$

where $S_a{}'$ denotes derivates with respect to the total argument.

3. THE LONGITUDINAL SPACE CHARGE FORCES FOURIER TRANSFORMS

The longitudinal space charge forces within the bunch are, from (4) and (6),

$$\tilde{F}_{z_n} = e\,\tilde{E}_{z_n} = -i\,\frac{k}{\gamma^2}\,e\,\left\{\alpha_n + P_n\,I_0\left(\frac{nr}{\gamma R}\right)\right\}$$

or, from (9),

$$\tilde{F}_{z_n} = -i\,\frac{k}{\gamma^2}\,e\,\tilde{\alpha}_n\left\{1 - (\pi_1 - \pi_2)\,I_0\left(\frac{nr}{\gamma R}\right)\right\}\ .$$

We split $\tilde{F}_{z_n}$ in two terms

$$\tilde{F}_{z_n} = \tilde{F}_{z_n}^{\ I} + \tilde{F}_{z_n}^{\ II}$$

with

$$\tilde{F}_{z_n}^{\ I} = -\,i\,\frac{k}{\gamma^2}\,e\,\alpha_n\left\{1 - \pi_1\,I_0\left(\frac{nr}{\gamma R}\right)\right\} \tag{13}$$

and

$$\tilde{F}_{z_n}^{\,II} = -i\,\frac{k}{\gamma^2}\,e\,\alpha_n\,\pi_2\,I_0\left(\frac{nr}{\gamma R}\right)\,. \tag{14}$$

The first term, $\tilde{F}_{z_n}^{\,I}$, does not depend on ζ_n and never vanishes. In the following it will be referred to as the reactive space charge forces. The second term, $\tilde{F}_{z_n}^{\,II}$, is sensitive on ζ_n, and is zero for the particular case of perfect conductive wall ($\zeta_n=0$). It will be referred to as the dissipative space charge forces. Let us refer now to the special case

$$\frac{nb}{\gamma R} \ll 1$$

for which we can use approximated expressions for the modified Bessel functions. We have

$$S_a\left(\frac{nr}{\gamma R}\right) \simeq 1 - \frac{1}{4}\left(\frac{na}{\gamma R}\right)^2\left\{1+2\,\ln\frac{r}{a} - \left(\frac{r}{a}\right)^2\right\}$$

$$S_a'\left(\frac{nr}{\gamma R}\right) \simeq \frac{1}{2}\left(\frac{nr}{\gamma R}\right)\left\{1-\left(\frac{a}{r}\right)^2\right\}$$

$$I_0(x) \simeq 1 + \frac{x^2}{4} + ----$$

$$I_1(x) \simeq \frac{x}{2} + ---- \,.$$

Thus, inserting (7) and (10) in (13) yields

$$\tilde{F}_{z_n}^{\,I} = -i\,\frac{k}{\gamma^2}\,\frac{Ne^2}{2\pi R}\,\tilde{f}_n\,g(r) \tag{15}$$

with

$$g(r) = 1 + 2\,\ln\frac{b}{a} - \left(\frac{r}{a}\right)^2 .$$

g(r) is a weak function of r for usual values of a and b. In the following it will be kept as constant and simply denoted by g.

For the computation of $\tilde{F}_{z_n}{}^{II}$ we insert Eqs. (10) and (12) in (11) and Eqs. (7) and (11) in (14).

$$\tilde{F}_{z_n}{}^{II} = -i\, \frac{k}{\gamma^2}\, \frac{Ne^2}{2\pi R}\, \mathcal{J}_n\, \tilde{f}_n \tag{16}$$

with

$$\mathcal{J}_n = -2i\zeta_n\beta\, \frac{\gamma^2 R}{nb} \cdot \frac{1 - \frac{1}{4}\left(\frac{nb}{\gamma R}\right)^2 g(0)}{1 - \frac{1}{4}\left(\frac{na}{\gamma R}\right)^2 g(b)} \cdot \frac{1 - \frac{1}{4}\left(\frac{na}{\gamma R}\right)^2 g(r)}{1 - i\zeta_n\beta\, \frac{nb}{2R}} \tag{17}$$

and

$$g(b) = 1 + 2\ \ln \frac{b}{a} - \left(\frac{b}{a}\right)^2 .$$

If g(b) does not take too large value, and $|\zeta_n|$ is enough small in order to verify the condition

$$|\zeta_n|\ \beta\ \frac{nb}{2R} << 1\ ,$$

(17) can be replaced by the following simpler equation

$$\mathcal{J}_n = -2i\zeta_n\ \beta\ \frac{\gamma^2 R}{nb} \quad . \tag{18}$$

Inspection of Eqs. (15) and (16) shows that $\tilde{F}_{z_n}{}^{I}$ has a linear γ^{-2} dependence that $\tilde{F}_{z_n}{}^{II}$ has not. That means the <u>reactive forces</u> are predominant at relative low energy but there is an energy above which the <u>dissipative forces</u> become more important. This energy is given by equating (15) and (16),

$$\gamma \approx \sqrt{\frac{|n| g b}{2 \beta R |\zeta_n|}} \; .$$

A second result is the following. The reactive forces are linear in the frequency, whereas the frequency response of the dissipative forces is the same of the material intrinsic impedance ζ_n.

