



## TRANSFORMATIONS USEFUL IN LINEAR BETATRON THEORY

S. C. Snowdon

May 8, 1969

## PURPOSE

To find a contact transformation in two degrees of freedom that effects the Courant-Snyder transformation simultaneously for both degrees of freedom. Subsequently, to find a contact transformation that transforms to action-angle variables in two degrees of freedom.

## EQUATIONS OF MOTION

Betatron motion in the linear approximation with median plane symmetry is uncoupled and given by:

$$\frac{d^2 q_x}{d s^2} + K_x(s) q_x = 0 \quad (1)$$

and

$$\frac{d^2 q_y}{d s^2} + K_y(s) q_y = 0 \quad (2)$$

## CONTACT TRANSFORMATION

Equations (1) and (2) may be derived from the hamiltonian

$$H = \frac{1}{2} p_x^2 + \frac{1}{2} K_x q_x^2 + \frac{1}{2} p_y^2 + \frac{1}{2} K_y q_y^2, \quad (3)$$

where  $(p_x, q_x)$  and  $(p_y, q_y)$  are canonically conjugate variables for each degree of freedom.

To execute the first transformation, choose a linear transformation represented by the following generator

$$F_2 (P_x, q_x, P_y, q_y; s) = \frac{1}{\sqrt{\beta_x}} P_x q_x + \frac{\beta_x'}{4\beta_x} q_x^2 + \frac{1}{\sqrt{\beta_y}} P_y q_y + \frac{\beta_y'}{4\beta_y} q_y^2, \quad (4)$$

where  $(P_x, Q_x)$  and  $(P_y, Q_y)$  are the new canonical momenta and coordinates. Also  $\beta_x$  and  $\beta_y$  are the periodic functions of  $s$  introduced by Courant and Snyder<sup>1</sup> which possess the properties:

$$\frac{1}{2} \beta_x'' \beta_x - \frac{1}{4} \beta_x'^2 + \beta_x K_x = 1, \quad (5)$$

and

$$\frac{1}{2} \beta_y'' \beta_y - \frac{1}{4} \beta_y'^2 + \beta_y K_y = 1. \quad (6)$$

Equation (4) results in the coordinate transformation

$$p_x = \frac{1}{\sqrt{\beta_x}} (P_x + \frac{1}{2} \beta_x' Q_x) \quad (7)$$

$$q_x = \sqrt{\beta_x} Q_x \quad (8)$$

$$p_y = \frac{1}{\sqrt{\beta_y}} (P_y + \frac{1}{2} \beta_y' Q_y) \quad (9)$$

$$q_y = \sqrt{\beta_y} Q_y, \quad (10)$$

and the new hamiltonian becomes

$$K = H + \frac{\partial F_2}{\partial s} = \frac{1}{2\beta_x} (P_x^2 + Q_x^2) + \frac{1}{2\beta_y} (P_y^2 + Q_y^2). \quad (11)$$

To see that the customary equations of motion result,

one finds from Eq. (11)

$$\frac{dQ_x}{ds} = \frac{P_x}{\beta_x}, \quad \frac{dQ_y}{ds} = \frac{P_y}{\beta_y}, \quad (12)$$

and

$$\frac{dP_x}{ds} = - \frac{Q_x}{\beta_x} \quad , \quad \frac{dP_y}{ds} = - \frac{Q_y}{\beta_y} \quad . \quad (13)$$

Then, if one introduces:

$$\xi = \frac{1}{v_x} \int \frac{ds}{\beta_x} \quad , \quad \eta = \frac{1}{v_y} \int \frac{ds}{\beta_y} \quad (14)$$

as is done by Courant and Snyder<sup>1</sup>, one has

$$\frac{d^2 Q_x}{d\xi^2} + v_x^2 Q_x = 0 \quad , \quad (15)$$

and

$$\frac{d^2 Q_y}{d\eta^2} + v_y^2 Q_y = 0 \quad . \quad (16)$$

#### ACTION-ANGLE VARIABLES

Finally, one may remove the s-dependence of the hamiltonian in Eq. (11) by transforming to the action-angle variables

$(\rho_x, \phi_x, \rho_y, \phi_y)$ . Consider first only one degree of freedom.

Take the generator of the form

$$F_1(Q, \phi; s) = \frac{1}{2} Q^2 f(\phi, s) \quad . \quad (17)$$

Then

$$P = Qf, \quad \rho = - \frac{1}{2} Q^2 \frac{\partial f}{\partial \phi} \quad , \quad (18)$$

and the new hamiltonian is

$$W = \rho \frac{\left( \frac{f^2 + 1}{\beta} + \frac{\partial f}{\partial s} \right)}{\left( - \frac{\partial f}{\partial \phi} \right)} \quad . \quad (19)$$

Since the desire is to have  $W$  independent of  $s$ , choose  $f(\phi, s)$  such that

$$\frac{f^2+1}{\beta} + \frac{\partial f}{\partial s} + C \frac{\partial f}{\partial \phi} = 0. \quad (20)$$

The characteristics of Eq. (20) are solutions of

$$\frac{ds}{\beta} = \frac{d\phi}{C\beta} = - \frac{df}{1+f^2} \quad (21)$$

which gives

$$\phi - Cs = C_1 ; \text{Cot}^{-1}f - \int \frac{ds}{\beta} = C_2 . \quad (22)$$

