# UNIFORM FIELD FROM DISTRIBUTION OF CURRENTS ON AN ELLIPSE

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## Purpose

To determine the distribution of currents on the surface of an ellipse that will produce a uniform field within the ellipse. Extention to a gradient field is also given.

## Reference

1. W. R. Smythe, Static and Dynamic Electricity, McGraw-Hill Book Co., Inc., New York (1950).

## Coordinate System

Conformal transformations[1] suggest that the variables (u,v) are useful when dealing with problems having an elliptical boundary.

$$x = a \sin u \,\mathrm{Ch}\, v \qquad y = a \cos u \,\mathrm{Sh}\, v \tag{1}$$

The factor for displacement is

$$h_x = h_y \equiv h = a\sqrt{\mathrm{Sh}^2 v + \cos^2 u} \tag{2}$$

## Potentials (Uniform Field)

Since one desires a uniform inside the ellipse one chooses

$$\Phi_i = -B_0 y = -B_0 a \cos u \,\mathrm{Sh}\, v \tag{3}$$

The transformation of Eq. (1), being conformal gives for the Laplace equation

$$\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0. \tag{4}$$

A solution appropriate to the external region is

$$\Phi_0 = A\ a \cos u\ e^{-v} \tag{5}$$

Boundary Conditions

The normal component of the flux density is continuous. This gives

$$\left(\frac{\partial \Phi_i}{\partial v}\right)_{v = v_0} = \left(\frac{\partial \Phi_0}{\partial v}\right)_{v = v_0}, \tag{6}$$

where from Eq. (1) $v = v_0$ is seen to generate an ellipse. Thus

$$A = B_0 e^{v_0} \text{Ch } v_0. \tag{7}$$

Applying the Ampere circuital law to a small region spanning $v = v_0$ and assuming a surface current of density $\sigma$ exists on the surface gives

$$\left(H_{u0} - H_{ui}\right)_{v = v_0} h \Delta u = 4 \pi \sigma h \Delta u. \text{ (emu)} \tag{8}$$

or

$$\sigma = \frac{1}{4\pi}\left(\frac{\partial \Phi_i}{h \partial u} - \frac{\partial \Phi_0}{h \partial u}\right)_{v = v_0} \tag{9}$$

or

$$\sigma = \frac{B_0}{4\pi} \cdot e^{v_0} \cdot \frac{\text{Sin } u}{\sqrt{\text{Sh}^2 v_0 + \cos^2 u}} \quad \text{(emu)} \tag{10}$$

Potentials (Gradient Field)

One desires to have inside the ellipse

$$\Phi_i = -B'_0 x\, y = -\frac{a^2 B'_0}{4} \sin 2u \text{ Sh } 2v \tag{11}$$

Outside the ellipse there are no sources. Hence

$$\Phi_0 = 1/4\, a^2 A \sin 2u\, e^{-2v} \tag{12}$$

Matching the normal fields on the elliptical boundary gives

$$A = B'_0 e^{2v_0} \mathrm{Ch}\, 2\, v_0 . \tag{13}$$

The surface current density is then determined from the discontinuity in the tangential component of the field.

$$\sigma = - \frac{aB'_0}{8\pi} e^{2v_0} \frac{\mathrm{Cos}\, 2\, u}{\sqrt{\mathrm{Sh}^2 v_0 + \cos^2 u}} \qquad \text{(emu)} \tag{14}$$