4. APPLICATION TO THE RESISTIVE WALL

Let us consider the case of wall material with finite conductivity σ. σ is supposed to be so high as to be always much larger than any frequency $\omega = n\Omega_0$ of interest. If the magnetic permeability is close to 1, we can define the material resistivity in the following way

$$\mathcal{R} = \sqrt{\frac{\omega}{8\pi\sigma}}$$

and the resistive wall characteristic impedance as

$$\zeta = (1-i)\, \mathcal{R} \; . \tag{19}$$

Comparison between the reactive and the dissipative forces can be easily done by inspecting the ratio $\mathcal{F}_n/g$. From (18) and (19) we have

$$\frac{\mathcal{F}_n}{g} = -(1+i) \frac{G}{\sqrt{n}} \quad , \tag{20}$$

with

$$G = \frac{\gamma^2}{g} \sqrt{\frac{\beta^3 c R}{2\pi\sigma b^2}} \; . \tag{21}$$

The resistive forces could be predominant at small frequencies, but their contribution is reduced at very high frequencies.

The total space charge forces are obtained by antitransforming $\tilde{F}_{z_n}^{\ I}$ and $\tilde{F}_{z_n}^{\ II}$.

We have

$$F_z^{\ I} = \sum_n \tilde{F}_{z_n}^{\ I} e^{in\frac{x}{R}} \qquad (22)$$

$$F_z^{\ II} = \sum_n \tilde{F}_{z_n}^{\ II} e^{in\frac{x}{R}} \qquad (23)$$

From (15) and (22) it is easy to obtain

$$F_z^{\ I} = -Ne^2(1-\beta^2)\ (1+2\ \ln \frac{b}{a})\ \frac{df}{dx} \qquad (24)$$

The reactive forces are linearly depending on the local particle density.

From (16) and (23) we have

$$F_z^{\ II} = -(1-i)\ \frac{Ne^2}{2\pi R} \sqrt{\frac{\beta^3 c/b^2}{2\pi R\sigma}} \sum_n \sqrt{n}\ \tilde{f}_n\ e^{in\frac{x}{R}} .$$

It can be shown that

$$\frac{d}{dx}\ B(x) = \frac{d}{dx} \int_x^\infty \frac{f(z)}{\sqrt{z-x}}\ dz$$

$$= -(1-i)\sqrt{\frac{\pi}{2R}} \sum_n \sqrt{n}\ \tilde{f}_n\ e^{in\frac{x}{R}}$$

so that

$$F_z^{II} = Ne^2 \frac{\beta}{\pi b} \sqrt{\frac{\beta c}{\sigma}} \frac{d}{dx} B(x) \qquad (25)$$

This force depends on the amount of charge ahead of the point in consideration.

Usually a parabolic particle distribution was taken for f(x) which leads to an expression for F_z^I linear in x.

Let us continue to use this approximation. We have

$$f(x) = \frac{6}{L^3} \left[\left(\frac{L}{2}\right)^2 - x^2\right]$$

which inserted in (24) gives

$$F_z^I = 12 \frac{Ne^2}{L^3} (1-\beta^2) (1+2 \ln\frac{b}{a}) x.$$

Neglecting the effect of several bunches, we have for B(x)

$$B(x) = \frac{6}{L^3} \int_x^{L/2} \frac{L^2/4-z^2}{\sqrt{z-x}} dz$$

$$= \frac{4}{L^3} \sqrt{\frac{L}{2} - x} \; \frac{3L^2-8x^2-2xL}{5}$$

from which

$$\frac{dB(x)}{dx} = - \frac{2}{L^3} \frac{2xL+L^2-8x^2}{\sqrt{\frac{L}{2} - x}}$$

Inserting this in (25) we obtain

$$F_z^{II} = -Ne^2 \frac{\beta}{\pi b} \frac{2}{L^3} \sqrt{\frac{\beta c}{\sigma}} \frac{2xL+L^2-8x^2}{\sqrt{\frac{L}{2} - x}}$$

5. APPLICATION TO THE NAL CYCLIC ACCELERATORS

a) Booster

$a = 0.5$ cm

$b = 3$ cm

$g = 4.6$

$R = 75.4717$ m

$\sigma = 5 \cdot 10^{16}$ sec^{-1}

from which and Eq. (21)

$$G = \gamma^2 \sqrt{\beta^3}\ 1.95 \times 10^{-3}$$

and in particular

$G = 0.001$ at injection, $\gamma = 1.2132$

$G = 0.055$ at transition, $\gamma = 5.373$

$G = 0.175$ at ejection, $\gamma = 9.5264$.