A general solution of Eq. (20) is then

$$f(\phi, s) = \text{Cot} \left[ g(\phi - Cs) + \int \frac{ds}{\beta} \right] , \quad (23)$$

where  $g$  is an arbitrary function. Any definite choice of  $g$  selects a particular canonical set  $(\rho, \phi)$  that is suitable as an action-angle pair. To see that this is true, let  $(\rho_1, \phi_1)$  be the pair generated by  $F_1$  of Eqs. (17) and (22). Transform to the pair  $(\rho_2, \phi_2)$  using the generator

$$F_2(\rho_2, \phi_1, s) = \rho_2 \left[ Cs + g(\phi_1 - Cs) \right] . \quad (24)$$

Then

$$\rho_1 = \rho_2 g'(\phi_1 - Cs) , \quad (25)$$

$$\phi_2 = Cs + g(\phi_1 - Cs) , \quad (26)$$

and the new hamiltonian

$$U = C \rho_2 g' + \rho_2 \left[ C - Cg' \right] = C \rho_2 . \quad (27)$$

But, from Eqs. (17) and (22)

$$P = \sqrt{\frac{2\rho_1}{g}} \text{Cos} \left( g + \int \frac{ds}{\beta} \right) \quad (28)$$

$$Q = \sqrt{\frac{2\rho_1}{g}} \text{Sin} \left( g + \int \frac{ds}{\beta} \right) , \quad (29)$$

which, using Eqs. (25) and (26) becomes

$$P = \sqrt{2\rho_2} \text{Cos} \left( \phi_2 - Cs + \int \frac{ds}{\beta} \right) \quad (30)$$

$$Q = \sqrt{2\rho_2} \text{Sin} \left( \phi_2 - Cs + \int \frac{ds}{\beta} \right) . \quad (31)$$

Thus there is no loss of generality in choosing

$$g = \phi - Cs \quad (32)$$

in Eq. (23).

The relation of the constant C to the tune  $\nu$  may be found by noticing that, since the hamiltonian is

$$W = C\rho , \quad (33)$$

$d\phi/ds = C$  or  $2\pi\nu = Cx$  circumference. If the circumference is written as  $2\pi R$  where R is the average radius

$$C = \frac{\nu}{R} . \quad (34)$$

Thus

$$W = \frac{\nu}{R} \rho . \quad (35)$$

With two degrees of freedom the corresponding generator becomes

$$F_1(Q_x, \phi_x, Q_y, \phi_y; s) = \frac{1}{2} Q_x^2 \text{Cot} \psi_x + \frac{1}{2} Q_y^2 \text{Cot} \psi_y, \quad (36)$$

where

$$\psi_x = \phi_x - \frac{\nu_x}{R} s + \int \frac{ds}{\beta} , \quad (37)$$

and

$$\psi_Y = \phi_Y - \frac{v_Y}{R} s + \int \frac{ds}{\beta} . \quad (38)$$

This generator results in the transformation

$$P_X = \sqrt{2\rho_X} \cos \psi_X , \quad (39)$$

$$Q_X = \sqrt{2\rho_X} \sin \psi_X , \quad (40)$$

$$P_Y = \sqrt{2\rho_Y} \cos \psi_Y , \quad (41)$$

$$Q_Y = \sqrt{2\rho_Y} \sin \psi_Y , \quad (42)$$

and the new hamiltonian becomes

$$W = \frac{v_X}{R} \rho_X + \frac{v_Y}{R} \rho_Y . \quad (43)$$

In summary the hamiltonian representing linear betatron motion with median plane symmetry has been successively transformed with the generators  $F_2$  and  $F_1$  to yield the hamiltonian in Eq. (43) which is independent of  $s$ . The overall transformation of the momenta and coordinates is

$$p_X = \sqrt{\frac{2\rho_X}{\beta_X}} \left( \cos \psi_X + \frac{\beta'_X}{2} \sin \psi_X \right) , \quad (44)$$

$$q_X = \sqrt{2\beta_X \rho_X} \sin \psi_X , \quad (45)$$

$$p_Y = \sqrt{\frac{2\rho_Y}{\beta_Y}} \left( \cos \psi_Y + \frac{\beta'_Y}{2} \sin \psi_Y \right) , \quad (46)$$

$$q_Y = \sqrt{2\beta_Y \rho_Y} \sin \psi_Y . \quad (47)$$

The hamiltonian  $W$ , being independent of  $s$  is an invariant. This invariant expressed in terms of the original canonical momenta and coordinates is

$$\begin{aligned}
W = & \frac{v_x}{2R\beta_x} \left\{ q_x^2 + \left( -\frac{\beta'_x}{2} q_x + \beta_x p_x \right)^2 \right\} \\
& + \frac{v_y}{2R\beta_y} \left\{ q_y^2 + \left( -\frac{\beta'_y}{2} q_y + \beta_y p_y \right)^2 \right\}. \quad (48)
\end{aligned}$$

Since  $p_x = q'_x$  and  $p_y = q'_y$ , Eq. (48) is seen to have the form given by Courant and Snyder<sup>1</sup>. In fact, as pointed out by Courant<sup>2</sup>, the transformation from H to W which gives rise to the momenta and coordinate transformation of Eqs. (44) to (47) may be accomplished by a single generator.

$$\begin{aligned}
F_1 (q_x, \phi_x, q_y, \phi_y; s) = & \frac{q_x^2}{2\beta_x} (\cot \psi_x + \frac{1}{2} \beta'_x) \\
& + \frac{q_y^2}{2\beta_y} (\cot \psi_y + \frac{1}{2} \beta'_y). \quad (49)
\end{aligned}$$

#### REFERENCES

1. E. D. Courant and H. S. Snyder, *Annals of Physics* 3, 1 (1958).
2. E. D. Courant, private communication.