Inserting these values in (20), it results that the resistive forces are always much smaller than the reactive forces, at any energy and at any frequency of interest. So that they can be neglected in any further investigation where the total longitudinal space charge forces are required.

b) Main Ring

$a = 0.5$ cm

$b = 3$ cm

$g = 4.6$

$R = 1000.$ m

$\sigma = 5 \cdot 10^{16}$ sec^{-1}

from which

$$G = \gamma^2 \sqrt{\beta^3}\ 7.11 \times 10^{-3}$$

and in particular

$G = 0.64$ at injection, $\gamma = 9.5264$

$G = 2.75$ at transition, $\gamma = 19.612$

$G = 327.$ at ejection, $\gamma = 214.2$

The resistive forces are very small at the injection in the Main Ring at any frequency of interest. At the transition energy they contribute weakly and only at low frequencies. But they are comparable and even larger than the usual space charge forces in presence of perfectly conductive wall, at the top energy of 200 GeV.

In Fig. 1 the longitudinal electric field per unit of charge is plotted for the three cases of injection, transition and ejection energy. The bunch half-length L/2 and γ are listed beside each plot. The internal scaled longitudinal coordinate 2x/L is in the abscissa, the electric field per unit charge E/Ne in the ordinate. The continuous line corresponds to the reactive beam-wall field, the dashed line to the resistive field component.

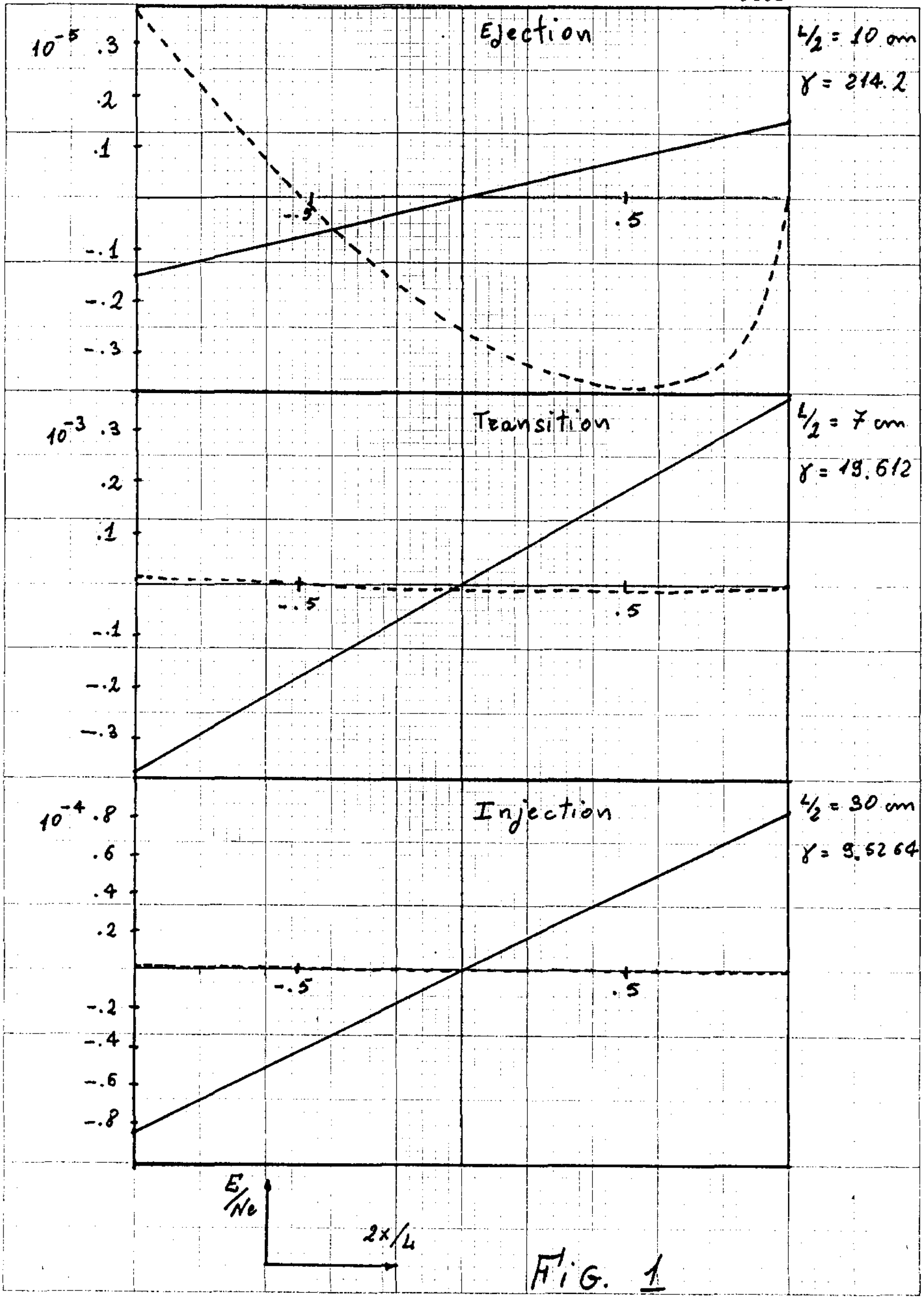
Ejection
10^{-5}
L/2 = 10 cm
γ = 214.2
Transition
10^{-3}
L/2 = 7 cm
γ = 19.612
Injection
10^{-4}
L/2 = 30 cm
γ = 9.5264
E/Ne
2x/L

FIG. 1